New Elements of Analysis on the Levi-Civita Field

By

Khodr Mahmoud Shamseddine

AN ABSTRACT OF A DISSERTATION

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Professor Martin Berz

ABSTRACT

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New elements of analysis on the Levi-Civita field $\mathcal{R}$ are presented. First we prove general results about skeleton groups and field automorphisms that will enhance the understanding of the structure of the field. We show that while the identity map is the only field automorphism on R, there can be nontrivial automorphisms on non-Archimedean field extensions of R like $\mathcal{R}$. We also show that every automorphism on $\mathcal{R}$ is order preserving and that if P is such an automorphism and r a real number then $P(r)$ is approximately equal to r; moreover, if q is a rational number, then $P(q) = q$.

After reviewing the algebraic, order, and topological structures of the field $\mathcal{R}$ [3, 5, 7], we review two types of convergence and prove new results about the convergence of the sums and products of sequences and infinite series. A weak convergence criterion [5] for power series is then enhanced and proved, and we show that power series can be reexpanded around any point of their domain of convergence. Knowledge of weak convergence of power series allows the extension to the new field and the study of all transcendental functions. This also will allow the extension of all the real functions that can be represented on a computer and is thus of great importance for the implementation of the $\mathcal{R}$ calculus on computers [38, 39, 40, 43].

We review two different definitions of continuity and differentiability [5, 10]. We

show that these smoothness criteria are preserved under addition, multiplication and composition of functions. We show with several examples that topological continuity and differentiability are not sufficient to assure that a function be bounded or satisfy any of the common theorems of real calculus on a closed interval of $\mathcal{R}$. We derive a result which allows for an easy check of the differentiability of functions. Then, based on the stronger concept of differentiability, we present a detailed study of a large class of functions for which we generalize the intermediate value theorem in [5] and prove an inverse function theorem.

Based on our knowledge of convergence of power series, we study a large class of functions which are given locally by power series with $\mathcal{R}$ coefficients and which generalize the normal functions discussed in [5]. We show that the so-called expandable functions [41, 42] form an algebra and have all the nice properties of real power series. In particular, they satisfy the intermediate value theorem, the maximum theorem and the mean value theorem. Moreover, they are infinitely often differentiable and integrable; and the derivative functions of all orders are themeselves expandable functions.

The existence of infinitely small numbers in the non-Archimedean field $\mathcal{R}$ allows the use of the old numerical algorithm for computing derivatives of real functions, but now with an error that in a rigorous way can be shown to become infinitely small (and hence irrelevant). Using calculus on $\mathcal{R}$, we formulate a necessary and sufficient condition for the derivatives of real functions representable on a computer to exist at any given real point, and we show how to accurately compute the derivatives up to very high orders if they exist, even when the coding exhibits branch points or nondifferentiable pieces [38, 39, 40, 43].

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To my parents Latifa and Mahmoud

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"When you work you are a flute through whose heart the whispering of the hours turns to music." Kahlil Gibran.

"A human being is a part of the whole called by us universe, a part limited in time and space. He experiences himself, his thoughts and feelings as something separated from the rest, a kind of optical delusion of his consciousness. This delusion is a kind of prison for us, restricting us to our personal desires and to affection for a few persons nearest to us. Our task must be to free ourselves from this prison by widening our circle of compassion to enhance all living creatures and the whole of nature in its beauty." Albert Einstein.

"Be the change you want to see in the world." Mahatma Gandhi.

"I ask you one thing: do not tire of giving, but do not give your leftovers. Give until it hurts, until you feel the pain." Mother Teresa.

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Contents

1 Introduction **1**

1.1 Motivation 1
1.2 Outline 4
1.3 Intuitive Remarks 8
1.4 Applications in Physics 9
1.5 Implementation 13
1.6 Notations 18

2 Skeleton Groups and Field Automorphisms **19**

2.1 Skeleton Groups 19
2.2 Field Automorphisms 23

3 The Non-Archimedean Field $\mathcal{R}$ **32**

3.1 Algebraic Structure 32
3.2 Differential Algebraic Structure 43
3.3 Order Structure 45
3.4 Topological Structure 49

4 Sequences and Series **57**

4.1 Strong Convergence 57
4.2 Weak Convergence 69
4.3 Power Series 82
4.4 Transcendental Functions 89

5 Calculus on $\mathcal{R}$ **98**

5.1 Topological Continuity and Topological Differentiability 99
5.2 Continuity and Differentiability 115
5.3 n-times Differentiability 128
5.4 Intermediate Value Theorem and Inverse Function Theorem 147

6 Expandable Functions **167**

6.1 Definition and Algebraic Properties 167

6.2 Calculus on the Expandable Functions 174

6.2.1 Intermediate Value Theorem 175

6.2.2 Maximum Theorem and Mimimum Theorem 185

6.2.3 Rolle's Theorem and the Mean Value Theorem 188

6.3 Integration 190

7 Computer Functions **194**

7.1 Introduction 194

7.2 Computer Functions of One Variable 197

7.3 Computation of Derivatives 205

7.4 Examples 211

7.5 Computer Functions of Many Variables 214

Chapter 1

Introduction

1.1 Motivation

The real numbers owe their fundamental role in Mathematics and the sciences to certain special properties. To begin, like all fields, they allow arithmetic calculation. Furthermore, they allow measurement; any result of even the finest measurement can be expressed as a real number. Additionally, they allow expression of geometric concepts, which (for example because of Pythagoras) requires the existence of roots, a property that at the same time is beneficial for algebra. Furthermore, they allow the introduction of certain transcendental functions such as exp, which are important in the sciences and arise from the concept of power series. In addition, they allow the formulation of an analysis involving differentiation and integration, a requirement for the expression of even simple laws of nature.

While the first two properties are readily satisfied by the rational numbers, the geometric requirements demand using at least the set of algebraic numbers. Transcendental functions, being the result of limiting processes, require Cauchy completeness, and it is easily shown that the field of real numbers is the smallest totally ordered field having this property. Because it is at such a basic level of our scientific language, hardly any thought is spent on the fundamental question of whether there may be other useful number systems having the required properties.

This question is perhaps even more intriguing in light of the observation that, while the field of real numbers R and its algebraic completion C as well as the vector space R^n have certainly proven extremely successful for the expression and rigorous mathematical formulation of many physical concepts, they have two shortcomings in interpreting intuitive scientific concepts. First, they do not permit a direct representation of improper functions such as those used for the description of point charges; of course, within the framework of distributions, these concepts can be accounted for in a rigorous fashion, but at the expense of the intuitive interpretation. Second, another intuitive concept of the fathers of analysis, and for that matter quite a number of modern scientists sacrificing rigor for intuition, the idea of derivatives as differential quotients, that is slopes of secants with infinitely small abscissa and ordinate differences cannot be formulated rigorously within the real numbers. Especially for the purpose of computational differentiation, the concept of "derivatives are differential quotients" would of course be a remedy to many problems, since it would replace any attempted limiting process involving the unavoidable cancellation of digits by computer-friendly algebra in a new number system.

The problems mentioned in the preceding paragraphs might be solved if, in addition to the real numbers, there were also "infinitely small" and "infinitely large" numbers; that is if the number system were non-Archimedean. Since any Archimedean Cauchy complete field is isomorphic to R, it is indeed the absence of such numbers that makes the real numbers unique. However, since the "fine structure" of the continuum is not observable by means of science, Archimedicity is not required by nature, and leaving it behind would possibly allow the treatment of the above two concepts. So it appears on the one hand legitimate and on the other hand intriguing to study such number systems, as long as the above mentioned essential properties of the real numbers are preserved.

There are simple ways to construct non-Archimedean extensions of the real numbers (see for example the books of Rudin [36], Hewitt and Stromberg [19], or Stromberg [44], or at a deeper level the works of Fuchs [16], Ebbinghaus et al. [15] or Lightstone and Robinson [29]), but such extensions usually quickly fail to satisfy one or several of the above criteria of a "useful" field, often already regarding the universal existence of roots.

An important idea for the problem of the infinite came from Schmieden and Laugwitz [37], which was then quickly applied to delta functions [21, 23] and distributions [22]. Certain equivalence classes of sequences of real numbers become the new number set, and, perhaps most interesting, logical statements are considered proved if they hold for "most" of the elements of the sequences. This approach lends itself to the introduction of a general scheme that allows the transfer of many properties of the real numbers to the new structure. This method supplies an elegant tool that, in particular, permits the determination of derivatives as differential quotients.

Unfortunately, the resulting structure has two shortcomings. On the one hand, while very large, it is not a field; there are zero divisors, and the ring is not totally ordered. On the other hand, the structure is already so large that individual numbers can never be represented by only a finite amount of information and are thus out of reach for computational problems. Robinson [34] recognized that the intuitive method can be generalized [25] by a nonconstructive process based on model theory to obtain a totally ordered field, and initiated the branch of Nonstandard Analysis. Some of the standard works describing this field are from Robinson [35], Stroyan and Luxemburg [45], and Davis [14]. In this discipline, the transfer of theorems about real numbers is extremely simple, although at the expense of a nonconstructive process invoking the axiom of choice, leading to an exceedingly large structure of numbers and theorems. The nonconstructiveness makes practical use difficult and leads to several oddities; for example, the fact that the sign of certain elements, although assured to be either positive or negative, can not be decided.

Another approach to a theory of infinitely small numbers originated in game theory and was pioneered by John Conway in his marvel "On Numbers and Games" [13]. A humorous and totally nonstandard yet at the same time very insightful account of these numbers can also be found in Donald Knuth's mathematical novelette "Surreal Numbers: How Two Ex-Students Turned to Pure Mathematics and Found Total Happiness" [20]. Other important accounts on surreal numbers are by Alling [1] and Gonshor [17].

1.2 Outline

In this dissertation, new elements of analysis on a different non-Archimedean extension of the real numbers are discussed. The numbers $\mathcal{R}$ were first discovered by the brilliant young Levi-Civita [27, 28] who succeeded in showing that they form a totally ordered field that is Cauchy complete. He concluded by showing that any power series with real coefficients converges for infinitely small arguments and used this to extend real differentiable functions to the field. His number system has subsequently been rediscovered independently by a handful of people, including M. Berz [3, 5, 7], and the subject appeared in the work of Ostrowski [32], Neder [30], and later in the work of Laugwitz [24]. Two modern and rather complete accounts of Levi-Civita's work can be found in the book by Lightstone and Robinson [29], which ends with the proof of Cauchy completeness, and in Laugwitz's account on Levi-Civita's work [26], which also contains a summary of properties of Levi-Civita fields.

In Chapter 2, we prove general results about skeleton groups and field automorphisms, which will serve as an introduction to the field $\mathcal{R}$ and will help in understanding more its structure reviewed in Chapter 3. We show that if a non-Archimedean field extension of R has roots of positive elements, then its skeleton group must contain the rational numbers Q and we show that the skeleton group of $\mathcal{R}$ is Q. This already says something about the uniqueness of $\mathcal{R}$ as a non-Archimedean field extension of R, which is reviewed in Chapter 3 below. We show that every automorphism on $\mathcal{R}$ is order preserving and that, contrary to the real case, there exist nontrivial field automorphisms; however, we show that if P is an automorphism on $\mathcal{R}$ and if q is a rational number and r a real number then $P(q) = q$ and $P(r)$ is approximately equal to r. The results mentioned above are not unique to $\mathcal{R}$ and hold in the same way for any non-Archimedean field extension of R which has roots of positive elements.

In Chapter 3, we review some of the work done by M. Berz in [3, 5, 7, 9]. We begin with questions about the algebraic, order and topological structures of the field and show that $\mathcal{R}$ admits nth roots of positive elements; more so, the field obtained by adjoining the imaginary unit is algebraically closed. It is shown that $\mathcal{R}$ is the smallest totally ordered non-Archimedean field extension of R which is Cauchy complete in the order topology, in which positive elements have roots and in which there exists an infinitely small positive number d such that the sequence (d^n) is null in the order topology. A new topology, complementing the order topology, is introduced, which is useful for the study of power series in Chapter 4.

In the following chapters, we extend the previous work and formulate new aspects of analysis on the Levi-Civita field $\mathcal{R}$. We start in Chapter 4 with a review of convergence of sequences and series with respect to the order and weak topologies which leads to the proof that $\mathcal{R}$ is Cauchy complete in the order topology while it is not in the weak topology [5, 7]. We prove new results on convergence; especially those dealing with the sums and products of sequences and series. We then review and enhance a weak convergence criterion for power series. This allows the extension to $\mathcal{R}$ and a detailed study of all transcendental functions in Section 4.4. This also will allow for the direct use of a large class of functions in Chapters 6 and 7, in particular all the functional dependencies that can be formulated on a von Neuman computer [38, 39, 40, 43].

In Chapter 5, we start by extending and generalizing the calculus developed in [5, 7]. After reviewing topological continuity and topological differentiability, we show that like in any metric space, the family of topologically continuous or differentiable functions at a point or on a domain is closed under addition, multiplication and composition of functions and that if the derivative exists, it must vanish at a local maximum or minimum. Unlike in R, however, we show with examples that these smoothness criteria are not strong enough to guarantee that a function topologically continuous or differentiable on a closed interval of $\mathcal{R}$ satisfy the common theorems of real calculus or even be bounded.

We then review stronger definitions of continuity and differentiability based on the concept of the derivate [10]. We show that these are preserved under operations on functions; and we derive a chain rule and a tool for easily checking the differentiability of functions on intervals of $\mathcal{R}$. Also in this chapter, we generalize the central result in [3, 5, 7, 10] that derivatives are differential quotients after all. This offers a pretty way of doing computational differentiation; see Chapter 7 and [38, 39, 40, 43]. We then review the definition of high order differentiability, the remainder formulas and the domain of strong convergence of the Taylor series for infinitely often differentiable functions [10]. We also study weak convergence of the Taylor series and use that to prove more results about power series; in particular, we show that they can be reexpanded around any point of their domain of convergence. This entails that the functions studied in Chapter 6 are translation invariant. Finally, based on the concept of differentiability mentioned above, we present a detailed study of a large class of functions which will be shown to satisfy an intermediate value theorem and an inverse function theorem.

Using the properties of the weak convergence discussed in Chapter 4 and the smoothness criteria discussed in Chapter 5, we study in Chapter 6 a large class of functions on $\mathcal{R}$ that contains all the continuations of power series from R to $\mathcal{R}$. We show that these expandable functions [41, 42] are closed under composition and arithmetic operations. We also show that for these functions, all the common theorems from real calculus including the intermediate value theorem, the maximum theorem, Rolle's theorem and the mean value theorem hold. Moreover, the expandable functions are infinitely often differentiable and integrable; and the derivative functions of any orders are again expandable functions.

Finally, in Chapter 7, we discuss one of the important applications of the field $\mathcal{R}$; namely, the computation of derivatives of functions representable on a computer [38, 39, 40, 43]. We show that using the calculus on the non-Archimedean field $\mathcal{R}$, it is possible to rigorously decide whether a function representable on a computer is differentiable or not at any given point, and if it is, to accurately determine its derivative, even if the coding exhibits nondifferentiable pieces. Details of an implementation of the method and examples for its use for typical pathological problems are given. Execution times for both standard problems as well as exceptions where conventional methods fail are compared with those obtained using the conventional algorithms.

1.3 Intuitive Remarks

In this section, we present general remarks about the Levi-Civita field $\mathcal{R}$ that will enhance the intuitive understanding of the structure of the field discussed in Chapter 3. Motivated by the need for differentials for applications in Physics (see Section 1.4), we will obtain the differential d in Definition 3.5 and show that every element x of $\mathcal{R}$ can be written as a formal power series of d

$$x = \sum_{q \in Q} x_q d^q, \tag{1.1}$$

in which the powers of d, also called the support points of x, form a left-finite set of rational numbers; that is, below any given rational number t only finitely many powers smaller than t appear in the series in Equation (1.1). We remark here that Equation (1.1) is directly connected to the Hahn theorem [18] which holds for a general totally ordered non-Archimedean field F and which states that every element of F can be written as a formal power series in which the exponents form a well-ordered subset of the skeleton group S_F of the field F; see Chapter 2 for the definition of skeleton groups. That is, every subset of that set of exponents has a minimum. As we will see in Chapter 2, the skeleton group of $\mathcal{R}$ is $S_{\mathcal{R}} = Q$, the field of rational numbers.

Because of left-finiteness, the set of rational powers in Equation (1.1) can be arranged as an ascending divergent sequence, and we can rewrite Equation (1.1) as

$$x = \sum_{n=1}^{\infty} x_n d^{q_n} \text{ with } q_{j_1} < q_{j_2} \text{ if } j_1 < j_2, \tag{1.2}$$

where convergence occurs with respect to the order topology of $\mathcal{R}$. We remark here that Equation (1.2) is proved directly in Chapter 3 without reference to the Hahn theorem, but using the left-finiteness of the support points and the properties of the order topology. John Conway also proved directly a similar result for his surreal numbers [13].

The fact that the powers of d are rational numbers, rather than integers, is necessary for the positive elements of $\mathcal{R}$ to have roots in $\mathcal{R}$; and this is used to show that $\mathcal{R}$ is algebraically closed [5]. The left-finiteness of the supports of the $\mathcal{R}$ numbers allows us to define multiplication of two given numbers x and y by multiplying, term by term, the corresponding rational power series of x and y and obtain a new element of $\mathcal{R}$. In the rational power series representation of the product, the coefficient of d^q for any given rational number q is obtained as the sum of finitely many terms: Let $x = \sum_{r \in Q} x_r d^r$ and $y = \sum_{s \in Q} y_s d^s$ be given in $\mathcal{R}$, and let $z = x \cdot y$; then according to Definition 3.6, we have that $z = \sum_{q \in Q} z_q d^q$ where for each q

$$z_q = \sum_{r+s=q} x_r y_s. \tag{1.3}$$

Because of the left-finiteness of the supports of x and y, the sum in Equation (1.3) is finite for each q. Moreover, the left-finiteness is a necessary condition for the implementation of these numbers on a computer as we will discuss in Section 1.5; this also follows directly from the Hahn theorem [18].

1.4 Applications in Physics

As we have mentioned in Section 1.1, the existence of infinitely small and infinitely large numbers in $\mathcal{R}$ allows us to have well-behaved delta functions; for example the functions $\delta_1, \delta_2 : \mathcal{R} \to \mathcal{R}$, given by

$$\delta_1(x) = \begin{cases} d^{-1} & \text{if } |x| \leq d/2 \\ 0 & \text{if } |x| > d/2 \end{cases}$$

and

$$\delta_2(x) = \begin{cases} 0 & \text{if } |x| \text{ is infinitely larger than } d \\ \frac{1}{\sqrt{\pi} d} \exp(-x^2/d^2) & \text{otherwise} \end{cases},$$

where d is again the differential introduced in Definition 3.5, are piecewise expandable (see Chapter 6) delta functions; they both assume infinitely large values at 0, they vanish at all other real points and their integrals are equal to one. We also note that we can replace d in the definitions of δ_1 and δ_2 with any other infinitely small number $a \in \mathcal{R}^+$, and obtain delta functions that behave in a similar way as δ_1 and δ_2. Thus, each of the two delta functions defined here generates a whole family of delta functions.

The major motivation for this work is the application of the existence of differentials in $\mathcal{R}$ to the computation of derivatives of complicated real functions up to very high orders. Using the calculus on $\mathcal{R}$ developed in Chapter 5, we derive in Chapter 7 a necessary and sufficient condition for the derivatives of real functions to exist at any given real point and show how to find the derivatives whenever they exist. Given a real-valued function f that is obtained from the intrinsic functions and the step function through a finite number of arithmetic operations and compositions, and given a real point r, we show that we can extend f to $\mathcal{R}$ and define it at $r \pm d$. Then representing $f(r \pm d)$ as expansions in powers of d allows us to isolate the (real) derivatives of f at r as coefficients in the expansions. We show that for a given positive integer m, f is m-times differentiable (in the real sense) at the real point r if and only if there exist real numbers $\alpha_1, \ldots, \alpha_m$ such that, up to the power m of d, $f(r-d)$ and $f(r+d)$ are given by

$$\begin{aligned} f(r-d) &=_m f(r) + \sum_{j=1}^{m} (-1)^j \alpha_j d^j \text{ and} \\ f(r+d) &=_m f(r) + \sum_{j=1}^{m} \alpha_j d^j, \end{aligned} \tag{1.4}$$

in which case the derivatives of f at r are given by

$$f^{(j)}(r) = j!\alpha_j \text{ for all } j \in \{1, \ldots, m\}.$$

Remark 1.1 *Another necessary and sufficient condition for f to be m-times differentiable (again in the real sense) at r is that for all $n \in \{1, \ldots, m\}$, we have* *that* $d^{-n}\left(\sum_{j=0}^{n}(-1)^{n-j}\begin{pmatrix} n \\ j \end{pmatrix} f\left(r+jd\right)\right)$ *and* $d^{-n}\left(\sum_{j=0}^{n}(-1)^{j}\begin{pmatrix} n \\ j \end{pmatrix} f\left(r-jd\right)\right)$ *are both at most finite in absolute value, and their real parts agree. In this case, the real derivatives are given by the respective real parts of the difference quotients. That is,*

$$f^{(n)}(r)=\Re\left(\frac{\sum_{j=0}^{n}(-1)^{n-j}\begin{pmatrix} n \\ j \end{pmatrix} f\left(r+jd\right)}{d^{n}}\right)$$

for all $n\in\{1,\ldots,m\}$*. In particular, if* f *is differentiable at* r*, then*

$$f'(r)=\Re\left(\frac{f(r+d)-f(r)}{d}\right).$$

Equation (1.4) or Remark 1.1 allow us to compute the real derivatives of a real-valued function at a real point to full machine precision and with no numerical penalties, because d is infinitely small. The proof of Remark 1.1 and of all the statements made in this section are found in Chapter 7.

The ability to calculate derivatives to high orders is important in many areas of Physics, e.g. in Beam Physics [9], non-linear Dynamics and Celestial Mechanics. In order to be able to use the theory developed in Chapter 7, the arithmetic operations and all the transcendental functions which are extended to $\mathcal{R}$ in Section 4.4 were implemented in COSY INFINITY [6, 8, 12]. Because of the theoretical work in Chapter 7, COSY INFINITY can now compute derivatives of very complicated functions to very high orders even if the coding contains if-else or other nondifferentiable pieces that do not affect the final result. Formula manipulators such as Mathematica, Automatic Differentiation methods [11] and even previous versions of COSY INFINITY [4] were not able to handle such cases. Even when Mathematica works, our method is much faster since no symbolic differentiation is required before the numerical evaluation of the derivatives. Moreover, the results obtained are accurate up to machine precision; this represents a clear advantage over traditional numerical differentiation methods in which case finite errors result from digit cancellation in the floating point representation and for high orders the errors usually become too large for the results to be of any practical use.

The practical usefulness of the existence of differentials is obtained at the cost that the field is disconnected in the order topology. The disconnection occurs because if x and y are any two positive elements of $\mathcal{R}$ and if x is infinitely smaller than y, then for all positive integers n we still have that nx is infinitely smaller than y. This disconnection between the different orders of magnitude makes it hard to extend the real calculus to $\mathcal{R}$ and explains the need for all the mathematical work in Chapters 2 through 6 before we get to the applications in Chapter 7. In particular, there are topologically continuous functions on closed intervals that do not satisfy the intermediate value theorem, the maximum theorem, or have multiple primitive functions that do not differ by a constant. For example consider the simple function $f : [0, 1] \to \mathcal{R}$, given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is infinitely small} \\ 1 & \text{if } x \text{ is finite} \end{cases} .$$

Then f is topologically continuous and differentiable on $[0, 1]$, but f does not assume the value d on $[0, 1]$ even though $f(0) < d < f(1)$. Moreover, even though $f'(x) = 0$ for all $x \in [0, 1]$, f is not constant on $[0, 1]$. This is due to the fact that the behavior of f in the infinitely small part of the domain, where it is constant and equal to zero, is totally disconnected from its behavior in the finite part of the domain, where it is constant and equal to 1. More examples are found at the end of Section 5.1.

The difficulties mentioned in the previous paragraph are not specific to $\mathcal{R}$ and are common to all non-Archimedean structures, which explains the previously rather limited results that could be derived in Non-Archimedean Analysis. In Chapter 5 and Chapter 6, we provide elegant solutions to the problems mentioned above; and we use that together with the results developed in the previous chapters to show the practical usefulness of the Levi-Civita field in Chapter 7. We first enhance the definitions of continuity and differentiability in Section 5.2 and obtain in Section 5.4 an intermediate value theorem and an inverse function theorem based on the new stronger definitions of continuity and differentiability. We then show in Chapter 6 that all the common theorems of real calculus hold for a large class of functions which are given locally by power series with $\mathcal{R}$ coefficients. In particular, we show that they satisfy the intermediate value theorem even though they do not satisfy the requirements of the general intermediate value theorem discussed in Section 5.4. Moreover, they satisfy the maximum theorem, Rolle's theorem and the mean value theorem; and they are infinitely often differentiable and integrable.

1.5 Implementation

Besides allowing illuminating theoretical conclusions, the strength of the $\mathcal{R}$ numbers is that they can be used in practice, and even in a computer environment. In this respect, they differ from the nonconstructive structures in Nonstandard Analysis [25, 35].

An implementation of the $\mathcal{R}$ numbers is not as direct as one of the Differential Algebras [2, 9] since $\mathcal{R}$ is infinite dimensional. However, it is still possible to implement the structure in a very useful way. Since there are only finitely many support points below every bound, it is possible to pick any such bound and store all the support points to the left of it together with the respective coefficients of the corresponding powers of d. Each $\mathcal{R}$ number is represented by its support points, the respective coefficients of the powers of d, and finally the value of the upper bound of the support points. That is, if we limit ourselves to n terms in the expansion, we can represent the series expansion of a given $\mathcal{R}$ number x by n pairs of numbers, the first n powers of d in the expansion and the n corresponding coefficients in the following way

$$\begin{aligned} x &=_M \left\{ \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ q_1 & q_2 & \cdots & q_n \end{array} \right\} \\ &=_M x_1 d^{q_1} + x_2 d^{q_2} + \cdots + x_n d^{q_n}, \end{aligned}$$

where M is the upper bound below which all the support points $q_1, q_2, \cdots, q_n$ of x are to be stored. In particular, a real number r will be represented as follows

$$\begin{aligned} r &=_M \left\{ \begin{array}{c} r \\ 0 \end{array} \right\} \\ &=_M rd^0 \end{aligned}$$

if $M \geq 0$ and $r + d$ is represented as follows

$$\begin{aligned} r + d &=_M \left\{ \begin{array}{cc} r & 1 \\ 0 & 1 \end{array} \right\} \\ &=_M rd^0 + 1d^1 \end{aligned}$$

if $M \geq 1$.

The sum of two such numbers can then be computed for all values to the left of the minimum of the two upper bounds; so the minimum of the upper bounds is the upper bound of the support points of the sum. In a similar way it is possible to find a bound below which the product of two such numbers can be computed from the bounds of the two numbers. Altogether, the bound to which each individual number is known is carried along through all arithmetic.

For the purpose of the implementation of the elementary functions, we make use of the addition theorems, e.g. Theorem 4.13 and Theorem 4.14, proved in Section 4.4 to truncate the series at a certain depth M. The elementary functions are defined for any number $x \in \mathcal{R}$ that is at most finite in absolute value. Any such x can be written as $x = r + s$, where $r = \Re(x)$ and where $|s|$ is infinitely small. Thus using the addition theorems for the elementary functions, we separate the function value at x into a real part and an infinitely small part which can be represented by a power series in s with positive exponents. This power series in s can be rewritten as a power series in d which converges fast and is truncated at the desired depth M. The following examples illustrate how the truncation is done for the division and the sine function; similar schemes are followed for the other transcendental functions.

Example 1.1 *Let $x = r + d$, where r is real and let $M = 10$.*

Then

$$\begin{aligned} \frac{1}{x} &= \frac{1}{r+d} \\ &= \frac{1}{r}\sum_{j=0}^{\infty}(-1)^j\left(\frac{d}{r}\right)^j \\ &=_{10} \frac{1}{r} - \frac{1}{r^2}d + \frac{1}{r^3}d^2 - \cdots + \frac{1}{r^{11}}d^{10}. \end{aligned}$$

Example 1.2 *Let $x = r + d$, where r is real and let $M = 10$.*

Then

$$\begin{aligned} \sin(x) &= \sin(r+d) \\ &= \sin(r)\cos(d) + \cos(r)\sin(d) \\ &=_{10} \sin(r)\left(1 - \frac{1}{2!}d^2 + \frac{1}{4!}d^4 - \frac{1}{6!}d^6 + \frac{1}{8!}d^8 - \frac{1}{10!}d^{10}\right) \\ &\quad + \cos(r)\left(d - \frac{1}{3!}d^3 + \frac{1}{5!}d^5 - \frac{1}{7!}d^7 + \frac{1}{9!}d^9\right) \\ &= \sin(r) + \cos(r)d - \frac{\sin(r)}{2!}d^2 - \frac{\cos(r)}{3!}d^3 + \cdots - \frac{\sin(r)}{10!}d^{10}. \end{aligned}$$

The real functions sin and cos already exist on the computer; and since the power series expansions of $\cos(d)$ and $\sin(d)$ converge very fast for the infinitely small d, the truncation is done easily as shown in the example above. Similar arguments hold for the other transcendental functions.

Using the rules outlined above, all the arithmetic operations and all the transcendental functions have been implemented in COSY INFINITY [6, 8, 12]. This allows us to apply the theoretical results of Chapter 7 for the computation of derivatives of real functions, in which case the upper bound of the support points in the final result must be greater than or equal to the order m of differentiability we would like to check. This final upper bound is calculated in terms of the upper bounds of the intermediate calculations, which we may have to change to get the desired final upper bound of the support points and hence obtain the desired information up to that depth. The number of support points of $f(r+d)$ or $f(r-d)$ smaller than or equal to m may depend on the complexity of the function f; however, using Equation (1.4), we can decide the differentiability of f at r up to order m and obtain the derivatives up to machine precision just by looking at the first $m+1$ support points of $f(r-d)$, the first $m+1$ support points of $f(r+d)$ and the corresponding coefficients. We give three simple examples here and refer the reader to the examples and the details of the method discussed in Chapter 7.

Example 1.3 *Let* $f_1 : [-1, 1] \to R$ *be given by* $f_1(x) = \exp(x)$

Then evaluating $f_1(\pm d)$ up to the power m of d, where m is a positive integer, yields

$$\begin{aligned} f_1(-d) &=_m 1 + \sum_{j=1}^{m} (-1)^j \frac{1}{j!} d^j \text{ and} \\ f_1(d) &=_m 1 + \sum_{j=1}^{m} \frac{1}{j!} d^j. \end{aligned}$$

Since $f_1(0) = 1$, we obtain that f_1 is m-times differentiable at 0 with derivatives

$$f_1^{(j)}(0) = 1 \text{ for all } j \leq m.$$

Example 1.4 *Let* $f_2 : [-1, 1] \to R$ *be given by*

$$f_2(x) = |x|^{7/2} \sin(x).$$

Then, evaluating $f(\pm d)$ up to the power 8 of d gives

$$\begin{aligned} f_2(-d) &=_8 -d^{9/2} + \frac{d^{13/2}}{6} \text{ and} \\ f_2(d) &=_8 d^{9/2} - \frac{d^{13/2}}{6}. \end{aligned}$$

Since $f_2(0) = 0$, we obtain using Equation (1.4) that f_2 is four-times differentiable at 0 with derivatives equal to 0; but f_2 is not m-times differentiable at 0 for any $m \geq 5$.

Example 1.5 *Let* $f_3 : [-1, 1] \to R$ *be given by*

$$f_3(x) = \begin{cases} |x|^{-1/2} \sin^3(x) & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0 \end{cases}.$$

Then, evaluating $f_3(\pm d)$ up to the power 2 of d yields

$$f_3(\pm d) =_2 0 = f(0),$$

from which we obtain that f_3 is twice differentiable at 0 with derivatives

$$f_3'(0) = f_3^{(2)}(0) = 0.$$

To check whether f_3 is three-times differentiable at 0, we need to evaluate $f_3(\pm d)$ up to the power 3 of d; if we do so, we obtain that

$$\begin{aligned} f_3(-d) &=_3 -d^{5/2} \\ f_3(d) &=_3 d^{5/2}, \end{aligned}$$

from which we infer, using Equation (1.4), that f_3 is not three-times differentiable at 0 since $5/2$ is a noninteger number smaller than 3 and the coefficient of $d^{5/2}$ in the expansion of $f_3(d)$ is not zero. The same conclusion follows also from Remark 1.1, since the difference quotient of order 3,

$$\frac{f_3(3d) - 3f_3(2d) + 3f_3(d) - f_3(0)}{d^3},$$

is of the same order of magnitude as $d^{-1/2}$ and hence it is infinitely large in absolute value.

1.6 Notations

Throughout this dissertation, we will adopt the following notations:

Z, Z^+, Z^-: the set of all integers, the set of positive integers and the set of negative integers, respectively;

Q, Q^+, Q^-: the field of rational numbers, the set of positive rational numbers and the set of negative rational numbers, respectively;

R, R^+, R^-: the field of real numbers, the set of positive real numbers and the set of negative real numbers, respectively;

L: the field of the formal Laurent series; and

$\mathcal{R}$ (read R-script): the Levi-Civita field.

Chapter 2

Skeleton Groups and Field Automorphisms

In this chapter, we prove general results about skeleton groups and field automorphisms which will be useful for understanding the structure of the Levi-Civita field $\mathcal{R}$, which will be introduced in Chapter 3.

2.1 Skeleton Groups

Let F be a totally ordered field, and let $a, b \in F^* = F \setminus \{0\}$ be given. We say that $a \sim b$ if and only if there exist $n, m \in Z^+$ such that $n|a| > |b|$ and $m|b| > |a|$, where $|\cdot|$ is the usual absolute value on F, defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Then $\sim$ is an equivalence relation.

Let S_F denote the set of all equivalence classes. Then

$$S_F = \{[a] : a \in F^*\}.$$

Let $a, a_1, b, b_1 \in F^*$ be such that $a \sim a_1$ and $b \sim b_1$. Then $a \cdot b \sim a_1 \cdot b_1$. Define $\oplus : S_F \times S_F \to S_F$ by

$$[a] \oplus [b] = [a \cdot b].$$

Then it is easy to verify that $(S_F, \oplus)$ is an abelian group whose additive neutral element is denoted by 0 and is given by $0 = [1]$, where 1 is the multiplicative neutral element of F. The additive inverse of an element $[a] \in S_F$ is denoted by $\ominus[a]$ and is given by $\ominus[a] = [a^{-1}]$.

Let $a, a_1, b, b_1 \in F^*$ be such that $a \sim a_1$ and $b \sim b_1$. Assume that, for all $n \in Z^+$, $n|a| < |b|$. Then $n|a_1| < |b_1|$ for all $n \in Z^+$. Define $<, \leq : S_F \to S_F$ by

$$\begin{aligned} [a] &< [b] \text{ if and only if } n \cdot |a| < |b| \text{ for all } n \in Z^+ \\ [a] &\leq [b] \text{ if and only if } [a] = [b] \text{ or } [a] < [b]. \end{aligned}$$

Then the relation $\leq$ defines a total ordering on $(S_F, \oplus)$. Thus $(S_F, \oplus, \leq)$ is a totally ordered abelian group; that is,

1. for all $[a], [b] \in (S_F, \oplus)$, $[a] \leq [b]$ or $[b] \leq [a]$, and $[a] = [b]$ if and only if $[a] \leq [b]$ and $[b] \leq [a]$,

2. for all $[a], [b], [c] \in S_F$, $[a] \leq [b] \Rightarrow [a] \oplus [c] \leq [b] \oplus [c]$.

In the following, the totally ordered group $(S_F, \oplus, \leq)$ will be simply denoted by S_F and will be referred to as the skeleton group of F.

One can easily verify that the skeleton group of R is given by

$$S_R = \{0\} = \{[1]\};$$

and the skeleton group of L is given by

$$S_L = Z,$$

where L is the field of the formal Laurent series. After having introduced the totally ordered field $\mathcal{R}$ in Section 3.1 and Section 3.3, one can also verify that the skeleton group of $\mathcal{R}$ is

$$S_{\mathcal{R}} = Q.$$

Definition 2.1 *Let F be a totally ordered field. Then we say that F is non-Archimedean if and only if the skeleton group S_F of F contains more than one element.*

Theorem 2.1 *Let F be a totally ordered non-Archimedean field. Then $Z \subset S_F$.*

Proof. Since F is non-Archimedean, there exists an element $d \in F^*$ such that $[d] \neq 0 = [1]$. Since $[d] = [-d]$, we may assume that $d > 0$, where 0 is the additive neutral element of F. Since $\ominus[d] = [d^{-1}] \in S_F$, we may assume that $[d] < 0$, i.e. that d is infinitely small. Consider the subset $Z_F = \{[d^n] : n \in Z\}$ of S_F. For $m > n$, we have that

$$\begin{aligned}
\ominus[d^n] \oplus [d^m] &= [d^{-n} \cdot d^m] = [d^{m-n}] = [\underbrace{d \cdot d \cdot \ldots \cdot d}_{(m-n) \text{ times}}] \\
&= \underbrace{[d] \oplus [d] \oplus \cdots \oplus [d]}_{(m-n) \text{ times}} \\
&< \underbrace{[d] \oplus [d] \oplus \cdots \oplus [d]}_{(m-n-1) \text{ times}} \\
&\vdots \\
&< [d] < 0.
\end{aligned}$$

Thus, $m \neq n \Rightarrow [d^m] \neq [d^n]$ for all $m, n \in Z$. The map $P : Z_F \to Z$ given by

$$P([d^n]) = -n$$

is an order preserving isomorphism; that is,

1. P is bijective,

2. P is compatible with the groups' operations, i.e. for all $m, n \in Z$,

$$P([d^m] \oplus [d^n]) = P([d^m]) + P([d^n]),$$

and

3. P is compatible with the groups' order relations, i.e. for all $m, n \in Z$,

$$[d^m] < [d^n] \Leftrightarrow P([d^m]) < P([d^n]).$$

Thus Z is isomorphic to a subset of S_F, or simply $Z \subset S_F$.

Theorem 2.2 *Let F be a totally ordered non-Archimedean field which admits roots of positive elements. Then $Q \subset S_F$.*

Proof. Since F is non-Archimedean, there exists an element $d \in F^*$ such that $[d] < 0$. Let $q > 0$ in Q be given; write $q = m/n$ where $m, n \in Z^+$. As in the proof of Theorem 2.1, $[d^m] < 0$. Using the fact that $[d^m] = [d^{n \cdot q}] = \underbrace{[d^q] \oplus [d^q] \oplus \cdots \oplus [d^q]}_{n \text{ times}}$, we obtain that $[d^q] < 0$. In particular, $[d^q] \neq 0$.

Now let $q_1 \neq q_2$ be given in Q. We may assume that $q_2 > q_1$. Then $q_2 - q_1 > 0$, and hence

$$\ominus[d^{q_1}] \oplus [d^{q_2}] = [d^{q_2 - q_1}] < 0.$$

Thus, $q_1 \neq q_2 \Rightarrow [d^{q_1}] \neq [d^{q_2}]$.

Let $Q_F = \{[d^q] : q \in Q\}$. Then Q_F is a subgroup of S_F, and the map $P : Q_F \to Q$, given by

$$P([d^q]) = -q, \tag{2.1}$$

is an order preserving group isomorphism from Q_F onto Q.

Remark 2.1 *Define* $\otimes : Q_F \times Q_F \to Q_F$ *by*

$$[d^{q_1}] \otimes [d^{q_2}] = [d^{q_1 \cdot q_2}].$$

Then $(Q_F, \oplus, \otimes, \leq)$ *is a totally ordered field, and the map* P *given in Equation (2.1) becomes a field isomorphism of* Q_F *onto* Q.

2.2 Field Automorphisms

Definition 2.2 *Let* F *be a set, and let* $P : F \to F$ *be given. Then we say that* P *is an automorphism on* F *if and only if* P *is an isomorphism from* F *onto itself.*

Lemma 2.1 *Let* S *and* T *be fields, and let* $P : S \to T$ *be a field isomorphism. Then* P *has an inverse* $P^{-1} : T \to S$ *which is itself a field isomorphism from* T *onto* S.

Proof. Since P is bijective, P^{-1} exists and it is bijective. Let $+_S$ and $+_T$ denote the addition operations in S and T, respectively; and let $\times_S$ and $\times_T$ denote the operations of multiplication in S and T, respectively. Now let $y_1, y_2 \in T$ be given, and let $x_1 = P^{-1}(y_1)$ and $x_2 = P^{-1}(y_2)$. Then

$$\begin{aligned} P^{-1}(y_1 +_T y_2) &= P^{-1}\left(P(x_1) +_T P(x_2)\right) \\ &= P^{-1}\left(P(x_1 +_S x_2)\right) \text{ since } P \text{ is an isomorphism} \\ &= x_1 +_S x_2 = P^{-1}(y_1) +_S P^{-1}(y_2), \end{aligned}$$

and

$$\begin{aligned} P^{-1}(y_1 \times_T y_2) &= P^{-1}\left(P(x_1) \times_T P(x_2)\right) \\ &= P^{-1}\left(P(x_1 \times_S x_2)\right) \text{ since } P \text{ is an isomorphism} \\ &= x_1 \times_S x_2 = P^{-1}(y_1) \times_S P^{-1}(y_2). \end{aligned}$$

Thus P^{-1} is a field isomorphism from T onto S.

Theorem 2.3 *Let S and T be totally ordered fields, and let $P : S \to T$ be an order preserving field isomorphism. Then P^{-1} is an order preserving field isomorphism from T onto S.*

Proof. Using Lemma 2.1, it remains to show that $P^{-1} : T \to S$ is order preserving. So let $y_1, y_2 \in T$ be such that $y_1 \leq_T y_2$, and let $x_1 = P^{-1}(y_1)$ and $x_2 = P^{-1}(y_2)$. We need to show that $x_1 \leq_S x_2$. Suppose not; then $x_2 <_S x_1$. Since P is order preserving, we obtain that $y_2 = P(x_2) <_T P(x_1) = y_1$, a contradiction.

Theorem 2.4 *Let F be a totally ordered field, and let $P : F \to F$ be an automorphism on F. Then $P(q) = q$ for all $q \in Q$.*

Proof. Since F is a totally ordered field, $Q \subset F$. For any $x \in F$, we have that $P(x) = P(0 + x) = P(0) + P(x)$; and hence

$$P(0) = 0.$$

Also for any $x \in F$, we have that $P(x) = P(1 \cdot x) = P(1) \cdot P(x)$; and hence

$$P(1) = 1. \tag{2.2}$$

Now let $q > 0$ in Q be given; write $q = m/n$ where $m, n \in Z^+$. Then $m = n \cdot q$. Thus

$$P(m) = P(n) \cdot P(q). \tag{2.3}$$

Using Equation (2.2), we obtain that for all $l \in Z^+$,

$$\begin{aligned} P(l) &= P(\underbrace{1 + 1 + \cdots + 1}_{l \text{ times}}) \\ &= \underbrace{P(1) + P(1) + \cdots + (1)}_{l \text{ times}} \end{aligned}$$

$$
\begin{aligned}
&= \underbrace{1 + 1 + \cdots + 1}_{l \text{ times}} \\
&= l.
\end{aligned}
$$

Thus, $P(m) = m$ and $P(n) = n$. Substituting into Equation (2.3), we obtain that $P(q) = q$. Finally, let $q < 0$ in Q be given; then $-q > 0$. Thus $P(-q) = -q$. Since $0 = P(0) = P(-q+q) = P(-q) + P(q)$, we have that $P(q) = -P(-q) = q$. Hence

$$P(q) = q \text{ for all } q \in Q.$$

Corollary 2.1 *The identity map $I : Q \to Q$ is the only field automorphism on Q.*

Theorem 2.5 *Let F be a totally ordered Archimedean field. Then the identity map $I : F \to F$ is the only order preserving field automorphism on F.*

Proof. Assume not. Then there exists a nontrivial order preserving field automorphism P on F. Thus there exists $x \in F \setminus Q$ such that $P(x) \neq x$. Since $P(-x) = -P(x) \neq -x$, we may assume without loss of generality that $x > 0$. Since $P(x^{-1}) = (P(x))^{-1} \neq x^{-1}$, we may assume that $x > 1$.

By Theorem 2.3, P^{-1} is also an order preserving field automorphism on F, and $P^{-1}(x) \neq P^{-1}(P(x)) = x$. If $P(x) < x$, then $x = P^{-1}(P(x)) < P^{-1}(x)$. So we may assume without loss of generality that

$$1 < x < P(x).$$

Since F is Archimedean, there exist $n, N \in Z^+$ such that

$$0 < \frac{1}{P(x) - x} < N \text{ and } n \leq Nx < n + 1.$$

Then

$$n + 1 \leq Nx + 1 < Nx + N(P(x) - x) = NP(x).$$

Thus,

$$Nx < n + 1 < NP(x),$$

and since $N > 0$, we finally obtain that

$$x < \frac{n+1}{N} < P(x). \tag{2.4}$$

Applying P to the first part of Equation (2.4), we obtain that

$$P(x) < P\left(\frac{n+1}{N}\right) = \frac{n+1}{N},$$

which contradicts Equation (2.4) itself. Hence the identity map is the only order preserving field automorphism on F.

Lemma 2.2 *Let F be a totally ordered field which admits roots of positive elements, and let P be a field automorphism on F. Then P is order preserving.*

Proof. It suffices to show that $P(a) > 0$ for all $a > 0$ in F; so let $a > 0$ in F be given. Let $b > 0$ in F be such that $b^2 = a$. Hence

$$P(a) = P(b^2) = (P(b))^2 \geq 0.$$

Since $a \neq 0$ and since P is one to one on F, we obtain that $P(a) \neq 0$. Thus, $P(a) > 0$.

Combining the results of Theorem 2.5 and Lemma 2.2, we obtain the following result.

Corollary 2.2 *The identity map is the only field automorphism on R.*

Theorem 2.6 *Let F be a totally ordered non-Archimedean field, let S_F be the skeleton group of F, and let P be an order preserving field automorphism on F. Then the map $\Gamma : S_F \to S_F$, given by $\Gamma([x]) = [P(x)]$, is a well-defined order preserving group automorphism on S_F.*

Proof. Let $x \in F^*$ be given; then $P(x) \neq 0$. If $x < 0$, then $P(x) < 0$ and hence $P(|x|) = P(-x) = -P(x) = |P(x)|$. On the other hand, if $0 < x$, then $0 < P(x)$ and hence $P(|x|) = P(x) = |P(x)|$. So for all $x \in F^*$, $P(|x|) = |P(x)|$.

To show that Γ is a well-defined map, we need to show that

$$[x] = [y] \Rightarrow [P(x)] = [P(y)].$$

So let $x, y \in F^*$ be such that $[x] = [y]$. Then there exist $m, n \in Z^+$ such that $|y| < m \cdot |x|$ and $|x| < n \cdot |y|$. Thus,

$$|P(y)| = P(|y|) < P(m \cdot |x|) = P(m) \cdot P(|x|) = m \cdot |P(x)|$$

and $|P(x)| < n \cdot |P(y)|$. Hence $[P(x)] = [P(y)]$.

Now we show that Γ is one to one. So let $[x], [y] \in S_F$ be such that $\Gamma([x]) = \Gamma([y])$. We need to show that $[x] = [y]$. Since $[P(x)] = [P(y)]$, there exist $k, l \in Z^+$ such that $|P(y)| < k \cdot |P(x)|$ and $|P(x)| < l \cdot |P(y)|$. From $|P(y)| < k \cdot |P(x)|$ we obtain that

$$P(|y|) < P(k) \cdot P(|x|) = P(k \cdot |x|); \text{ and hence } |y| < k \cdot |x|.$$

Similarly, $|P(x)| < l \cdot |P(y)|$ entails that $|x| < l \cdot |y|$. Hence $[x] = [y]$.

To show that Γ is surjective, let $[y] \in S_F$ be given. We need to find $[x] \in S_F$ such that $[y] = \Gamma([x])$. Since P is surjective, there exists $x \in F$ such that $y = P(x)$. Since $y \neq 0$, we have also that $x \neq 0$. Hence

$$[x] \in S_F \text{ and } \Gamma([x]) = [P(x)] = [y].$$

For any $[x], [y] \in S_F$, we have that

$$\begin{aligned} \Gamma([x] \oplus [y]) &= \Gamma([x \cdot y]) = [P(x \cdot y)] \\ &= [P(x) \cdot P(y)] = [P(x)] \oplus [P(y)] \\ &= \Gamma([x]) \oplus \Gamma([y]) \end{aligned}$$

It remains to show that Γ preserves order in S_F. So let $[x], [y] \in S_F$ be such that $[x] \leq [y]$; we need to show that $\Gamma([x]) \leq \Gamma([y])$. If $[x] = [y]$, then $\Gamma([x]) = \Gamma([y])$. Suppose $[x] < [y]$; then for all $n \in Z^+$, $n \cdot |x| < |y|$. It follows that

$$n \cdot |P(x)| = P(n) \cdot P(|x|) = P(n \cdot |x|) < P(|y|) = |P(y)| \text{ for all } n \in Z^+.$$

Thus $[P(x)] < [P(y)]$; i.e. $\Gamma([x]) < \Gamma([y])$.

Corollary 2.3 *Let F, P, and Γ be as in Theorem 2.6. Define $\Lambda : S_F \to S_F$ by $\Lambda([x]) = [P^{-1}(x)]$. Then Λ is an order preserving group automorphism on S_F, and $\Lambda = \Gamma^{-1}$.*

Corollary 2.4 *Let F be a totally ordered non-Archimedean field extension of R, and let P be an order preserving field automorphism on F. Then, for all $r \in R^*$, $[P(r)] = 0$ i.e $P(r) \sim 1$.*

Proof. Let $r \in R^*$ be given; then $[r] = [1] = 0$. Hence, by Theorem 2.6, we have that $[P(r)] = [P(1)] = [1] = 0$.

Corollary 2.5 *Let P be an order preserving field automorphism on L. Then, for all $x \in L^*$, $[P(x)] = [x]$ i.e. $P(x) \sim x$.*

Proof. By Theorem 2.6, the map $\Gamma : S_L = Z \to Z$, given by $\Gamma([x]) = [P(x)]$, is an order preserving group automorphism on $Z = S_L$. We need to show that $\Gamma = I$, the identity map on Z. By Theorem 2.6, we have that

$$[x] = [y] \quad \Rightarrow \quad [P(x)] = [P(y)] \text{ and } [P^{-1}(x)] = [P^{-1}(y)] \tag{2.5}$$

$$[x] < [y] \quad \Rightarrow \quad [P(x)] < [P(y)] \text{ and } [P^{-1}(x)] < [P^{-1}(y)]. \tag{2.6}$$

We have that

$$\Gamma(0) = \Gamma([1]) = [P(1)] = [1] = 0. \tag{2.7}$$

Let d be the L number representing the formal Laurent series x. Since $[d] = -1 < 0 = [1]$, we have by Equation (2.6) that

$$[P(d)] < [P(1)] = [1] = 0 \text{ and } [P^{-1}(d)] < [P^{-1}(1)] = [1] = 0.$$

Since $d > 0$, we have that $P(d) > 0$ and $P^{-1}(d) > 0$. If $P(d) = d$, then $[P(d)] = [d]$. If $d < P(d)$, then $[d] \leq [P(d)]$, and hence $-1 \leq [P(d)] < 0$. Since $[P(d)]$ is an integer, $[P(d)] = -1 = [d]$. If $P(d) < d$, then $d < P^{-1}(d)$. Thus $-1 = [d] \leq [P^{-1}(d)] < 0$ and hence $[P^{-1}(d)] = -1 = [d]$. Using Equation (2.5), we obtain that $[P(P^{-1}(d))] = [P(d)]$, and hence $[P(d)] = [d] = -1$. Thus,

$$\Gamma(-1) = -1.$$

Now let $n \in Z^-$ be given. Then

$$\begin{aligned}
\Gamma(n) &= \Gamma([d^{-n}]), \text{ where } -n \in Z^+ \\
&= \Gamma(\underbrace{[d] + [d] + \cdots + [d]}_{-n \text{ times}}) \\
&= \underbrace{\Gamma([d]) + \Gamma([d]) + \cdots + \Gamma([d])}_{-n \text{ times}} \\
&= \underbrace{-1 - 1 - \ldots - 1}_{-n \text{ times}} = -n \cdot (-1) \\
&= n.
\end{aligned}$$

Finally, let $n \in Z^+$ be given; then $-n \in Z^-$. Since $\Gamma(-n) + \Gamma(n) = \Gamma(-n+n) = \Gamma(0) = 0$,

$$\Gamma(n) = -\Gamma(-n) = -(-n) = n.$$

Therefore,

$$\Gamma(n) = n \text{ for all } n \in Z; \text{ and hence } \Gamma = I.$$

Corollary 2.6 *Let P be an order preserving field automorphism on L. Then $P(r) = r + \sum_{k=1}^{\infty} r_k d^k$ for all $r \in R$, where $r_k \in R$ for all $k \in Z^+$.*

Proof. Let $r > 1$ be given in R. Then $1 = P(1) < P(r)$. If $P(r) = r$, we are done. Assume $P(r) \neq r$; then $P(r) - r \neq 0$. Since $[P(r)] = [r] = 0$, we have that $[P(r) - r] \leq 0$. Assume $[P(r) - r] = 0$; then $[P(r) - r] = [1]$, and hence $[P^{-1}(P(r) - r)] = [P^{-1}(1)] = [1]$. Thus $[P^{-1}(r) - r] = 0$. If $P(r) < r$, then $r < P^{-1}(r)$; so we may assume without loss of generality that $r < P(r)$. Therefore $1 < r < P(r)$. Since $[P(r)] = 0$, there exists $N \in Z^+$ such that $P(r) < N$. Hence

$$1 < r < P(r) < N < N \cdot r < N \cdot P(r). \tag{2.8}$$

Since $r < P(r)$ and $[P(r) - r] = 0$, we have that

$$P(r) - r = t \cdot r + \sum_{k=1}^{\infty} r_k d^k, \text{ with } t > 0 \text{ in } R.$$

Let ϵ be a rational number satisfying $0 < \epsilon < t$. There exists $k \in Z^+$ such that $N < k \cdot \epsilon$. Thus $\epsilon \cdot r < P(r) - r$, and hence $(1 + \epsilon) \cdot r < P(r)$. It follows that

$$(1+\epsilon)^2 \cdot r < (1+\epsilon) \cdot P(r) = P((1+\epsilon)) \cdot P(r) = P((1+\epsilon) \cdot r) < P(P(r)) = P^2(r).$$

Using induction, we obtain that

$$(1+\epsilon)^m < P^m(r) \text{ for all } m \in Z^+.$$

In particular,

$$(1+N) \cdot r < (1 + k \cdot \epsilon) \cdot r < (1+\epsilon)^k \cdot r < P^k(r),$$

from which we obtain that

$$N \cdot r < P^k(r). \tag{2.9}$$

By Equation (2.8), $P(r) < N$. Hence $P^2(r) < P(N) = N$. Using induction, it follows that $P^m(r) < N$ for all $m \in Z^+$. In particular, $P^k(r) < N < N \cdot r$, which contradicts Equation (2.9). So if $r > 1$ and $P(r) \neq r$, then $[P(r) - r] < 0$. It follows that

$$P(r) = r + \sum_{k=1}^{\infty} r_k d^k.$$

Hence the result is true for all $r > 1$ in R.

Now let $r \in R$ be such that $0 < r < 1$; then $r^{-1} > 1$. Thus,

$$P(r^{-1}) = r^{-1} + \sum_{k=1}^{\infty} s_k d^k = r^{-1} \cdot \left(1 + r \cdot \sum_{k=1}^{\infty} s_k d^k\right) = r^{-1} \cdot (1+s),$$

where $|s|$ is infinitely small. Thus $(1+s)^{-1} = (1+s')$ where $|s'|$ is also infinitely small. It follows that

$$P(r) = \left(P\left(r^{-1}\right)\right)^{-1} = r \cdot (1+s)^{-1} = r + r \cdot s',$$

which proves the result for $0 < r < 1$.

Since $P(0) = 0$ and $P(1) = 1$, the result is true for all $r \geq 0$ in R. To show it is true for $r < 0$ in R, we make use of the fact that $P(r) = -P(-r)$.

Example 2.1 *Define $P : L \to L$ as follows: for $x \in L$, write $x = \sum_{k \geq k_x} a_k d^k$ and set $P(x) = \sum_{k \geq k_x} 2^k a_k d^k$, where $k_x = -[x]$. Then P is an order preserving field automorphism on L.*

After this study of the properties of skeleton groups and field automorphisms, we will now move on to introduce the Levi-Civita field $\mathcal{R}$; and we will prove more results about order preserving field automorphisms on $\mathcal{R}$ in Section 3.3.

Chapter 3

The Non-Archimedean Field $\mathcal{R}$

In this chapter, we review the algebraic structure, the order structure and the topological structure of the non-Archimedean Levi-Civita field $\mathcal{R}$, which are found in [3, 5, 7]. We also review the differential algebraic structure of the field, which is useful for the concept of differentiability [43].

3.1 Algebraic Structure

We begin the discussion by introducing a specific family of sets.

Definition 3.1 (The Family of Left-Finite Sets) *A subset M of Q is called left-finite if and only if for every number $r \in Q$ there are only finitely many elements of M that are smaller than r. The set of all left-finite subsets of Q will be denoted by $\mathcal{F}$.*

The next lemma gives some insight into the structure of left-finite sets.

Lemma 3.1 *Let $M \in \mathcal{F}$ be given. If $M \neq \emptyset$, the elements of M can be arranged in ascending order, and there exists a minimum of M. If M is infinite, the resulting strictly increasing sequence is divergent.*

Proof. A finite totally ordered set can always be arranged in ascending order; so we may assume that M is infinite.

For each $n \in Z^+$, set $M_n = \{x \in M : x \leq n\}$. Then, for all n, M_n is finite by the left-finiteness of M, and we have that $M = \cup_n M_n$. Hence, we first arrange the finitely many elements of M_0 in ascending order, append the finitely many elements of M_1 not in M_0 in ascending order, and continue inductively.

If the resulting strictly increasing sequence were bounded, there would also be a rational bound below which there would be infinitely many elements of M, contrary to the assumption that M is left-finite. Therefore, we conclude that the sequence is divergent.

Lemma 3.2 *Let $M, N \in \mathcal{F}$. Then the following are true.*

- $X \subset M \Rightarrow X \in \mathcal{F}$.
- $M \cup N \in \mathcal{F}$.
- $M \cap N \in \mathcal{F}$.
- $M + N = \{x + y : x \in M \text{ and } y \in N\} \in \mathcal{F}$, *and for every* $x \in M + N$, *there are only finitely many pairs* $(a, b) \in M \times N$ *such that* $x = a + b$.

Proof. The first three statements follow directly from the definition. For the proof of the fourth statement, let x_M, x_N denote the smallest elements in M and N respectively; these exist by Lemma 3.1. Let $r \in Q$ be given. Set

$$M^u = \{x \in M | x < r - x_N\}, \qquad N^u = \{x \in N | x < r - x_M\},$$

$$M^o = M \setminus M^u, \qquad N^o = N \setminus N^u.$$

Then we have that

$$\begin{aligned} M+N &= (M^u \cup M^o) + (N^u \cup N^o) \\ &= (M^u + N^u) \cup (M^o + N^u) \cup (M^u + N^o) \cup (M^o + N^o) \\ &= (M^u + N^u) \cup (M^o + N) \cup (M + N^o). \end{aligned}$$

By definition of M^o and N^o, we have that $(M^o + N)$ and $(M + N^o)$ do not contain any elements smaller than r. Thus all elements of $M + N$ that are smaller than r must actually be contained in $M^u + N^u$. Since both M^u and N^u are finite because of the left-finiteness of M and N, we obtain that $M^u + N^u$ is also finite. Thus there are only finitely many elements in $M + N$ that are smaller than r.

Finally, let $x \in M + N$ be given. Set $r = x + 1$ and define M^u, N^u as in the preceding paragraph. Then we have that $x \notin (M^o + N)$ and $x \notin (M + N^o)$. Hence all pairs $(a, b) \in M \times N$ that satisfy $x = a + b$ lie in the finite set $M^u \times N^u$.

Having discussed the family of left-finite sets, we introduce the following set of functions from Q into R.

Definition 3.2 (The Set $\mathcal{R}$) *We define*

$$\mathcal{R} = \{f : Q \to R : \{x | f(x) \neq 0\} \in \mathcal{F}\}.$$

Hence, the elements of $\mathcal{R}$ are those real-valued functions on Q that are nonzero only on a left-finite set, that is, they have left-finite support.

Remark 3.1 *Since the desired field $\mathcal{R}$ is to be non-Archimedean and have roots of positive elements (see Chapter 1), we infer using Theorem 2.2 that Q is the minimal domain of definition of the elements of $\mathcal{R}$ in Definition 3.2. This already tells us something about the uniqueness of $\mathcal{R}$; see Theorem 3.11.*

In the following, we denote elements of $\mathcal{R}$ by x, y, etc. and identify their values at $q \in Q$ with brackets, like in $x[q]$. This avoids confusion when we later consider functions on $\mathcal{R}$. Since the elements of $\mathcal{R}$ are functions with left-finite support, it is convenient to use the properties of left-finite sets in Lemma 3.1 for their description.

Definition 3.3 (Notation for Elements of $\mathcal{R}$) *An element x of $\mathcal{R}$ is uniquely characterized by an ascending (finite or infinite) sequence (q_n) of support points and a corresponding sequence $(x[q_n])$ of function values. We refer to the pair of sequences $((q_n), (x[q_n]))$ as the table of x.*

Already at this point it is worth noting that for questions of implementation, it is usually sufficient to store only the first few of the support points and remember carefully up to what "depth" a given number in $\mathcal{R}$ is known.

For subsequent discussion, it is convenient to introduce the following terminology.

Definition 3.4 (supp, λ, $\sim$, $\approx$, $=_r$) *For $x, y \in \mathcal{R}$, we define*

$\text{supp}(x) = \{q \in Q : x[q] \neq 0\}$ *and call it the support of x.*

$\lambda(x) = \min(\text{supp}(x))$ *for $x \neq 0$ (which exists because of left-finiteness) and*

$\lambda(0) = +\infty$.

Comparing two elements, we say

$x \sim y$ *if and only if* $\lambda(x) = \lambda(y)$*;*

$x \approx y$ *if and only if* $\lambda(x) = \lambda(y)$ *and* $x[\lambda(x)] = y[\lambda(y)]$*;*

$x =_r y$ *if and only if* $x[q] = y[q]$ *for all* $q \leq r$*.*

At this point, these definitions may feel somewhat arbitrary; but after having introduced the concept of ordering on $\mathcal{R}$, we will see that λ describes "orders of infinite largeness or smallness", the relation "$\approx$" corresponds to agreement up to infinitely small relative error, while "$\sim$" corresponds to agreement of order of magnitude and is thus the same as the $\sim$ introduced in Section 2.1.

Lemma 3.3 *The relations $\sim$, $\approx$ and $=_r$ are equivalence relations. They satisfy*

$$x \approx y \Rightarrow x \sim y;$$

and

$$\textit{if } a > b \textit{ in } Q, \textit{ then } x =_a y \Rightarrow x =_b y.$$

Definition 3.5 (The Number d) *We define the number $d \in \mathcal{R}$ as follows.*

$$d[q] = \begin{cases} 1 & \textit{if } q = 1 \\ 0 & \textit{else} \end{cases} .$$

Apparently, the number d admits an n-th root for all $n \in Z^+$, denoted by $d^{1/n}$ and given by

$$d^{1/n}[q] = \begin{cases} 1 & \text{if } q = \frac{1}{n} \\ 0 & \text{else} \end{cases} .$$

Also d has a multiplicative inverse denoted by d^{-1} and given by

$$d^{-1}[q] = \begin{cases} 1 & \text{if } q = -1 \\ 0 & \text{else} \end{cases} .$$

As we shall see, d plays the role of an infinitesimal and thus satisfies what Rall suspected about the number $(0, 1)$ in his arithmetic of differentiation [33].

We now define arithmetic on $\mathcal{R}$.

Definition 3.6 (Addition and Multiplication on $\mathcal{R}$) *We define addition on $\mathcal{R}$ componentwise:*

$$(x + y)[q] = x[q] + y[q] \textit{ for all } q \in Q.$$

Multiplication is defined as follows. For $q \in Q$, we set

$$(x \cdot y)[q] = \sum_{\substack{q_x, q_y \in Q, \\ q_x + q_y = q}} x[q_x] \cdot y[q_y].$$

We remark that $\mathcal{R}$ is closed under addition since $\text{supp}(x+y) \subset \text{supp}(x) \cup \text{supp}(y)$, so by Lemma 3.2, we have that $\text{supp}(x+y)$ is left-finite. Lemma 3.2 also shows that only finitely many terms contribute to the sum in the definition of the product. Furthermore, the product defined above is itself an element of $\mathcal{R}$ since the sets of support points satisfy $\text{supp}(x \cdot y) \subset \text{supp}(x) + \text{supp}(y)$; so that application of Lemma 3.2 shows that $\text{supp}(x \cdot y) \in \mathcal{F}$.

It turns out that the operations $+$ and $\cdot$ we just defined on $\mathcal{R}$ make $(\mathcal{R},+,\cdot)$ into a field (see Theorem 3.4 below).

Theorem 3.1 *$(\mathcal{R},+,\cdot)$ is a commutative ring with a unit.*

Proof. The proof is straightforward, and we leave it as an exercise for the reader to fill in the details.

As it turns out, $\mathcal{R}$ can be viewed as an extension of R.

Theorem 3.2 (Embedding of R into $\mathcal{R}$) *R can be embedded into $\mathcal{R}$ under the preservation of its arithmetic structure.*

Proof. Let $x \in R$. Define $\Pi : R \to \mathcal{R}$ by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}.$$

Then Π is one to one, and direct calculation shows that

$$\begin{aligned} \Pi(x+y) &= \Pi(x) + \Pi(y) \text{ and} \\ \Pi(x \cdot y) &= \Pi(x) \cdot \Pi(y). \end{aligned}$$

So R is embedded as a subfield in the ring $\mathcal{R}$. However, the embedding is not surjective, since only elements with support $\{0\}$ are actually reached.

Remark 3.2 *In the following, we identify an element $x \in R$ with its image $\Pi(x) \in \mathcal{R}$ under the embedding.*

We also make the following observation.

Remark 3.3 *Let x_1 and x_2 be real numbers. Then if both x_1 and x_2 are nonzero, we have that $x_1 \sim x_2$. Furthermore, $x_1 \approx x_2$ is equivalent to $x_1 = x_2$.*

The only nontrivial step toward the proof that $\mathcal{R}$ is a field is the existence of multiplicative inverses of nonzero elements. For this purpose, we prove a new theorem that will be of key importance for a variety of proofs and applications.

Theorem 3.3 (Fixed Point Theorem) *Let $q_M \in Q$ be given. Define $M \subset \mathcal{R}$ to be the set of all elements x of $\mathcal{R}$ such that $\lambda(x) \geq q_M$. Let $f : M \to \mathcal{R}$ satisfy $f(M) \subset M$. Suppose there exists $k > 0$ in Q such that for all $x_1, x_2 \in M$ and for all $q \in Q$, we have that*

$$x_1 =_q x_2 \ \Rightarrow \ f(x_1) =_{q+k} f(x_2).$$

Then there exists a unique solution $x \in M$ of the fixed point equation

$$x = f(x).$$

Proof. We choose an arbitrary $a_0 \in M$ and define recursively

$$a_n = f(a_{n-1}), \text{ for } n = 1, 2, \ldots .$$

Since f maps M into itself, this generates a sequence of elements of M. First we show that for all $n \in Z^+$, we have that

$$a_n[p] = a_{n-1}[p] \text{ for all } p < (n-1)k + q_M. \tag{3.1}$$

Since $a_0, a_1 \in M$, we have that $a_1[p] = 0 = a_0[p]$ for all $p < q_M$. So Equation (3.1) holds for $n = 1$. Assume it is true for $n = m$; we show it is true for $n = m + 1$. Thus, we have that

$$a_m[p] = a_{m-1}[p] \text{ for all } p < (m-1)k + q_M. \tag{3.2}$$

Let $t < mk + q_M$ be given. Then $t - k < (m-1)k + q_M$; and hence Equation (3.2) entails that

$$a_m =_{t-k} a_{m-1}.$$

Hence

$$a_{m+1} = f(a_m) =_t f(a_{m-1}) = a_m,$$

which entails that

$$a_{m+1}[p] = a_m[p] \text{ for all } p \leq t.$$

This is true for all $t < mk + q_M$; hence

$$a_{m+1}[p] = a_m[p] \text{ for all } p < mk + q_M.$$

Thus, Equation (3.1) is true for $n = m + 1$, and hence it is true for all $n \in Z^+$.

Next we define a function $x : Q \to R$ in the following way. For $q \in Q$ choose $n \geq 1$ such that $(n-1)k + q_M > q$. Set $x[q] := a_n[q]$; note that, by virtue of Equation (3.1), this is independent of the choice of n. Furthermore, we have that $x =_q a_n$. So in particular x is an element of $\mathcal{R}$ since for every $q \in Q$, the set of its support points smaller than q agrees with the set of support points smaller than q of one of the $a_n \in M$. Also, since $x[p] = 0$ for all $p < q_M$, we obtain that $x \in M$.

Now we show that x defined as above is a solution of the fixed point equation. For $q \in Q$ choose again $n \geq 1$ such that $(n-1)k + q_M > q$. Then it follows that $x =_q a_n =_q a_{n+1}$. By the contraction property of f, we thus obtain that $f(x) =_{q+k} f(a_n)$, which in turn implies that

$$x[q] = a_{n+1}[q] = f(a_n)[q] = f(x)[q].$$

Since this holds for all $q \in Q$, we have that x is a fixed point of f.

It remains to show that x is a unique fixed point of f in M. Assume that $y \in M$ is a fixed point of f. The contraction property of f is equivalent to

$$\lambda(f(x_1) - f(x_2)) \geq \lambda(x_1 - x_2) + k \text{ for all } x_1, x_2 \in M.$$

This implies that

$$\lambda(x - y) = \lambda(f(x) - f(y)) \geq \lambda(x - y) + k,$$

which is possible only if $y = x$, since $k > 0$.

Remark 3.4 *Without further knowledge about $\mathcal{R}$, the requirements and meaning of the fixed point theorem are not very intuitive. However, as we will see later, the assumption about f means that f is a contracting function with an infinitely small contraction factor. Furthermore, the sequence (a_n) that is constructed in the proof is indeed a Cauchy sequence, which is assured convergence because of the Cauchy completeness of $\mathcal{R}$ with respect to its order topology, as discussed in Chapter 4. However, while making the situation more transparent, the properties of ordering and Cauchy completeness are not required to formulate and prove the fixed point theorem, and so we refrained from invoking them here.*

It is also worthwhile to point out that, in spite of the iterative character of the fixed point theorem, for every $q \in Q$ the value of the fixed point x at q can be calculated in finitely many steps. This is of significant importance especially for practical purposes.

Using the fixed point theorem, we can now easily show the existence of multiplicative inverses.

Theorem 3.4 *($\mathcal{R}$,+,$\cdot$) is a field.*

Proof. It remains to show the existence of multiplicative inverses of nonzero elements. So let $x \in \mathcal{R} \setminus \{0\}$ be given. Set $q = \lambda(x)$, $a = x[q]$ and $x^* = 1/a \cdot d^{-q} \cdot x$. Then $\lambda(x^*) = 0$ and $x^*[0] = 1$. If an inverse of x^* exists, then $1/a \cdot d^{-q} \cdot (x^*)^{-1}$ is an inverse of x; so without loss of generality, we may assume that $\lambda(x) = 0$ and $x[0] = 1$.

If $x = 1$, there exists an inverse. Otherwise, x is of the form $x = 1 + y$ with $0 < k = \lambda(y) < +\infty$. It suffices to find $z \in \mathcal{R}$ such that $(1+z)\cdot(1+y) = 1$. This is equivalent to

$$z = -y \cdot z - y.$$

Setting $f(z) = -y \cdot z - y$ reduces the problem to finding a fixed point of f. Let $M = \{z \in \mathcal{R} : \lambda(z) \geq k\}$. Let $z \in M$ be given; then

$$\lambda(y \cdot z) > \lambda(y); \text{ and hence } \lambda(f(z)) = \lambda(y) = k.$$

Hence $f(M) \subset M$. Now let $z_1, z_2 \in M$ satisfying $z_1 =_q z_2$ be given. Since $\lambda(y) = k$, we obtain that $y \cdot z_1 =_{q+k} y \cdot z_2$, and hence

$$-y \cdot z_1 - y =_{q+k} -y \cdot z_2 - y.$$

Thus f satisfies the hypothesis of the fixed point theorem (Theorem 3.3), and consequently a fixed point of f exists. This finishes the proof of Theorem 3.4.

Now we examine the existence of roots in $\mathcal{R}$. Using the fixed point theorem, we show that, regarding this important property, the new field behaves just like R.

Theorem 3.5 *Let $x \in \mathcal{R}$ be nonzero, and set $q = \lambda(x)$. If n is even and $x[q]$ is positive, then x has two nth roots in $\mathcal{R}$. If n is even and $x[q]$ is negative, then x has no nth roots in $\mathcal{R}$. If n is odd, then x has a unique nth root in $\mathcal{R}$.*

Proof. Let x be a nonzero number and write $x = a \cdot d^q \cdot (1+y)$, where $a \in R$, $q \in Q$, and $\lambda(y) > 0$. Assume that w is an nth root of x. Since $q = \lambda(x) = \lambda(w^n) = n\lambda(w)$, we can write $w = b \cdot d^{q/n} \cdot (1+z)$, where $b \in R$ and $\lambda(z) > 0$. Raising to the nth power, we see that $b^n = a$ and $(1+z)^n = 1+y$ have to hold simultaneously. The first of these equations has a solution if and only if the corresponding roots exist in R. So it suffices to show that the equation

$$(1+z)^n = 1+y \tag{3.3}$$

has a unique solution with $\lambda(z) > 0$. But this equation is equivalent to $nz + z^2 \cdot P(z) = y$, where $P(z)$ is a polynomial with integer coefficients. The equation can be rewritten as a fixed point problem $z = f(z)$, where

$$f(z) = \frac{y}{n} - z^2 \cdot \frac{P(z)}{n}.$$

Let

$$M = \{z \in \mathcal{R} : \lambda(z) \geq \lambda(y)\}.$$

For all $z \in M$, we have that $\lambda(z) \geq \lambda(y) > 0$. Thus, $\lambda(P(z)) \geq 0$, and hence

$$\lambda(z^2 \cdot P(z)) = 2\lambda(z) + \lambda(P(z)) > \lambda(z) \geq \lambda(y).$$

Hence we obtain that $f(z) \approx y/n$; so $f(z) \in M$. Hence

$$f(M) \subset M.$$

Now let $z_1, z_2 \in M$ satisfying $z_1 =_q z_2$ be given. Then $\lambda(z_1) \geq \lambda(y)$, $\lambda(z_2) \geq \lambda(y)$, and the definition of multiplication shows that we obtain $z_1^2 =_{q+\lambda(y)} z_2^2$. By induction on m, we obtain that $z_1^m =_{q+\lambda(y)} z_2^m$ for all $m > 1$. In particular, this implies $z_1^2 \cdot P(z_1) =_{q+\lambda(y)} z_2^2 \cdot P(z_2)$ and finally $f(z_1) =_{q+\lambda(y)} f(z_2)$. So f and M satisfy the hypothesis of the fixed point theorem which provides a unique solution of $(1+z)^n = 1+y$ in M and hence in $\mathcal{R}$.

We remark that a crucial point to the proof was the existence of roots of the numbers d^q; hence we could not have chosen anything smaller than Q as the domain of the functions that are the elements of our new field.

We end this section by remarking that the field $\mathcal{C}$, obtained by adjoining the imaginary number i to $\mathcal{R}$, is algebraically closed. Although a rather deep result, it is obtained with limited effort using the fixed point theorem as well as the algebraic completeness of C (see [5]).

3.2 Differential Algebraic Structure

We introduce an operator ∂ on $\mathcal{R}$ and show that it is a derivation.

Definition 3.7 *Define $\partial : \mathcal{R} \to \mathcal{R}$ by*

$$(\partial x)[q] = (q+1)x[q+1].$$

Lemma 3.4 *∂ is a derivation on $\mathcal{R}$; that is*

$$\partial(x+y) = \partial x + \partial y \text{ and } \partial(x \cdot y) = (\partial x) \cdot y + x \cdot (\partial y) \text{ for all } x, y \in \mathcal{R}.$$

Thus, $(\mathcal{R}, +, \cdot, \partial)$ is a differential algebraic field. Furthermore, we have that

$$\begin{aligned} \lambda(\partial x) &= \lambda(x) - 1 \text{ if } \lambda(x) \neq 0, \infty \text{ and} \\ \partial 0 &= 0; \end{aligned}$$

but if $\lambda(x) = 0$, then $\lambda(\partial x)$ can be either greater than, equal to, or smaller than $\lambda(x)$.

Proof. Let $x, y \in \mathcal{R}$ and let $q \in Q$ be given. Then

$$\begin{aligned} (\partial(x+y))[q] &= (q+1)(x+y)[q+1] = (q+1)x[q+1] + (q+1)y[q+1] \\ &= (\partial x)[q] + (\partial y)[q]. \end{aligned}$$

This is true for all $q \in Q$; hence $\partial(x+y) = \partial x + \partial y$.

For all $q \in Q$, we also have that

$$\begin{aligned}
(\partial(x \cdot y))[q] &= (q+1)(x \cdot y)[q+1] \\
&= (q+1) \sum_{\substack{q_1 + q_2 = q+1 \\ q_1 \in \operatorname{supp}(x),\, q_2 \in \operatorname{supp}(y)}} x[q_1]y[q_2] \\
&= \sum_{\substack{q_1 + q_2 = q+1 \\ q_1 \in \operatorname{supp}(x),\, q_2 \in \operatorname{supp}(y)}} (q+1)x[q_1]y[q_2] \\
&= \sum_{\substack{q_1 + q_2 = q+1 \\ q_1 \in \operatorname{supp}(x),\, q_2 \in \operatorname{supp}(y)}} (q_1 x[q_1]y[q_2] + x[q_1]q_2 x[q_2]) \\
&= \sum_{\substack{q_1 + q_2 = q+1 \\ q_1 \in \operatorname{supp}(x),\, q_2 \in \operatorname{supp}(y)}} q_1 x[q_1]y[q_2] + \sum_{\substack{q_1 + q_2 = q+1 \\ q_1 \in \operatorname{supp}(x),\, q_2 \in \operatorname{supp}(y)}} x[q_1]q_2 y[q_2] \\
&= \sum_{\substack{s + t = q \\ s+1 \in \operatorname{supp}(x),\, t \in \operatorname{supp}(y)}} (s+1)x[s+1]y[t] + \\
&\quad \sum_{\substack{s + t = q \\ s \in \operatorname{supp}(x),\, t+1 \in \operatorname{supp}(y)}} x[s](t+1)y[t+1] \\
&= \sum_{\substack{s + t = q \\ s \in \operatorname{supp}(\partial x),\, t \in \operatorname{supp}(y)}} (\partial x)[s]y[t] + \sum_{\substack{s + t = q \\ s \in \operatorname{supp}(x),\, t \in \operatorname{supp}(\partial y)}} x[s](\partial y)[t] \\
&= ((\partial x) \cdot y)[q] + (x \cdot (\partial y))[q] \\
&= ((\partial x) \cdot y + x \cdot (\partial y))[q].
\end{aligned}$$

This is true for all $q \in Q$; and hence $\partial(x \cdot y) = (\partial x) \cdot y + x \cdot (\partial y)$.

Now let $x \in \mathcal{R}$ be given such that $\lambda(x) \neq 0, \infty$. Then for all $q < \lambda(x) - 1$, we have that $q + 1 < \lambda(x)$; and hence

$$(\partial x)[q] = (q+1)x[q+1] = 0.$$

Hence $\lambda(\partial x) \geq \lambda(x) - 1$; but

$$(\partial x)[\lambda(x) - 1] = \lambda(x)x[\lambda(x)] \neq 0.$$

Hence, $\lambda(\partial x) = \lambda(x) - 1$.

On the other hand, we have that

$$(\partial 0)[q] = (q+1)0[q+1] = 0 \text{ for all } q \in Q.$$

Thus,

$$\partial 0 = 0; \text{ and hence } \lambda(\partial 0) = \lambda(0) = \infty.$$

To prove the last statement, let

$$x_1 = 1,\ x_2 = 1 + d, \text{ and } x_3 = 1 + d^{1/2}; \text{ then } \lambda(x_j) = 0 \text{ for } j = 1, 2, 3.$$

We have that

$$\begin{aligned} \partial x_1 &= 0, \text{ and hence } \lambda(\partial x_1) > \lambda(x_1); \\ \partial x_2 &= 1, \text{ and hence } \lambda(\partial x_2) = \lambda(x_2); \\ \partial x_3 &= \frac{1}{2} d^{-1/2}, \text{ and hence } \lambda(\partial x_3) < \lambda(x_3). \end{aligned}$$

3.3 Order Structure

In the previous section we showed that $\mathcal{R}$ does not differ significantly from R as far as its algebraic properties are concerned. In this section we discuss the ordering.

The simplest way of introducing an order is to define a set of "positive" numbers.

Definition 3.8 (The Set $\mathcal{R}^+$) *Let $\mathcal{R}^+$ be the set of all nonvanishing elements x of $\mathcal{R}$ that satisfy $x[\lambda(x)] > 0$.*

Lemma 3.5 (Properties of $\mathcal{R}^+$) *The set $\mathcal{R}^+$ has the following properties.*

$\mathcal{R}^+ \cap (-\mathcal{R}^+) = \emptyset$, $\mathcal{R}^+ \cap \{0\} = \emptyset$, and $\mathcal{R}^+ \cup \{0\} \cup (-\mathcal{R}^+) = \mathcal{R}$;

$x, y \in \mathcal{R}^+ \ \Rightarrow\ x + y \in \mathcal{R}^+$ and $x \cdot y \in \mathcal{R}^+$.

The proofs follow rather directly from the respective definitions.

Having defined $\mathcal{R}^{+}$, we can now easily introduce an order in $\mathcal{R}$.

Definition 3.9 (Ordering in $\mathcal{R}$) *Let $x, y \in \mathcal{R}$ be distinct. We say $x > y$ if and only if $x - y \in \mathcal{R}^{+}$. Furthermore, we say $x < y$ if and only if $y > x$.*

With this definition of the order relation, $\mathcal{R}$ is a totally ordered field.

Theorem 3.6 (Properties of the Order) *With the order relation defined in Definition 3.9, $(\mathcal{R},+,\cdot)$ becomes a totally ordered field.*

Furthermore, the order is compatible with the algebraic structure of $\mathcal{R}$, that is, for any $x, y, z \in \mathcal{R}$, we have that

$$\begin{aligned} x > y \quad &\Rightarrow \quad x + z > y + z; \ \textit{and} \\ x > y \quad &\Rightarrow \quad x \cdot z > y \cdot z \ \textit{if} \ z > 0. \end{aligned}$$

Since the proof follows the same arguments as the corresponding ones for R, the details are omitted here. We immediately obtain that the embedding Π in Theorem 3.2 is compatible with the ordering, that is

$$x < y \Rightarrow \Pi(x) < \Pi(y).$$

Furthermore $\mathcal{C}$, like C, cannot be ordered.

Thus $\mathcal{R}$, like C, is a proper field extension of R. Note that this is not a contradiction of the well-known uniqueness of C as a field extension of R. The respective theorem of Frobenius asserts only the nonexistence of any (commutative) field on R^n for $n > 2$. However, regarded as an R-vector space, $\mathcal{R}$ is infinite dimensional.

Besides the usual order relations, some other notations are convenient.

Definition 3.10 ($\ll, \gg$) *Let $a, b \in \mathcal{R}$ be nonnegative. We say that a is infinitely smaller than b (and write $a \ll b$) if and only if $n \cdot a < b$ for all $n \in Z^+$; we say that a is infinitely larger than b (and write $a \gg b$) if and only if $b \ll a$. If $a \ll 1$, we say that a is infinitely small; if $1 \ll a$, we say that a is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Nonnegative numbers that are neither infinitely small nor infinitely large are also called finite.*

Corollary 3.1 *For all $a, b, c \in \mathcal{R}^+$, we have that*

$$\begin{aligned} a \ll b \quad &\Rightarrow \quad a < b, \text{ and} \\ a \ll b \text{ and } b \ll c \quad &\Rightarrow \quad a \ll c. \end{aligned}$$

Moreover, we observe that

$$d^q \ll 1 \text{ if and only if } q > 0, \text{ and } d^q \gg 1 \text{ if and only if } q < 0.$$

Corollary 3.2 *The field $\mathcal{R}$ is non-Archimedean.*

Proof. We have that $n \cdot d < 1$ for all $n \in Z^+$; and hence $d \not\sim 1$.

By Lemma 2.2, every field automorphism on $\mathcal{R}$ is order preserving. The following example shows that, while the identity map is the only field automorphism on R by Corollary 2.2, there are nontrivial field automorphisms on $\mathcal{R}$. However, Theorem 3.7 below shows that the image of a real number under a field automorphism is approximately equal to the number itself.

Example 3.1 *Define $P : \mathcal{R} \to \mathcal{R}$ as follows: For $x \in \mathcal{R}$, write $x = \sum_{q \in supp(x)} a_q d^q$ and set $P(x) = \sum_{q \in supp(x)} a_q d^{3q}$. Then P is a field automorphism on $\mathcal{R}$.*

Remark 3.5 *Note that, in Example 3.1,* $P(d) = d^3 \not\sim d$*, in contrast with Corollary 2.5 and Example 2.1.*

Theorem 3.7 *Let* P *be a field automorphism on* $\mathcal{R}$*. Then* $P(r) \approx r$ *for all* $r \in R$*.*

Proof. By Lemma 2.2, we have that P is order preserving. Then the proof is exactly the same as that of Corollary 2.6.

It is a crucial property of the field $\mathcal{R}$ that the differentials, especially the formerly defined number d, satisfy Leibniz's intuitive idea of derivatives as differential quotients. This will be discussed in great detail in Chapter 5; but already here we want to give a simple example.

Example 3.2 (Calculation of Derivatives with Differentials) *Let* $f : R \to R$ *be given by* $f(x) = x^2 - 2x$*.*

Obviously, f is differentiable on R, and we have that $f'(x) = 2x - 2$ for all $x \in R$. As we know, we can obtain certain approximations to the derivative at the position x by calculating the difference quotient

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

at x. Roughly speaking, the accuracy increases if Δx gets smaller. In our enlarged field $\mathcal{R}$, infinitely small quantities are available, and thus it is natural to calculate the difference quotient for such infinitely small numbers. For example, if we let $\Delta x = d$ and let $\bar{f}$ denote the continuation of f to $\mathcal{R}$, then we obtain that

$$\frac{\bar{f}(x + d) - f(x)}{d} = \frac{(x^2 + 2xd + d^2 - 2x - 2d) - (x^2 - 2x)}{d} = 2x - 2 + d.$$

We realize that the difference quotient differs from the exact value of the derivative by only an infinitely small error. If all we are interested in is the usual real derivative of the real function f, then this is given exactly by the "real part" of the difference quotient.

3.4 Topological Structure

In this section we examine the topological structures of $\mathcal{R}$ and the related sets. We will see that on $\mathcal{R}$, in contrast to R, several different nontrivial topologies can be defined, all of which have certain advantages.

We begin with the introduction of an absolute value; this is done as in any totally ordered field.

Definition 3.11 (Absolute Value on $\mathcal{R}$) *Let $x \in \mathcal{R}$ be given. We define the absolute value of x as follows.*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} .$$

Lemma 3.6 (Properties of the Absolute Value) *The mapping $|\cdot| : \mathcal{R} \to \mathcal{R}$ has the following properties.*

$$|x| = 0 \text{ if and only if } x = 0.$$

$$|x \cdot y| = |x| \cdot |y| \text{ for all } x, y \in \mathcal{R}.$$

$$|x + y| \leq |x| + |y| \text{ for all } x, y \in \mathcal{R}.$$

$$||x| - |y|| \leq |x - y| \text{ for all } x, y \in \mathcal{R}.$$

Proof. The proof follows the same lines as the proof of the corresponding result in R.

Just as in any totally ordered set, we can now introduce the so-called order topology.

Definition 3.12 (Order Topology) *We call a subset M of $\mathcal{R}$ open if and only if for any $x_0 \in M$ there exists an $\epsilon > 0$ in $\mathcal{R}$ such that $O(x_0, \epsilon)$, the set of points x with $|x - x_0| < \epsilon$, is a subset of M.*

Thus all ϵ-balls form a basis of the topology. We obtain the following theorem.

Theorem 3.8 (Properties of the Order Topology) *With the above topology, $\mathcal{R}$ is a nonconnected topological space. It is Hausdorff. There are no countable bases. The topology induced to R is the discrete topology. The topology is not locally compact.*

Proof. We first observe that for all $x_0 \in \mathcal{R}$ and for all $\epsilon > 0$ in $\mathcal{R}$, the balls $O(x_0, \epsilon)$ are open; and so is the whole space. Furthermore, all unions and finite intersections of open sets are obviously open. To show that $\mathcal{R}$ is not connected, let

$$\begin{aligned} M_1 &= \{x \in \mathcal{R} : x \leq 0 \text{ or } (x > 0 \text{ and } x \ll 1)\}; \text{ and} \\ M_2 &= \{x \in \mathcal{R} : x > 0 \text{ and } x \not\ll 1\}. \end{aligned}$$

Then M_1 and M_2 are open and disjoint; moreover, we have that $M_1 \cup M_2 = \mathcal{R}$.

For all $x, y \in \mathcal{R}$, $O(x, |x - y|/2)$ and $O(y, |x - y|/2)$ are open and disjoint, and they contain x and y, respectively. Hence $\mathcal{R}$ is Hausdorff.

There can not be any countable bases because the uncountably many open sets $M_X = O(X, d)$, with $X \in R$, are disjoint. The open sets induced on R by the sets M_X are just the single points. Thus, in the induced topology, all sets are open and the topology is therefore discrete.

To prove that the space is not locally compact, let $x \in \mathcal{R}$ be given and let U be a neighborhood of x. We show that the closure $\bar{U}$ of U is not compact. Let $\epsilon > 0$ in $\mathcal{R}$ be such that $O(x, \epsilon) \subset U$ and consider the sets

$$M_{-1} = \{y \in \mathcal{R} : y < x \text{ or } y - x \gg d \cdot \epsilon\};$$

$$M_n \quad = \quad (x + (n-1)d \cdot \epsilon, x + (n+1)d \cdot \epsilon) \text{ for } n = 0, 1, 2, \ldots.$$

Then it is easy to check that M_n is open for all $n \geq -1$ and

$$\cup_{n=-1}^{\infty} M_n = \mathcal{R}; \text{ in particular, } \bar{U} \subset \cup_{n=-1}^{\infty} M_n.$$

But it is impossible to select finitely many of the M_n's to cover $\bar{U}$ because each of the infinitely many elements $x + nd \cdot \epsilon$ of $\bar{U}$, $n = -1, 0, 1, 2, \ldots$, is contained only in the set M_n.

Remark 3.6 *A detailed study of the properties in Theorem 3.8 reveals that they hold in an identical way on any other non-Archimedean structure, and thus the above unusual properties are not specific to $\mathcal{R}$.*

Besides the absolute value, it is useful to introduce a semi-norm that is not based on the order. For this purpose, we regard $\mathcal{R}$ as a space of functions as in the beginning, and define the semi-norm as a mapping from $\mathcal{R}$ into R.

Definition 3.13 *Given $r \in Q$, we define a mapping $\| \cdot \|_r : \mathcal{R} \to R$ as follows.*

$$\|x\|_r = \sup\{|x[q]| : q \leq r\}. \tag{3.4}$$

Remark 3.7 *The supremum in Equation (3.4) is finite and it is even a maximum since, for any r, only finitely many of the $x[q]$'s considered do not vanish.*

Lemma 3.7 *For any $r \in Q$, the mapping $\| \cdot \|_r : \mathcal{R} \to R$ satisfies the following.*

$$\|0\|_r = 0 = \|d^t\|_r \text{ for all } t > r \text{ in } Q. \tag{3.5}$$

$$\|x\|_r = \|-x\|_r \text{ for all } x \in \mathcal{R}. \tag{3.6}$$

$$\|x\|_r \geq 0 \text{ for all } x \in \mathcal{R}. \tag{3.7}$$

$$\|x+y\|_r \leq \|x\|_r + \|y\|_r \text{ for all } x, y \in \mathcal{R}. \tag{3.8}$$

$$|\|x\|_r - \|y\|_r| \leq \|x-y\|_r \text{ for all } x, y \in \mathcal{R}. \tag{3.9}$$

Proof. Equations (3.5), (3.6), and (3.7) follow readily from the definition. To prove Equation (3.8), let $x, y \in \mathcal{R}$ and $r \in Q$ be given. Then

$$\begin{aligned} \|x\|_r &= \sup\{|x[q]| : q \leq r\}, \\ \|y\|_r &= \sup\{|y[q]| : q \leq r\}, \text{ and} \\ \|x+y\|_r &= \sup\{|(x+y)[q]| : q \leq r\}. \end{aligned}$$

Let $q_0 \in Q$ be such that $q_0 \leq r$ and $|(x+y)[q_0]| = \|x+y\|_r$. Then

$$\begin{aligned} \|x+y\|_r &= |(x+y)[q_0]| = |x[q_0] + y[q_0]| \\ &\leq |x[q_0]| + |y[q_0]| \\ &\leq \|x\|_r + \|y\|_r. \end{aligned}$$

We finally prove Equation (3.9): Let $x, y \in \mathcal{R}$ and $r \in Q$ be given. Then, using Equation (3.8), we have that $\|x\|_r \leq \|x-y\|_r + \|y\|_r$, from which we obtain that

$$\|x\|_r - \|y\|_r \leq \|x-y\|_r. \tag{3.10}$$

Interchanging x and y in Equation (3.10) and using Equation (3.6), we obtain that

$$\|y\|_r - \|x\|_r \leq \|y-x\|_r = \|x-y\|_r. \tag{3.11}$$

combining Equation (3.10) and Equation (3.11), we obtain Equation (3.9) and finish the proof of the lemma.

Remark 3.8 *From Equations (3.5), (3.6), (3.7), and (3.8), we infer that $\|\cdot\|_r$ is a semi-norm but not a norm, for any $r \in Q$.*

The topology induced by the family of these semi-norms will be called weak topology.

Definition 3.14 (Weak Topology) *We call a subset M of $\mathcal{R}$ open with respect to the weak topology if and only if for any $x_0 \in M$ there exists $\epsilon > 0$ in R such that $S(x_0, \epsilon) = \{x \in \mathcal{R} : ||x - x_0||_{1/\epsilon} < \epsilon\} \subset M$.*

We will see that the weak topology is the most useful topology for considering convergence in general; see Chapters 4, 5, 6, and 7. Moreover, it is of great importance for the implementation of the $\mathcal{R}$ calculus on computers; see Chapter 7.

Theorem 3.9 (Properties of the Weak Topology) *With the above definition of the weak topology, $\mathcal{R}$ is a topological space. It is Hausdorff with countable bases. The topology induced on R by the weak topology is the usual order topology on R.*

Proof. It is easy to check that for all $x_0 \in \mathcal{R}$ and for all $\epsilon > 0$ in R, the balls $S(x_0, \epsilon)$ are open; and so is the whole space. Furthermore, all unions and finite intersections of open sets are open. The balls $S(r, q)$ with $r, q \in Q$ form a countable basis of the topology. We obtain a Hausdorff space: Let $x, y \in \mathcal{R}$ be given, let $r = \lambda(x - y)$, and let

$$\epsilon = \min\left\{\frac{|(x-y)[r]|}{2}, \frac{1}{2|r|}\right\}.$$

Then $S(x, \epsilon)$ and $S(y, \epsilon)$ are disjoint and open, and they contain x and y, respectively. Finally, considering elements of R, their supports are all equal to $\{0\}$. Therefore, the open subsets of $\mathcal{R}$ in the weak topology correspond to the open subsets of R in its order topology.

In Chapter 4, we will study in details convergence of sequences and series in the strong and weak topologies, and we will show that $\mathcal{R}$ is Cauchy complete with respect to the strong topology while it is not with respect to the weak topology.

In addition to the two topologies discussed above, there is another topology which takes into account that, in any practical scenario, it will not be possible to detect infinitely small errors, nor will it possible to measure infinitely large quantities. We obtain this topology by a suitable continuation of the order topology on R.

Definition 3.15 (Measure Topology) *Given any open subset of R, we form a subset of $\mathcal{R}$ containing the elements of the original set as well as all the elements infinitely close to them. To the family of sets obtained this way, we add one more set, namely the one containing every element with infinitely large absolute value.*

Thus a basis of this topology consists of all ϵ-balls with real ϵ and the set of numbers with infinitely large absolute value.

Theorem 3.10 (Properties of the Measure Topology) *With the above topology, $\mathcal{R}$ is a nonconnected topological space with countable bases. It is not Hausdorff. The topology is locally compact and induces the usual order topology on R.*

Proof. We can directly show that the whole space as well as unions and finite intersections of open sets are open. Obviously, elements with infinitely small difference can not be separated; they are always simultaneously inside or outside of any given open set. Hence the space is not Hausdorff with respect to the measure topology. The rest follows by transferring the properties of the order topology on R.

Remark 3.9 (Comparison of the Topologies) *The order topology is a refinement of both the weak topology and the measure topology.*

To finish this section, we will show that the field $\mathcal{R}$ is indeed the smallest non-Archimedean extension of R satisfying the basic requirements demanded in Chapter 1, which gives it a unique position among all other field extensions.

Theorem 3.11 (Uniqueness of $\mathcal{R}$) *The field $\mathcal{R}$ is the smallest totally ordered non-Archimedean field extension of R that is complete with respect to the order topology, in which every positive number has an nth root, and in which there is a positive infinitely small element a such that (a^n) is a null sequence with respect to the order topology.*

Proof. Obviously, $\mathcal{R}$ satisfies the conditions above. So it remains to show that $\mathcal{R}$ can be embedded in any other field extension of R that has the properties mentioned above. So let $\mathcal{S}$ be such a field.

Let $\delta \in \mathcal{S}$ be positive and infinitely small such that (δ^n) is a null sequence with respect to the order topology in $\mathcal{S}$. Let $\delta^{1/n}$ be an n-th root of δ. Such a root exists according to the requirements. Now observe that

$$\left(\delta^{1/n}\right)^m = \left(\delta^{1/(np)}\right)^{mp}, \text{ for all } p \in Z^+.$$

So let $q = m/n$ be given in Q, and let

$$\delta^q = \left(\delta^{1/n}\right)^m.$$

This element is unique. Furthermore, δ^q is infinitely small for $q > 0$. Let $q_1 < q_2$ be given in Q. Then we clearly have that $\delta^{q_1} > \delta^{q_2}$. Now let $a \in R$ be given. Then we also have that $a \in \mathcal{S}$, and hence $a \cdot \delta^q \in \mathcal{S}$.

Now let $((q_i), x[q_i])$ be the table of an element x of $\mathcal{R}$. Consider the sequence

$$s_j = \sum_{i=1}^{j} x[q_i]\delta^{q_i}.$$

Then the sequence (s_j) converges in $\mathcal{S}$: Let $\epsilon > 0$ be given in $\mathcal{S}$. Since the sequence (δ^n) is a null sequence in $\mathcal{S}$, there exists $N \in Z^+$ such that

$$|\delta^\nu| < \epsilon \text{ for all } \nu \geq N.$$

Since the sequence (q_i) is strictly divergent, there exists $N_1 \in Z^+$ such that

$$q_i \geq N \text{ for all } i \geq N_1.$$

But then we have for arbitrary $j_1 > j_2 \geq N_1$ that

$$\begin{aligned} |s_{j_1} - s_{j_2}| &= |\sum_{i=j_2}^{j_1} x[q_i]\delta^{q_i}| \leq \sum_{i=j_2}^{j_1} |x[q_i]|\delta^{q_i} \leq \left(\sum_{i=j_2}^{j_1} |x[q_i]| \right) \delta^{q_{j_2+1}} \\ &\leq \left(\sum_{i=j_2}^{j_1} |x[q_i]| \right) \delta^{N+1} < \delta^N < \epsilon. \end{aligned}$$

Thus the sequence (s_j) is Cauchy, and hence it converges in $\mathcal{S}$ because of the Cauchy completeness of $\mathcal{S}$. We now assign to every element $\sum_{i=1}^{\infty} x[q_i] \cdot d^{q_i}$ of $\mathcal{R}$ the element $\sum_{i=1}^{\infty} x[q_i] \cdot \delta^{q_i}$ of $\mathcal{S}$. The mapping is one to one. Furthermore, we can easily verify that it is compatible with the algebraic operations and the order on $\mathcal{R}$.

Remark 3.10 *A field with the properties of $\mathcal{R}$ could also be obtained by successively extending a simpler non-Archimedean field, e.g. the well-known field of rational functions. To do this, we first would have to Cauchy complete the field. After that, the algebraic closure has to be done, for example by the method of Kronecker-Steinitz. This method, however, is nonconstructive, whereas the direct path followed here is entirely constructive.*

Remark 3.11 *In the proof of the uniqueness, we noted that δ was only required to be positive, infinitely small and such that (δ^n) is a null sequence in $\mathcal{S}$. But besides that, its actual magnitude was irrelevant. Thus, none of the infinitely small quantities is significantly different from the others. In particular, there exists a nontrivial field automorphism on $\mathcal{S}$. This remarkable property has no analogy in R where the identity map is the only field automorphism; see Corollary 2.2.*

Chapter 4

Sequences and Series

In this chapter, we review convergence of sequences and series with respect to the order and weak topologies following [3, 5, 7]. We also prove new results; in particular, those dealing with the convergence of sums and products of sequences and series. We then enhance and prove a weak convergence criterion for power series, Theorem 4.12, and use that to extend the transcendental functions to $\mathcal{R}$ and study their properties in Section 4.4.

4.1 Strong Convergence

We begin this section by studying a special property of sequences.

Definition 4.1 (Regularity) *A sequence (s_n) in $\mathcal{R}$ is called regular if and only if the union of the supports of all members of the sequence is a left-finite set, that is if and only if $\cup_{n=0}^{\infty}\mathrm{supp}(s_n) \in \mathcal{F}$.*

This property is not automatically assured, as becomes apparent from considering the sequence (d^{-n}). As the next theorem shows, the property of regularity is compatible with the common operations of sequences.

Lemma 4.1 (Properties of Regularity) *Let* (s_n) *and* (t_n) *be regular sequences in* $\mathcal{R}$*. Then the sequence of the sums, the sequence of the products, any rearrangement, as well as any subsequence of one of the sequences, and the merged sequence* $r_{2n} = s_n$*,* $r_{2n+1} = t_n$ *are regular.*

Proof. Let $A = \cup_{n=0}^{\infty}\text{supp}(s_n)$ and $B = \cup_{n=0}^{\infty}\text{supp}(t_n)$. Then, according to the requirements, we have that $A \in \mathcal{F}$ and $B \in \mathcal{F}$.

Every support point of the sequence of the sums is a support point of either one of the s_n or one of the t_n and is thus contained in $(A \cup B) \in \mathcal{F}$, using Lemma 3.2. Every support point of the sequence of the products is contained in $(A + B) \in \mathcal{F}$, again using Lemma 3.2.

The support points of any subsequence of (s_n) are contained in A , and the support points of the joined sequence (r_n) are contained in $A \cup B$.

Definition 4.2 (Strong Convergence) *Let* (s_n) *be a sequence in* $\mathcal{R}$*. We say that* (s_n) *is strongly convergent to the limit* $s \in \mathcal{R}$ *if and only if for every* $\epsilon > 0$ *in* $\mathcal{R}$ *there exists* $N \in Z^+$ *such that*

$$|s_n - s| < \epsilon \text{ for all } n \geq N.$$

Remark 4.1 *Like in any other metric space, it is easy to show that if the limit of a sequence exists then it is unique.*

Using the idea of strong convergence allows a simple representation of the elements of $\mathcal{R}$ that is indeed strongly reminiscent of the familiar expansion of real numbers in powers of ten, and that enjoys a similar usefulness for practical calculations.

Theorem 4.1 (Expansion in Powers of Differentials) *Let $((q_n), (x[q_n]))$ be the table of $x \in \mathcal{R}$ (see Definition 3.3). Then the sequence (x_n), given by $x_n = \sum_{i=1}^{n} x[q_i] \cdot d^{q_i}$ for all $n \in Z^+$, converges strongly to the limit x. Hence we can write*

$$x = \sum_{n=1}^{\infty} x[q_n] \cdot d^{q_n}.$$

Proof. Without loss of generality, we may assume that the set $\{q_n\}$ is infinite. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Choose $N_1 \in Z^+$ such that $d^{N_1} < \epsilon$. Since (q_n) diverges strictly according to Lemma 3.1, there exists $N \in Z^+$ such that

$$q_n > N_1 \text{ for all } n \geq N.$$

Hence we have that

$$(x_n - x)[t] = 0 \text{ for all } t \leq N_1 \text{ and for all } n \geq N.$$

Thus

$$|x_n - x| < \epsilon \text{ for all } n \geq N.$$

Therefore, (x_n) converges strongly to x.

Lemma 4.2 *Let (s_n) be a sequence converging strongly in $\mathcal{R}$ to s. Then $(|s_n|)$ converges strongly in $\mathcal{R}$ to $|s|$.*

Proof. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Then there exists $N \in Z^+$ such that $|s_m - s| < \epsilon$ for all $m \geq N$. Therefore,

$$||s_m| - |s|| \leq |s_m - s| < \epsilon \text{ for all } m \geq N.$$

Hence, $(|s_n|)$ converges strongly to $|s|$.

Remark 4.2 *The converse of Lemma 4.2 is not true.*

Proof. Consider the sequence (s_n) in $\mathcal{R}$, where, for $k = 0, 1, \ldots$, $s_{2k} = -1$ and $s_{2k+1} = 1$. Then $|s_n| = 1$ for all $n \geq 0$. Hence $(|s_n|)$ converges strongly to 1 in $\mathcal{R}$. However, (s_n) does not converge strongly in $\mathcal{R}$.

Definition 4.3 *Let (s_n) be a sequence in $\mathcal{R}$. Then we say that (s_n) is strongly Cauchy if and only if for all $\epsilon > 0$ in $\mathcal{R}$, there exists $N \in Z^+$ such that*

$$|s_m - s_l| < \epsilon \textit{ for all } m, l \geq N.$$

Like R, the new field $\mathcal{R}$ is Cauchy complete with respect to the order topology. That is, a sequence (s_n) in $\mathcal{R}$ converges strongly if and only if (s_n) is strongly Cauchy.

Theorem 4.2 *$\mathcal{R}$ is Cauchy complete with respect to the order topology.*

Proof. Let (s_n) be a sequence in $\mathcal{R}$ that converges strongly to $s \in \mathcal{R}$. We show that (s_n) is strongly Cauchy. So let $\epsilon > 0$ be given in $\mathcal{R}$. Then there exists $N \in Z^+$ such that

$$|s_n - s| < \frac{\epsilon}{2} \text{ for all } n \geq N.$$

For all $m, l \geq N$, we have that

$$|s_m - s_l| = |s_m - s - (s_l - s)| \leq |s_m - s| + |s_l - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence (s_n) is strongly Cauchy.

Now let (s_n) be a strongly Cauchy sequence in $\mathcal{R}$. We show that (s_n) converges strongly in $\mathcal{R}$. For all $r \in Q$, there exists $N_r \in Z^+$ such that

$$|s_m - s_l| < d^{r+1} \text{ for all } m, l \geq N_r. \tag{4.1}$$

Thus,

$$s_m[q] = s_{N_r}[q] \text{ for all } m \geq N_r \text{ and for all } q \leq r. \tag{4.2}$$

By Equation (4.1), we may assume that

$$N_{r_1} \leq N_{r_2} \text{ if } r_1 < r_2. \tag{4.3}$$

Define $s : Q \to R$ by $s[q] = s_{N_q}[q]$. First we show that $s \in \mathcal{R}$; that is, we show that $\mathrm{supp}(s)$ is left-finite. So let $r \in Q$ be given. Combining Equation (4.2) and Equation (4.3), we obtain that

$$s[q] = s_{N_q}[q] = s_{N_r}[q] \text{ for all } q \leq r.$$

Thus there are only finitely many $q \leq r$ such that $s[q] \neq 0$, and hence $\mathrm{supp}(s)$ is left-finite. Finally, we show that (s_n) converges strongly to s. So let $\epsilon > 0$ be given in $\mathcal{R}$. There exists $r \in Q$ such that $d^r < \epsilon$. Then

$$s_m[q] = s_{N_r}[q] = s[q] \text{ for all } m \geq N_r \text{ and for all } q \leq r.$$

Hence

$$|s_m - s| \ll d^r < \epsilon \text{ for all } m \geq N_r.$$

Thus, (s_n) converges strongly to s in $\mathcal{R}$.

Like in any other metric space, every Cauchy sequence is bounded.

Lemma 4.3 *Let (s_n) be a strongly Cauchy sequence in $\mathcal{R}$. Then (s_n) is bounded.*

Corollary 4.1 *Every strongly convergent sequence in $\mathcal{R}$ is bounded.*

Theorem 4.3 (Strong Convergence Criterion for sequences) *Let (s_n) be a sequence in $\mathcal{R}$. Then (s_n) converges strongly in $\mathcal{R}$ if and only if for all $r \in Q$ there exists $N \in Z^+$ such that*

$$s_m =_r s_l \text{ for all } m, l \geq N.$$

Proof. Let (s_n) be a strongly convergent sequence in $\mathcal{R}$, and let $r \in Q$ be given. Then there exists $N \in Z^+$ such that

$$|s_m - s_l| < d^{r+1} \text{ for all } m, l \geq N.$$

Hence $s_m =_r s_l$ for all $m, l \geq N$.

Now let (s_n) be a sequence in $\mathcal{R}$ such that for all $r \in Q$ there exists $N \in Z^+$ such that $s_m =_r s_l$ for all $m, l \geq N$. Let $\epsilon > 0$ in $\mathcal{R}$ be given; and let

$$r = \lambda(\epsilon).$$

Then there exists $N \in Z^+$ such that $s_m =_r s_l$ for all $m, l \geq N$; and hence

$$|s_m - s_l| \ll d^r \text{ for all } m, l \geq N.$$

Since $d^r \sim \epsilon$, we obtain that

$$|s_m - s_l| \ll \epsilon \text{ for all } m, l \geq N.$$

Hence (s_n) is a strongly Cauchy sequence; so by Theorem 4.2, the sequence (s_n) converges strongly in $\mathcal{R}$.

Lemma 4.4 *Let (s_n) be a strongly convergent sequence in $\mathcal{R}$. Then (s_n) is regular.*

Proof. Let s be the limit of (s_n) in $\mathcal{R}$; and let $r \in Q$ be given. Then there exists $N \in Z^+$ such that

$$|s_m - s| < d^{r+1} \text{ for all } m \geq N.$$

It follows that

$$s_m[q] = s[q] \text{ for all } m \geq N \text{ and for all } q \leq r.$$

Thus,

$$(-\infty, r] \cap \left(\cup_{n=0}^{\infty} \text{supp}(s_n)\right) = (-\infty, r] \cap \left(\cup_{n=0}^{N-1} \text{supp}(s_n)\right) \cup \left((-\infty, r] \cap \text{supp}(s)\right)$$

is finite, it being the union of two finite sets. So $\cup_{n=0}^{\infty}\text{supp}(s_n)$ is left-finite, and hence the sequence (s_n) is regular.

The following results: Theorem 4.4, Corollary 4.2, Corollary 4.3 and Corollary 4.4 do not hold in R; the non-Archimedicity of $\mathcal{R}$ is the key to their proofs.

Theorem 4.4 *Let (s_n) be a sequence in $\mathcal{R}$. Then (s_n) is strongly Cauchy if and only if $(s_{n+1} - s_n)$ is a null sequence.*

Proof. Let (s_n) be a Cauchy sequence in $\mathcal{R}$, and let $\epsilon > 0$ in $\mathcal{R}$ be given. Then there exists $N \in Z^+$ such that $|s_l - s_m| < \epsilon$ for all $l, m \geq N$. In particular, $|s_{m+1} - s_m| < \epsilon$ for all $m \geq N$. Hence, $\lim_{n\to\infty}(s_{n+1} - s_n) = 0$.

Now assume that $(s_{n+1} - s_n)$ is a null sequence in $\mathcal{R}$, and let $\epsilon > 0$ in $\mathcal{R}$ be given. Then there exists $N \in Z^+$ such that

$$|s_{m+1} - s_m| < d\epsilon \text{ for all } m \geq N.$$

Let $k, l \geq N$ be given. Without loss of generality, we may assume that $k > l$. Then we have that

$$\begin{aligned} |s_k - s_l| &= |s_k - s_{k-1} + s_{k-1} - s_{k-2} + \cdots + s_{l+1} - s_l| \\ &\leq |s_k - s_{k-1}| + |s_{k-1} - s_{k-2}| + \cdots + |s_{l+1} - s_l| \\ &< (k-l)d\epsilon \\ &< \epsilon \text{ since } (k-l)d \text{ is infinitely small.} \end{aligned}$$

Thus, (s_n) is strongly Cauchy in $\mathcal{R}$.

Corollary 4.2 *Let (s_n) be a sequence in $\mathcal{R}$. Then, (s_n) converges strongly if and only if $(s_{n+1} - s_n)$ is a null sequence with respect to the order topology.*

Corollary 4.3 *The infinite series* $\sum_{n=0}^{\infty} a_n$ *converges strongly in* $\mathcal{R}$ *if and only if the sequence* (a_n) *is a null sequence in* $\mathcal{R}$.

Corollary 4.4 *The series* $\sum_{n=0}^{\infty} a_n$ *converges strongly if and only if it converges absolutely strongly, that is if and only if* $\sum_{n=0}^{\infty} |a_n|$ *converges strongly.*

Proof. We have, using Corollary 4.3, that

$$\begin{aligned} \sum_{n=0}^{\infty} a_n \text{ converges strongly} \quad &\Leftrightarrow \quad \lim_{n\to\infty} a_n = 0 \\ &\Leftrightarrow \quad \lim_{n\to\infty} |a_n| = 0 \\ &\Leftrightarrow \quad \sum_{n=0}^{\infty} |a_n| \text{ converges strongly.} \end{aligned}$$

The proofs of Lemma 4.5, Corollary 4.5 and Lemma 4.6 follow the same lines as the those of the corresponding results in R; so we omit these proofs here.

Lemma 4.5 *Let* (s_n) *and* (t_n) *be two sequences converging strongly in* $\mathcal{R}$ *to* s *and* t, *respectively. Then, the sequence* $(s_n + t_n)$ *converges strongly to* $s + t$.

Corollary 4.5 *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be two infinite series converging strongly in* $\mathcal{R}$ *to* a *and* b, *respectively. Then, the series* $\sum_{n=0}^{\infty}(a_n + b_n)$ *converges strongly to* $a + b$.

Lemma 4.6 *Let* (s_n) *and* (t_n) *be two sequences in* $\mathcal{R}$ *converging strongly to* s *and* t, *respectively. Then the sequence* $(s_n t_n)$ *converges strongly to* st.

Again the non-Archimedicity of $\mathcal{R}$ gives us a nice result in Theorem 4.5, which does not hold in R without the additional requirement that one of the series converge absolutely.

Theorem 4.5 *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be two infinite series converging strongly in* $\mathcal{R}$ *to* a *and* b*, respectively. Then, the series* $\sum_{n=0}^{\infty} c_n$*, where* $c_n = \sum_{j=0}^{n} a_j b_{n-j}$*, converges strongly to* ab *in* $\mathcal{R}$*.*

Proof. First, we show that $\sum_{n=0}^{\infty} c_n$ converges strongly in $\mathcal{R}$. By Corollary 4.3, it suffices to show that $\lim_{n\to\infty} c_n = 0$. Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge strongly in $\mathcal{R}$, the sequences (a_n) and (b_n) are both strongly null in $\mathcal{R}$. Hence, by Corollary 4.1, (a_n) and (b_n) are both bounded. Therefore, there exists $B > 0$ in $\mathcal{R}$ such that

$$|a_n| < B \text{ and } |b_n| < B \text{ for all } n \geq 0.$$

Let $\epsilon > 0$ in $\mathcal{R}$ be given. Then, there exists $M \in Z^+$ such that

$$|a_m| < \frac{d\epsilon}{B} \text{ and } |b_m| < \frac{d\epsilon}{B} \text{ for all } m \geq M.$$

Let $N = 2M$. Then, for all $m \geq N$, we have that

$$\begin{aligned}
|c_m| &= |a_0 b_m + a_1 b_{m-1} + \cdots + a_{m-1} b_1 + a_m b_0| \\
&\leq |a_0 b_m| + |a_1 b_{m-1}| + \cdots + |a_{m-1} b_1| + |a_m b_0| \\
&= |a_0||b_m| + |a_1||b_{m-1}| + \cdots + |a_{m-1}||b_1| + |a_m||b_0| \\
&< B\frac{d\epsilon}{B} + B\frac{d\epsilon}{B} + \cdots + \frac{d\epsilon}{B}B + \frac{d\epsilon}{B}B \\
&= (m+1)d\epsilon \\
&< \epsilon.
\end{aligned}$$

So, for all $\epsilon > 0$ in $\mathcal{R}$, we can find $N \in Z^+$ such that $|c_m| < \epsilon$ for all $m \geq N$. Hence, $\lim_{n\to\infty} c_n = 0$ and thus $\sum_{n=0}^{\infty} c_n$ converges strongly in $\mathcal{R}$. It remains to show that $\sum_{n=0}^{\infty} c_n = ab$.

Consider the sequence of partial sums (s_{2n}), where

$$s_{2n} = c_0 + c_1 + \cdots + c_{2n}$$

$$\begin{aligned} &= \sum_{i+j=0}^{2n} a_i b_j \\ &= (a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n) \\ &\quad + a_0(b_{n+1} + \cdots + b_{2n}) + a_1(b_{n+1} + \cdots + b_{2n-1}) + \cdots + a_{n-1}b_{n+1} \\ &\quad + b_0(a_{n+1} + \cdots + a_{2n}) + b_1(a_{n+1} + \cdots + a_{2n-1}) + \cdots + b_{n-1}a_{n+1}. \end{aligned}$$

Note that

$$\begin{aligned} & |a_0(b_{n+1} + \cdots + b_{2n}) + a_1(b_{n+1} + \cdots + b_{2n-1}) + \cdots + a_{n-1}b_{n+1}| \\ \leq\ & |a_0||b_{n+1} + \cdots + b_{2n}| + |a_1||b_{n+1} + \cdots + b_{2n-1}| + \cdots + |a_{n-1}||b_{n+1}| \\ \leq\ & B(|b_{n+1} + \cdots + b_{2n}| + |b_{n+1} + \cdots + b_{2n-1}| + \cdots + |b_{n+1}|). \end{aligned}$$

Let $\epsilon > 0$ in $\mathcal{R}$ be given. Since (b_n) is a null sequence, there exists $N \in Z^+$ such that

$$|b_n| < \frac{d\epsilon}{B} \text{ for all } n \geq N.$$

Hence, for all $n \geq N$ and for all $p \in Z^+$, we have that

$$\begin{aligned} |b_{n+1} + \cdots + b_{n+p}| &\leq |b_{n+1}| + \cdots + |b_{n+p}| \\ &< \frac{d\epsilon}{B} + \cdots + \frac{d\epsilon}{B} \\ &= (pd) \cdot \frac{\epsilon}{B} = (pnd)\frac{\epsilon}{nB} \\ &< \frac{\epsilon}{nB}, \end{aligned}$$

where, in the last step, we made use of the fact that d is infinitely small and pn is an integer, so that $pnd < 1$. Therefore, for all $n \geq N$, we have that

$$\begin{aligned} & |a_0(b_{n+1} + \cdots + b_{2n}) + a_1(b_{n+1} + \cdots + b_{2n-1}) + \cdots + a_{n-1}b_{n+1}| \\ & < B(\underbrace{\frac{\epsilon}{nB} + \frac{\epsilon}{nB} + \cdots + \frac{\epsilon}{nB}}_{n \text{ times}}) = \epsilon. \end{aligned}$$

Therefore,

$$\lim_{n \to \infty} (a_0(b_{n+1} + \cdots + b_{2n}) + a_1(b_{n+1} + \cdots + b_{2n-1}) + \cdots + a_{n-1}b_{n+1}) = 0.$$

Similarly, we can show that

$$\lim_{n\to\infty}\left(b_0(a_{n+1}+\cdots+a_{2n})+b_1(a_{n+1}+\cdots+a_{2n-1})+\cdots+b_{n-1}a_{n+1}\right)=0.$$

Hence

$$\begin{aligned}\lim_{n\to\infty} s_{2n} &= \lim_{n\to\infty}\left((a_0+a_1+\cdots+a_n)(b_0+b_1+\cdots+b_n)\right)+\\ &\quad \lim_{n\to\infty}\left(a_0(b_{n+1}+\cdots+b_{2n})+a_1(b_{n+1}+\cdots+b_{2n-1})+\cdots+a_{n-1}b_{n+1}\right)+\\ &\quad \lim_{n\to\infty}\left(b_0(a_{n+1}+\cdots+a_{2n})+b_1(a_{n+1}+\cdots+a_{2n-1})+\cdots+b_{n-1}a_{n+1}\right)\\ &= \lim_{n\to\infty}\left((a_0+a_1+\cdots+a_n)(b_0+b_1+\cdots+b_n)\right).\end{aligned}$$

Let (A_n) and (B_n) denote the sequences of partial sums of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, respectively. Then $A_n = a_0 + a_1 + \cdots + a_n$, $\lim_{n\to\infty} A_n = a$; $B_n = b_0 + b_1 + \cdots + b_n$, and $\lim_{n\to\infty} B_n = b$. Therefore,

$$\begin{aligned}\lim_{n\to\infty} s_{2n} &= \lim_{n\to\infty}(A_n B_n)\\ &= \left(\lim_{n\to\infty} A_n\right)\left(\lim_{n\to\infty} B_n\right) \text{ using Lemma 4.6}\\ &= ab.\end{aligned}$$

Since $\sum_{n=0}^{\infty} c_n$ converges strongly, it has one and only one limit. Hence

$$\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = ab = \lim_{n\to\infty} s_n;$$

so

$$\sum_{n=0}^{\infty} c_n = ab = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Lemma 4.7 *Let (s_n) be a sequence in $\mathcal{R}$ converging strongly to s. Assume there exist $j \in Z^+$ and $t \in Q$ such that $\lambda(s_m) = t$ for all $m \geq j$. Then $\lambda(s) = t$; in particular, $s \neq 0$.*

Proof. Let $\epsilon = d^{t+1}$. Then there exists $M \in Z^+$ such that $|s_m - s| < \epsilon$ for all $m \geq M$. Let $N = \max\{j, M\}$. Then we have that $|s_m - s| < \epsilon$ and $\lambda(s_m) = t$ for all $m \geq N$. From $|s_N - s| < \epsilon$, we have that

$$\lambda(s_N - s) \geq \lambda(\epsilon) = t + 1 > t. \tag{4.4}$$

But we know that

$$\lambda(s_N - s) = \min\{\lambda(s_N), \lambda(s)\} \leq t \text{ if } \lambda(s) \neq \lambda(s_N). \tag{4.5}$$

From Equation (4.4) and Equation (4.5), we infer that $\lambda(s) = \lambda(s_N) = t$.

Lemma 4.8 *Let (s_n) be a sequence in $\mathcal{R}$. Assume there exists $j \in Z^+$ such that $s_m = 0$ for all $m \geq j$. Then (s_n) converges strongly to 0.*

Proof. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Then

$$|s_m - 0| = |s_m| = 0 < \epsilon \text{ for all } m \geq j;$$

so $\lim_{n\to\infty} s_n = 0$.

Combining the results of Lemma 4.7 and Lemma 4.8, we obtain the following result.

Corollary 4.6 *Let (s_n) be a sequence in $\mathcal{R}$ converging strongly to s. Assume there exists $j \in Z^+$ such that $\lambda(s_m) = t$ for all $m \geq j$, where t can be either finite or infinite. Then $\lambda(s) = t$.*

Proof. If t is finite, then we are done, by Lemma 4.7. Otherwise, $s_m = 0$ for all $m \geq j$. Then, using Lemma 4.8, we have that (s_n) converges strongly to $s = 0$. Hence $\lambda(s) = \infty = t$.

The following lemma is a consequence of the fact that the topology induced on R by the order topology in $\mathcal{R}$ is the discrete topology in R (see Theorem 3.8).

Lemma 4.9 *Let (s_n) be a sequence in $\mathcal{R}$ whose members are purely real. Then (s_n) is strongly Cauchy if and only if there exists $j \in Z^+$ such that $s_m = s_j$ for all $m \geq j$.*

Proof. Let (s_n) be a strongly Cauchy sequence in $\mathcal{R}$ with purely real members. Then there exists $j \in Z^+$ such that

$$|s_m - s_l| < d \text{ for all } m, l \geq j. \tag{4.6}$$

Since the members of (s_n) are purely real, we obtain from Equation (4.6) that $|s_m - s_l| = 0$ for all $m, l \geq j$. Hence, $s_m = s_j$ for all $m \geq j$.

Conversely, let (s_n) be a sequence in $\mathcal{R}$ whose members are purely real, and assume there exists $j \in Z^+$ such that $s_m = s_j$ for all $m \geq j$. Then, given $\epsilon > 0$ in $\mathcal{R}$, we have that

$$|s_m - s_l| = 0 < \epsilon \text{ for all } m, l \geq j,$$

and hence (s_n) is strongly Cauchy.

Corollary 4.7 *Let (s_n) be a sequence in $\mathcal{R}$ whose members are purely real. Then (s_n) converges strongly if and only if there exists $j \in Z^+$ such that $s_m = s_j$ for all $m \geq j$.*

As we see, the concept of strong convergence provides very nice properties, and moreover strong convergence can be checked easily by virtue of Theorem 4.3 and Corollary 4.3. However, for some applications it is not sufficient, and it is advantageous to study a new kind of convergence.

4.2 Weak Convergence

Definition 4.4 *A sequence (s_n) in $\mathcal{R}$ is said to be weakly convergent if and only if there exists $s \in \mathcal{R}$ such that for all $\epsilon > 0$ in R, there exists $N \in Z^+$ such that*

$$\|s_m - s\|_{1/\epsilon} < \epsilon \textit{ for all } m \geq N.$$

If that is the case, we call s the weak limit of the sequence (s_n), and we write

$$s = wk\text{-}\lim_{n\to\infty} s_n.$$

Lemma 4.10 *Let (s_n) be a sequence in $\mathcal{R}$ that is weakly convergent. Then (s_n) has exactly one weak limit in $\mathcal{R}$.*

Proof. Let s and t be two weak limits of (s_n) in $\mathcal{R}$. We need to show that $s = t$. Let $\epsilon > 0$ in R be given. There exists $N \in Z^+$ such that

$$\|s_m - s\|_{2/\epsilon} < \epsilon/2 \text{ and } \|s_m - t\|_{2/\epsilon} < \epsilon/2 \text{ for all } m \geq N.$$

It follows that

$$\begin{aligned} \|s - t\|_{2/\epsilon} &= \|s - s_N + s_N - t\|_{2/\epsilon} \\ &\leq \|s_N - s\|_{2/\epsilon} + \|s_N - t\|_{2/\epsilon} \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned} \tag{4.7}$$

Let $r \in Q$ be given. Choose $\epsilon > 0$ in R, small enough such that $r < 2/\epsilon$. Then, using Equation (4.7), we have that

$$|(s-t)[r]| < \epsilon.$$

Since ϵ can be chosen arbitrarily small in R, we deduce that $|(s-t)[r]| = 0$. Therefore,

$$(s-t)[r] = 0.$$

Since r was an arbitrary rational number, we have that $s - t = 0$ and hence $s = t$.

Lemma 4.11 *Let $r_1, r_2 \in Q$ be such that $r_1 < r_2$. Then*

$$\|x\|_{r_1} \leq \|x\|_{r_2} \text{ for all } x \in \mathcal{R}.$$

Proof. Let $x \in \mathcal{R}$ be given. Then

$$\begin{aligned}
\|x\|_{r_2} &= \sup\{|x[q]| : q \leq r_2\} = \max\{|x[q]| : q \leq r_2\} \\
&= \max\left\{\max\{|x[q]| : q \leq r_1\}, \max\{|x[q]| : r_1 < q \leq r_2\}\right\} \\
&= \max\left\{\|x\|_{r_1}, \max\{|x[q]| : r_1 < q \leq r_2\}\right\} \\
&\geq \|x\|_{r_1}.
\end{aligned}$$

Lemma 4.12 *Let (s_n) be a sequence in $\mathcal{R}$ converging weakly to s. Then, for all $r \in Q$, the sequence $(\|s_n\|_r)$ converges in R to $\|s\|_r$.*

Proof. Let $r \in Q$ and $\epsilon > 0$ in R be given. Let $\epsilon_1 > 0$ in R be such that

$$\epsilon_1 < \min\left\{\frac{1}{1+|r|}, \epsilon\right\}.$$

Since (s_n) converges weakly to s in $\mathcal{R}$, there exists $N \in Z^+$ such that

$$\|s_m - s\|_{1/\epsilon_1} < \epsilon_1 \text{ for all } m \geq N.$$

Since $r < 1 + |r| < 1/\epsilon_1$, we have, using Lemma 4.11, that

$$\|s_m - s\|_r \leq \|s_m - s\|_{1/\epsilon_1} < \epsilon_1 < \epsilon \text{ for all } m \geq N.$$

Finally, using Equation (3.9), we obtain that

$$|\|s_m\|_r - \|s\|_r| < \epsilon \text{ for all } m \geq N,$$

which shows that $(\|s_n\|_r)$ converges in R to $\|s\|_r$.

Remark 4.3 *The converse of Lemma 4.12 is not true.*

Proof. Let (s_n) be the sequence in $\mathcal{R}$ whose terms are given by $s_{2k} = 1$ and $s_{2k+1} = -1$. Then, for all n, we have that

$$\|s_n\|_r = \|1\|_r = \begin{cases} 1 & \text{if } r \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Hence, for all $r \in Q$, $\|s_n\|_r = \|1\|_r$ for all n; so the sequence $(\|s_n\|_r)$ converges in R to $\|1\|_r$, for all $r \in Q$. However, the sequence (s_n) does not converge weakly in $\mathcal{R}$.

Theorem 4.6 (Convergence Criterion for Weak Convergence) *Let the sequence (s_n) converge weakly to the limit s. Then, the sequence $(s_n[q])$ converges to $s[q]$ in R, for all $q \in Q$, and the convergence is uniform on every subset of Q bounded above. Let on the other hand (s_n) be regular, and let the sequence $(s_n[q])$ converge in R to $s[q]$ for all $q \in Q$. Then (s_n) converges weakly in $\mathcal{R}$ to s.*

Proof. Let (s_n) be a sequence in $\mathcal{R}$ converging weakly to s. Let $r \in Q$ and $\epsilon > 0$ in R be given. Let

$$\epsilon_1 < \min\left\{\frac{1}{1+|r|}, \epsilon\right\}$$

be given. Then

$$1/\epsilon_1 > 1/\epsilon \text{ and } 1/\epsilon_1 > 1 + |r| > r.$$

Choose $N \in Z^+$ such that

$$\|s_m - s\|_{1/\epsilon_1} < \epsilon_1 \text{ for all } m \geq N. \tag{4.8}$$

Since $r < 1/\epsilon_1$, we have, using Lemma 4.11, that

$$\|s_m - s\|_r \leq \|s_m - s\|_{1/\epsilon_1} \text{ for all } m. \tag{4.9}$$

It then follows from Equation (4.8) and Equation (4.9) that

$$\|s_m - s\|_r < \epsilon_1 < \epsilon \text{ for all } m \geq N.$$

Hence

$$|s_m[q] - s[q]| = |(s_m - s)[q]| < \epsilon \text{ for all } q \leq r \text{ and for all } m \geq N, \tag{4.10}$$

which entails that, for all $r \in Q$, and for all $\epsilon > 0$ in R, there exists $N \in Z^+$ such that $|s_m[r] - s[r]| < \epsilon$ for all $m \geq N$. Therefore, for all $r \in Q$, the sequence $(s_n[r])$ converges in R to $s[r]$. Moreover, from Equation (4.10), we see that the convergence is uniform on the subset $\{q \in Q, q \leq r\}$ bounded above by r.

On the other hand, let (s_n) be a regular sequence in $\mathcal{R}$ and let $(s_n[q])$ converge in R to $s[q]$ for all q in Q, where $s : Q \to R$ is a real-valued function on Q. We need to show that $s \in \mathcal{R}$ and that (s_n) converges weakly to s. Let $q \in Q$ be given. Since $(s_n[q])$ converges to $s[q]$, we have that $s[q] = 0$ if $s_n[q] = 0$ for all n. Thus, $s[q] \neq 0$ only if there exists $m \in Z^+$ such that $s_m[q] \neq 0$. Therefore, every support point of s agrees at least with one support point of one member of the sequence, and hence is contained in $S = \cup_{n=0}^{\infty} \text{supp}(s_n)$, which is left-finite since (s_n) is regular. Hence $\text{supp}(s)$ is a subset of S and is itself left-finite. So $s \in \mathcal{R}$.

Now let $\epsilon > 0$ in R be given, and let r in Q be such that $r > 1/\epsilon$. We first show that the sequence $(s_n[q])$ converges uniformly to $s[q]$ on $\{q \in Q, q \leq r\}$. Since (s_n) is regular, any point at which s can differ from any s_n has to be in S. Thus, there are only finitely many points to be studied below r, say $q_1, \ldots, q_k$. For $j = 1, \ldots, k$, find $N_j \in Z^+$ such that

$$|s_m[q_j] - s[q_j]| < \epsilon \text{ for all } m \geq N_j.$$

Let $N = \max\{N_j : j = 1, \ldots, k\}$. Then $|s_m[q] - s[q]| < \epsilon$ for all $m \geq N$ and for all $q \leq r$. In particular, $|(s_m - s)[q]| = |s_m[q] - s[q]| < \epsilon$ for all $m \geq N$ and for all $q \leq 1/\epsilon$. It follows that

$$\|s_m - s\|_{1/\epsilon} = \max\{|(s_m - s)[q]| : q \leq 1/\epsilon\} < \epsilon \text{ for all } m \geq N,$$

which shows that (s_n) converges weakly to s.

Definition 4.5 *Let (s_n) be a sequence in $\mathcal{R}$. Then we say that (s_n) is weakly Cauchy in $\mathcal{R}$ if and only if for all $\epsilon > 0$ in R, there exists $N \in Z^+$ such that*

$$\|s_m - s_l\|_{1/\epsilon} < \epsilon \textit{ for all } m, l \geq N.$$

Lemma 4.13 *Let (s_n) be a sequence in $\mathcal{R}$ that converges weakly. Then (s_n) is weakly Cauchy.*

Proof. Let s be the weak limit of (s_n) in $\mathcal{R}$, and let $\epsilon > 0$ in R be given. Then there exists $N \in Z^+$ such that $\|s_m - s\|_{2/\epsilon} < \epsilon/2$ forall $m \geq N$. Let $m, l \geq N$ be given. Then,

$$\begin{aligned}
\|s_m - s_l\|_{2/\epsilon} &= \|s_m + s - s - s_l\|_{2/\epsilon} \\
&\leq \|s_m - s\|_{2/\epsilon} + \|s_l - s\|_{2/\epsilon} \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Using Lemma 4.11, we have that

$$\|s_m - s_l\|_{1/\epsilon} \leq \|s_m - s_l\|_{2/\epsilon} < \epsilon \text{ forall } m, l \geq N,$$

from which we infer that (s_n) is weakly Cauchy.

The converse of Lemma 4.13 is not true, as the following theorem will show.

Theorem 4.7 *$\mathcal{R}$ is not Cauchy complete with respect to the weak topology.*

Proof. It suffices to find a sequence (s_n) in $\mathcal{R}$ that is weakly Cauchy but not weakly convergent. Consider the sequence (s_n), where

$$s_n = \sum_{j=1}^{n} \frac{d^{-j}}{j}.$$

For all $n \in Z^+$, we have that $\text{supp}(s_n) = \{-j : 1 \leq j \leq n\} \in \mathcal{F}$. Hence $s_n \in \mathcal{R}$ for all n.

Let ϵ in R be given. Choose $N \in Z^+$ such that $N > 1/\epsilon$. Then $\epsilon > 1/N \geq 1/n$ for all $n \geq N$. Let $m > l \geq N$ be given. Then,

$$s_m - s_l = \sum_{j=l+1}^{m} \frac{d^{-j}}{j}.$$

Hence,

$$\|s_m - s_l\|_{1/\epsilon} = \frac{1}{l+1} < \epsilon.$$

So, for all $m, l \geq N$, $\|s_m - s_l\|_{1/\epsilon} < \epsilon$. Hence (s_n) is weakly Cauchy.

Assume that (s_n) converges weakly to s in $\mathcal{R}$. Then, by Theorem 4.6, the sequence $(s_n[q])$ converges in R to $s[q]$ for all $q \in Q$. Let $q \in Z^-$ be given. Then $q = -k$ for some $k \in Z^+$. Therefore,

$$s_n[q] = s_n[-k] = \begin{cases} \frac{1}{k} = -\frac{1}{q} & \text{if } n \geq k \\ 0 & \text{if } n < k \end{cases}.$$

Hence, the sequence $(s_n[q])$ converges in R to $-\frac{1}{q}$. So

$$s[q] = -\frac{1}{q} \neq 0 \text{ for all } q \in Z^-,$$

from which we infer that $\operatorname{supp}(s)$ is not left-finite. This contradicts the assumption that $s \in \mathcal{R}$. Hence (s_n) does not converge weakly in $\mathcal{R}$.

Lemma 4.14 *Let (s_n) be a sequence in $\mathcal{R}$ that is weakly Cauchy. Then (s_n) is weakly bounded , that is there exists $B > 0$ in R such that $\|s_n\|_{1/B} < B$ for all n.*

Proof. Since (s_n) is weakly Cauchy, there exists $N \in Z^+$ such that

$$\|s_n - s_m\|_1 < 1 \text{ for all } n, m \geq N.$$

In particular, we have that $\|s_n - s_N\|_1 < 1$ for all $n \geq N$, from which we obtain, using Equation (3.8), that $\|s_n\|_1 < 1 + \|s_N\|_1$ for all $n \geq N$. Let $B_1 = 1 + \|s_N\|_1$. Then $1/B_1 \leq 1$, and hence $\|s_n\|_{1/B_1} \leq \|s_n\|_1 < B_1$ for all $n \geq N$. Let $B_2 =$

$\max\left\{\|s_n\|_{1/B_1} : n \leq N-1\right\} + 1$, and let $B = \max\{B_1, B_2\}$. Then, for all $n \geq N$, we have that

$$\|s_n\|_{1/B} \leq \|s_n\|_{1/B_1} < B_1 \leq B. \tag{4.11}$$

Moreover, for all $n \leq N-1$, we have that

$$\|s_n\|_{1/B} \leq \|s_n\|_{1/B_1} < B_2 \leq B. \tag{4.12}$$

Combining Equation (4.11) and Equation (4.12), we finally obtain that $\|s_n\|_{1/B} < B$ for all n.

Lemma 4.15 *Let (s_n) be a sequence in $\mathcal{R}$ that is weakly Cauchy. Then for all $r \in Q$, there exists $B_r \in R^+$ such that $\|s_n\|_r < B_r$ for all n.*

Proof. Let $r \in Q$ be given. Then there exists $N \in Z^+$ such that

$$\|s_n - s_m\|_{1+|r|} < \frac{1}{1+|r|} \text{ for all } n, m \geq N.$$

In particular,

$$\|s_n - s_N\|_{1+|r|} < \frac{1}{1+|r|} \leq 1 \text{ for all } n \geq N.$$

Using Equation (3.8), we have that $\|s_n\|_{1+|r|} < 1 + \|s_N\|_{1+|r|}$ for all $n \geq N$. Therefore, using Lemma 4.11, we have that $\|s_n\|_r < 1 + \|s_N\|_{1+|r|}$ for all $n \geq N$. Let

$$B_{1,r} = 1 + \|s_N\|_{1+|r|} \text{ and } B_{2,r} = \max\left\{\|s_n\|_{1+|r|} : n \leq N-1\right\} + 1,$$

and let $B_r = \max\{B_{1,r}, B_{2,r}\}$. Then

$$\|s_n\|_r < 1 + \|s_N\|_{1+|r|} = B_{1,r} \leq B_r \text{ for all } n \geq N, \text{ and} \tag{4.13}$$

$$\|s_n\|_r < B_{2,r} \leq B_r \text{ for all } n \leq N-1. \tag{4.14}$$

Combining Equation (4.13) and Equation (4.14), we finally obtain that $\|s_n\|_r < B_r$ for all n. This finishes the proof of the theorem.

The following lemma is a direct result of the fact that the topology induced on R by the weak topology in $\mathcal{R}$ is the usual order topology in R (see Theorem 3.9).

Lemma 4.16 *Let (s_n) be a purely real sequence converging to s in R. Then, regarded as a sequence in $\mathcal{R}$, (s_n) converges weakly to s. On the other hand, let (s_n) be a sequence in $\mathcal{R}$ with purely real members, converging weakly to s. Then s is purely real, and the sequence (s_n) converges in R to s.*

Proof. Let (s_n) be a purely real sequence converging to s in R. We now view (s_n) as a sequence in $\mathcal{R}$. Let $\epsilon > 0$ in R be given. There exists $N \in Z^+$ such that

$$|s_n - s| < \epsilon \text{ for all } n \geq N. \tag{4.15}$$

Since, for all n, s_n and s are purely real, we have that

$$\begin{aligned} \|s_n - s\|_{1/\epsilon} &= |(s_n - s)[0]| \\ &= |s_n - s|. \end{aligned} \tag{4.16}$$

Combining Equation (4.15) and Equation (4.16), we have that $\|s_n - s\|_{1/\epsilon} < \epsilon$ for all $n \geq N$. Hence (s_n) converges weakly to s in $\mathcal{R}$.

Now, let (s_n) be a sequence in $\mathcal{R}$ with purely real members, converging weakly to $s \in \mathcal{R}$. By Theorem 4.6, the sequence $(s_n[q])$ converges to $s[q]$ for all $q \in Q$. Since $s_n[q] = 0$ for all $q \neq 0$ and for all n, we have that $s[q] = 0$ for all $q \neq 0$ and hence s is purely real. To show that (s_n) converges to s in R, let $\epsilon > 0$ in R be given. Then there exists $N \in Z^+$ such that $|s_n - s| = \|s_n - s\|_{1/\epsilon} < \epsilon$ for all $n \geq N$, which finishes the proof of the theorem.

Lemma 4.17 *Let (s_n) and (t_n) be two sequences in $\mathcal{R}$ converging weakly to s and t, respectively. Then the sequence $(s_n + t_n)$ converges weakly to $s + t$.*

Proof. Let $\epsilon > 0$ be given in R. Then there exists $N \in Z^+$ such that

$$\|s_n - s\|_{2/\epsilon} < \epsilon/2 \text{ and } \|t_n - t\|_{2/\epsilon} < \epsilon/2 \text{ for all } n \geq N.$$

Therefore, for all $n \geq N$, we have that

$$\begin{aligned} \|(s_n + t_n) - (s + t)\|_{1/\epsilon} &\leq \|(s_n + t_n) - (s + t)\|_{2/\epsilon} \\ &\leq \|s_n - s\|_{2/\epsilon} + \|t_n - t\|_{2/\epsilon} \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence $(s_n + t_n)$ converges weakly to $s + t$.

Corollary 4.8 *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two infinite series in $\mathcal{R}$ converging weakly to a and b, respectively. Then the infinite series $\sum_{n=0}^{\infty}(a_n + b_n)$ converges weakly to $a + b$.*

Proof. Let (A_n) and (B_n) be the sequences of partial sums of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, respectively. Then (A_n) and (B_n) converge weakly to a and b, respectively. Therefore, by Lemma 4.17, $(A_n + B_n)$ converges weakly to $a + b$. But $(A_n + B_n)$ is the sequence of partial sums of $\sum_{n=0}^{\infty}(a_n + b_n)$. Hence $\sum_{n=0}^{\infty}(a_n + b_n)$ converges weakly to $a + b$. We can thus write

$$\sum_{n=0}^{\infty}(a_n + b_n) = a + b = \sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n.$$

Theorem 4.8 *Let (s_n) and (t_n) be two regular sequences in $\mathcal{R}$ converging weakly to s and t, respectively. Then the sequence $(s_n t_n)$ converges weakly to st.*

Proof. Since (s_n) and (t_n) are both regular, so is $(s_n t_n)$, by Lemma 4.1. To show that $(s_n t_n)$ converges weakly to st, it remains to show that the sequence $((s_n t_n)[q])$ converges in R to $(st)[q]$ for all $q \in Q$, using Theorem 4.6. Let $A = \cup_{n=0}^{\infty} \text{supp}(a_n)$

and $B = \cup_{n=0}^{\infty}\mathrm{supp}(b_n)$. Then $A, B \in \mathcal{F}$. Let $q \in Q$ be given. Then, for all n, we have that

$$(s_n t_n)[q] = \sum_{\substack{q_1 + q_2 = q \\ q_1 \in A,\, q_2 \in B}} s_n[q_1] t_n[q_2]. \tag{4.17}$$

Since A and B are left-finite, only finitely many terms contribute to the sum in Equation (4.17); and we have that

$$\begin{aligned}
\lim_{n\to\infty}(s_n t_n)[q] &= \lim_{n\to\infty}\left(\sum_{\substack{q_1 + q_2 = q \\ q_1 \in A,\, q_2 \in B}} s_n[q_1] t_n[q_2]\right) \\
&= \sum_{\substack{q_1 + q_2 = q \\ q_1 \in A,\, q_2 \in B}} \left(\lim_{n\to\infty}\left(s_n[q_1] t_n[q_2]\right)\right) \\
&= \sum_{\substack{q_1 + q_2 = q \\ q_1 \in A,\, q_2 \in B}} \left(\left(\lim_{n\to\infty} s_n[q_1]\right)\left(\lim_{n\to\infty} t_n[q_2]\right)\right) \\
&= \sum_{\substack{q_1 + q_2 = q \\ q_1 \in A,\, q_2 \in B}} \left(s[q_1] t[q_2]\right) \\
&= (st)[q].
\end{aligned}$$

This finishes the proof of the theorem.

Theorem 4.9 *If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are regular, $\sum_{n=0}^{\infty} a_n$ converges absolutely weakly to a, and $\sum_{n=0}^{\infty} b_n$ converges weakly to b, then $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$, converges weakly to ab.*

Proof. Let (A_n), (B_n), and (C_n) be the sequences of partial sums of $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, and $\sum_{n=0}^{\infty} c_n$, respectively. Then (A_n) and (B_n) are both regular, (A_n) converges absolutely weakly to a and (B_n) converges weakly to b. Since (A_n) and (B_n) are both regular, so is (C_n). It remains to show that $(C_n[q])$ converges in R to $(ab)[q]$ for all $q \in Q$.

Since (A_n) converges absolutely weakly to a, $(A_n[t])$ converges absolutely in R to $a[t]$ for all $t \in Q$. Similarly, $(B_n[t])$ converges in R to $b[t]$ for all $t \in Q$. Let $A = \cup_{n=0}^{\infty}\mathrm{supp}(a_n)$ and $B = \cup_{n=0}^{\infty}\mathrm{supp}(b_n)$, and let $q \in Q$ be given. Then

$$\begin{aligned}
C_n[q] &= \left(\sum_{m=0}^{n} c_m\right)[q] = \sum_{m=0}^{n} c_m[q] \\
&= \sum_{m=0}^{n}\left(\left(\sum_{j=0}^{m} a_j b_{m-j}\right)[q]\right) \\
&= \sum_{m=0}^{n}\left(\sum_{j=0}^{m}(a_j b_{m-j})[q]\right) \\
&= \sum_{m=0}^{n}\sum_{j=0}^{m}\left(\sum_{\substack{q_1+q_2=q \\ q_1\in A,\, q_2\in B}} a_j[q_1]b_{m-j}[q_2]\right) \\
&= \sum_{\substack{q_1+q_2=q \\ q_1\in A,\, q_2\in B}}\left(\sum_{m=0}^{n}\sum_{j=0}^{m} a_j[q_1]b_{m-j}[q_2]\right) \text{ because of regularity} \\
&= \sum_{\substack{q_1+q_2=q \\ q_1\in A,\, q_2\in B}}\left(\sum_{m=0}^{n}\left(\sum_{j=0}^{m} a_j[q_1]b_{m-j}[q_2]\right)\right).
\end{aligned}$$

Since $\sum_{n=0}^{\infty} a_n[q_1]$ converges absolutely to $a[q_1]$ and since $\sum_{n=0}^{\infty} b_n[q_2]$ converges to $b[q_2]$, we have that

$$\lim_{n\to\infty}\left(\sum_{m=0}^{n}\left(\sum_{j=0}^{m} a_j[q_1]b_{m-j}[q_2]\right)\right)$$

exists in R and is equal to $a[q_1]b[q_2]$. Since the sum over the q's is finite because of left-finiteness of A and B, we have also that

$$\lim_{n\to\infty}\left(\sum_{\substack{q_1+q_2=q \\ q_1\in A,\, q_2\in B}}\left(\sum_{m=0}^{n}\left(\sum_{j=0}^{m} a_j[q_1]b_{m-j}[q_2]\right)\right)\right)$$

exists in R and is equal to

$$\sum_{\substack{q_1 + q_2 = q \\ q_1 \in A, q_2 \in B}} \left(\lim_{n\to\infty} \left(\sum_{m=0}^{n} \left(\sum_{j=0}^{m} a_j[q_1] b_{m-j}[q_2] \right) \right) \right).$$

Hence $\lim_{n\to\infty} C_n[q]$ exists in R and we have that

$$\lim_{n\to\infty} C_n[q] = \sum_{\substack{q_1 + q_2 = q \\ q_1 \in A, q_2 \in B}} a[q_1]b[q_2] = (ab)[q].$$

Since (C_n) is regular and since $\lim_{n\to\infty} C_n[q] = (ab)[q]$ for all $q \in Q$, (C_n) converges weakly in $\mathcal{R}$ to ab. Therefore, $\sum_{n=0}^{\infty} c_n$ converges weakly to ab, and we can write

$$\sum_{n=0}^{\infty} c_n = ab = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right).$$

The relationship between strong convergence and weak convergence is provided by the following theorem.

Theorem 4.10 *Strong convergence implies weak convergence to the same limit.*

Proof. Let (s_n) be a sequence in $\mathcal{R}$ converging strongly to s. Then, by Lemma 4.4, we have that (s_n) is regular. To show that (s_n) converges weakly to s, it suffices by Theorem 4.6 to show that the sequence $(s_n[q])$ converges in R to $s[q]$ for all $q \in Q$. So let $q \in Q$ be given. Since (s_n) converges strongly to s in $\mathcal{R}$, there exists $N_q \in Z^+$ such that

$$|s_m - s| < d^{q+1} \text{ for all } m \geq N_q.$$

Thus,

$$s_m[q] = s[q] \text{ for all } m \geq N_q,$$

which entails that the sequence $(s_n[q])$ converges in R to $s[q]$.

4.3 Power Series

We now discuss a very important class of sequences, namely, the power series. We first study general criteria for power series with $\mathcal{R}$ coefficients to converge strongly or weakly. Once their convergence properties are established, they will allow the extension of many important real functions, and they will also provide the key for an exhaustive study of differentiability of all functions that can be represented on a computer (see Chapter 7; also see [38, 39, 40]). Also based on our knowledge of the convergence properties of power series with $\mathcal{R}$ coefficients, we will be able to study in Chapter 6 a large class of functions which will prove to have all the nice smoothness properties that real power series have in R; (see also [41, 42]). We begin our discussion of power series with an observation.

Lemma 4.18 *Let $M \in \mathcal{F}$ be given. Define*

$$M_\Sigma = \{q_1 + ... + q_n : n \in Z^+, \text{ and } q_1, ..., q_n \in M\};$$

then $M_\Sigma \in \mathcal{F}$ if and only if $\min(M) \geq 0$.

Proof. Let $g = \min(M)$. First assume that $g < 0$. Clearly, all multiples of g are in M_Σ. In other words, M_Σ contains infinitely many elements smaller than zero and is therefore not left-finite. Now assume that $g \geq 0$. For $g = 0$, we start the discussion by considering $\bar{M} = M \setminus \{0\}$, which has a minimum greater than zero. But since M differs from $\bar{M}$ only by containing zero, and since inclusion of zero does not change a sum, we obviously have that $\bar{M}_\Sigma = M_\Sigma$. It therefore suffices to consider the case when $g > 0$. Now let $r \in Q$ be given; we show that there are only finitely many elements in M_Σ that are smaller than r. Since all elements in M_Σ are greater than or equal to the minimum g, the property holds for $r < g$. Now let $r \geq g$ be given, and let $n = \text{integer}(r/g)$ be the greatest integer less than or equal to r/g. Let $q < r$ in M_Σ be given. Then at most n terms can sum up to q, since any sum with more than n terms exceeds r and thus q. Furthermore, the sum can contain only finitely many different elements of M, namely those below r. But this means that there are only finitely many ways of forming sums, and thus only finitely many results of summations below r.

Corollary 4.9 *The sequence (x^n) is regular if and only if $\lambda(x) \geq 0$.*

Let (a_n) be a sequence in $\mathcal{R}$. Then the sequences $(a_n x^n)$ and $(\sum_{j=0}^{n} a_j x^j)$ are regular if (a_n) is regular and $\lambda(x) \geq 0$.

Proof. First observe that the set $\cup_{n=1}^{\infty} \text{supp}(x^n)$ is identical with the set M_Σ in the previous lemma if we set $M = \text{supp}(x)$. This is left-finite if and only if $\text{supp}(x)$ has a minimum greater than or equal to zero; that is if and only if $\lambda(x) \geq 0$.

The second part is an immediate consequence of Lemma 4.1, which asserts that the product of regular sequences is regular.

Theorem 4.11 (Strong Convergence Criterion for Power Series) *Let (a_n) be a sequence in $\mathcal{R}$, and let*

$$\lambda_0 = -\liminf_{n\to\infty} \left(\frac{\lambda(a_n)}{n}\right) = \limsup_{n\to\infty} \left(\frac{-\lambda(a_n)}{n}\right) \quad \text{in } R.$$

Let $x_0 \in \mathcal{R}$ be fixed and let $x \in \mathcal{R}$ be given. Then the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges strongly in $\mathcal{R}$ if $\lambda(x-x_0) > \lambda_0$ and is strongly divergent if $\lambda(x-x_0) < \lambda_0$ or if $\lambda(x-x_0) = \lambda_0$ and $-\lambda(a_n)/n > \lambda_0$ for infinitely many n.

Proof. First assume that $\lambda(x-x_0) > \lambda_0$. To show that $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges strongly in $\mathcal{R}$, using Corollary 4.3, it suffices to show that the sequence $(a_n (x-x_0)^n)$ is a null sequence with respect to the order topology. Since $\lambda(x-x_0) > \lambda_0$, there exists $t > 0$ in Q such that

$$\lambda(x - x_0) - t > \lambda_0 = \limsup_{n\to\infty} \left(\frac{-\lambda(a_n)}{n} \right).$$

Hence there exists $N \in Z^+$ such that

$$\lambda(x - x_0) - t > \frac{-\lambda(a_n)}{n} \text{ for all } n \geq N.$$

Thus,

$$\lambda(a_n(x - x_0)^n) = \lambda(a_n) + n\lambda(x - x_0) > nt \text{ for all } n \geq N.$$

Since $t > 0$, we obtain that $(a_n(x - x_0)^n)$ is a null sequence with respect to the order topology.

Now assume that $\lambda(x - x_0) < \lambda_0$. To show that $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is strongly divergent in $\mathcal{R}$, it suffices to show that the sequence $(a_n(x - x_0)^n)$ is not a null sequence with respect to the order topology. Since $\lambda(x - x_0) < \lambda_0$, for all $N \in Z^+$ there exists $n > N$ such that

$$\lambda(x - x_0) < \frac{-\lambda(a_n)}{n}.$$

Hence, for all $N \in Z^+$, there exists $n > N$ such that

$$\lambda(a_n(x - x_0)^n) < 0,$$

which entails that the sequence $(a_n(x - x_0)^n)$ is not a null sequence with respect to the order topology.

Finally, assume that $\lambda(x - x_0) = \lambda_0$ and $-\lambda(a_n)/n > \lambda_0$ for infinitely many n. Then for all $N \in Z^+$, there exists $n > N$ such that

$$\frac{-\lambda(a_n)}{n} > \lambda_0 = \lambda(x - x_0).$$

Thus, for each $N \in Z^+$, there exists $n > N$ such that $\lambda(a_n(x - x_0)^n) < 0$. Therefore, the sequence $(a_n(x - x_0)^n)$ is not a null sequence with respect to the order topology;

and hence $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is strongly divergent in $\mathcal{R}$. This finishes the proof of the theorem.

The following two examples show that for the case when $\lambda(x-x_0)=\lambda_0$ and $-\lambda(a_n)/n \geq \lambda_0$ for only finitely many n, the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ can either converge or diverge strongly. For this case, Theorem 4.12 provides a test for weak convergence.

Example 4.1 *For each $n \geq 0$, let $a_n = d$; and let $x_0 = 0$ and $x = 1$.*

Then

$$\lambda_0 = \limsup_{n\to\infty}\left(\frac{-\lambda(a_n)}{n}\right) = \limsup_{n\to\infty}\left(-\frac{1}{n}\right) = 0 = \lambda(x).$$

Moreover, we have that

$$\frac{-\lambda(a_n)}{n} = -\frac{1}{n} < \lambda_0 \text{ for all } n \geq 0;$$

and $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} d$ is strongly divergent in $\mathcal{R}$.

Example 4.2 *For each n, let $q_n \in Q$ be such that $\sqrt{n}/2 < q_n < \sqrt{n}$, let $a_n = d^{q_n}$; and let $x_0 = 0$ and $x = 1$.*

Then

$$\lambda_0 = \limsup_{n\to\infty}\left(\frac{-\lambda(a_n)}{n}\right) = \limsup_{n\to\infty}\left(-\frac{q_n}{n}\right) = 0 = \lambda(x).$$

Moreover, we have that

$$\frac{-\lambda(a_n)}{n} = -\frac{q_n}{n} < 0 = \lambda_0 \text{ for all } n \geq 0;$$

and $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} d^{q_n}$ converges strongly in $\mathcal{R}$ since the sequence (d^{q_n}) is a null sequence with respect to the order topology.

Theorem 4.12 (Weak Convergence Criterion for Power Series) *Let (a_n) be a sequence in $\mathcal{R}$, and let*

$$\lambda_0 = -\liminf_{n\to\infty}\left(\frac{\lambda(a_n)}{n}\right) = \limsup_{n\to\infty}\left(\frac{-\lambda(a_n)}{n}\right) \ \text{ in } R.$$

Let $x_0 \in \mathcal{R}$ be fixed, and let $x \in \mathcal{R}$ be such that $\lambda(x-x_0)=\lambda_0$. For each $n \geq 0$, let $b_n = a_n d^{n\lambda_0}$. Suppose that the sequence (b_n) is regular and write $\cup_{n=0}^{\infty} supp(b_n) = \{q_1, q_2, \ldots\}$; with $q_{j_1} < q_{j_2}$ if $j_1 < j_2$. For each n, write $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$, where $b_{n_j} = b_n[q_j]$. Let

$$r = \frac{1}{\sup\left\{\limsup_{n\to\infty} |b_{n_j}|^{1/n} : j \geq 1\right\}}.$$

Then $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely weakly in $\mathcal{R}$ if $|(x-x_0)[\lambda_0]| < r$ and is weakly divergent in $\mathcal{R}$ if $|(x-x_0)[\lambda_0]| > r$.

Proof. Letting $y = d^{-\lambda_0}(x-x_0)$, we obtain that

$$\lambda(y) = 0 = \limsup_{n\to\infty}\left(\frac{-\lambda(b_n)}{n}\right), \text{ and } a_n(x-x_0)^n = b_n y^n \text{ for all } n \geq 0.$$

So without loss of generality, we may assume that

$$x_0 = 0;\ \ \lambda_0 = 0 = \lambda(x);\ \text{ and } b_n = a_n \text{ for all } n \geq 0.$$

Let $X = \Re(x)$; then $X \neq 0$. First assume that $|X| < r$.

First Claim: For all $j \geq 1$, we have that $\sum_{n=0}^{\infty} a_{n_j} X^n$ converges in R.

Proof of the first claim: Since $|X| < r$, we have that

$$\frac{1}{|X|} > \sup\left\{\limsup_{n\to\infty} |a_{n_j}|^{1/n} : j \geq 1\right\};$$

and hence

$$\frac{1}{|X|} > \limsup_{n\to\infty} |a_{n_j}|^{1/n} \text{ for all } j \geq 1.$$

Thus,

$$|X| < \frac{1}{\limsup_{n\to\infty} |a_{n_j}|^{1/n}} \text{ for all } j \geq 1.$$

Hence $\sum_{n=0}^{\infty} a_{n_j} X^n$ converges in R for all $j \geq 1$.

Second claim: For all $j \geq 1$, $\sum_{n=0}^{\infty} a_{n_j} x^n$ converges weakly in $\mathcal{R}$.

Proof of the second claim: Let $j \geq 1$ be given. For each n, let $A_{n_j}(x) = \sum_{i=0}^{n} a_{i_j} x^i$. So we need to show that the sequence $(A_{n_j}(x))_{n\geq 0}$ is weakly convergent. Using Corollary 4.9, the sequence is regular since $\lambda(x) \geq 0$ and since the sequence (a_{i_j}) is purely real and hence regular. Thus, it suffices to show that the sequence $(A_{n_j}(x)[t])$ converges in R for all $t \in Q$. Let $s = x - X$. If $s = 0$, then we are done. So we may assume that $s \neq 0$. Let $t \in Q$ be given; and choose $m \in Z^+$ such that $m\lambda(s) > t$. Then $(X+s)^n$ evaluated at t yields:

$$\begin{aligned} ((X+s)^n)[t] &= \left(\sum_{l=0}^{n} s^l \frac{n!}{(n-l)!l!} X^{n-l}\right)[t] \\ &= \sum_{l=0}^{\min\{m,n\}} s^l[t] \frac{n!}{(n-l)!l!} X^{n-l}. \end{aligned}$$

For the last equality, we used the fact that s^l vanishes at t for $l > m$. So we get the following chain of inequalities for any $\nu_2 > \nu_1 > m$:

$$\begin{aligned} \sum_{n=\nu_1}^{\nu_2} \left|a_{n_j}(X+s)^n[t]\right| &= \sum_{n=\nu_1}^{\nu_2} \left|a_{n_j}\right| \left|\sum_{l=0}^{\min\{m,n\}} s^l[t] \frac{n!}{(n-l)!l!} X^{n-l}\right| \\ &\leq \sum_{n=\nu_1}^{\nu_2} \sum_{l=0}^{m} \left|a_{n_j}\right| \left|s^l[t]\right| \frac{n!}{(n-l)!l!} |X|^{n-l} \\ &\leq \left(\sum_{l=0}^{m} \frac{\left|s^l[t]\right| |X|^{m-l}}{l!}\right) \left(\sum_{n=\nu_1}^{\nu_2} \left|a_{n_j}\right| n^m |X|^{n-m}\right). \end{aligned}$$

Note that the right sum contains only real terms. As $|X| < r$, the series converges; the additional factor n^m does not influence convergence since $\lim_{n\to\infty} \sqrt[n]{n^m} = 1$. As the left hand term does not depend on ν_1 and ν_2, we therefore obtain absolute convergence at t. This finishes the proof of the second claim.

Third claim: $\sum_{n=0}^{\infty} a_n x^n$ converges weakly in $\mathcal{R}$.

Proof of the third claim: By the result of the second claim, we have that $\sum_{n=0}^{\infty} a_{n_j} x^n$ converges weakly in $\mathcal{R}$ for all $j \geq 1$. For each j, let $f_j(x) = \sum_{n=0}^{\infty} a_{n_j} x^n$; then

$$\lambda\left(f_j(x)\right) \geq 0 \text{ for all } j \geq 1.$$

Thus $\sum_{j=1}^{\infty} d^{q_j} f_j(x)$ converges strongly (and hence weakly) in $\mathcal{R}$. Now let $t \in Q$ be given. Then there exists $m \in Z^+$ such that $q_j > t$ for all $j \geq m$. Thus,

$$\begin{aligned}
\left(\sum_{j=1}^{\infty} d^{q_j} f_j(x)\right)[t] &= \sum_{j=1}^{\infty} \left(d^{q_j} f_j(x)\right)[t] = \sum_{j=1}^{\infty} \left(\sum_{t_1+t_2=t} d^{q_j}[t_1] f_j(x)[t_2]\right) \\
&= \sum_{j=1}^{m} \left(\sum_{t_1+t_2=t} d^{q_j}[t_1] f_j(x)[t_2]\right) = \sum_{j=1}^{m} \sum_{t_1+t_2=t} d^{q_j}[t_1] \left(\sum_{n=0}^{\infty} a_{n_j} x^n\right)[t_2] \\
&= \sum_{j=1}^{m} \sum_{t_1+t_2=t} d^{q_j}[t_1] \sum_{n=0}^{\infty} a_{n_j} x^n[t_2] = \sum_{n=0}^{\infty} \sum_{j=1}^{m} a_{n_j} \left(\sum_{t_1+t_2=t} d^{q_j}[t_1] x^n[t_2]\right) \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} a_{n_j} \left(\sum_{t_1+t_2=t} d^{q_j}[t_1] x^n[t_2]\right) = \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} a_{n_j} \left(d^{q_j} x^n\right)[t] \\
&= \left(\sum_{n=0}^{\infty} \sum_{j=1}^{\infty} a_{n_j} d^{q_j} x^n\right)[t] = \left(\sum_{n=0}^{\infty} \left(\sum_{j=1}^{\infty} a_{n_j} d^{q_j}\right) x^n\right)[t] \\
&= \left(\sum_{n=0}^{\infty} a_n x^n\right)[t].
\end{aligned}$$

This is true for all $t \in Q$. Thus, $\sum_{n=0}^{\infty} a_n x^n$ converges weakly to $\sum_{j=1}^{\infty} d^{q_j} f_j(x)$.

Now assume that $|X| > r$. Then

$$\frac{1}{|X|} < \sup\left\{\limsup_{n\to\infty} |a_{n_j}|^{1/n} : j \geq 1\right\}.$$

Hence there exists $j_0 \in Z^+$ such that

$$\frac{1}{|X|} < \limsup_{n\to\infty} |a_{n_{j_0}}|^{1/n}.$$

Thus,

$$|X| > \frac{1}{\limsup_{n\to\infty} |a_{n_{j_0}}|^{1/n}};$$

and hence $\sum_{n=0}^{\infty} a_{n_{j_0}} X^n$ diverges in R. Therefore, $\left(\sum_{n=0}^{\infty} a_n x^n\right)[q_{j_0}]$ diverges in R; and hence $\sum_{n=0}^{\infty} a_n x^n$ is weakly divergent in $\mathcal{R}$.

Corollary 4.10 (Power Series with Purely Real Coefficients) *Let $\sum_{n=0}^{\infty} a_n X^n$, $a_n \in R$, be a power series with classical radius of convergence equal to η. Let $x \in \mathcal{R}$, and let $A_n(x) = \sum_{i=0}^{n} a_i x^i \in \mathcal{R}$. Then, for $|x| < \eta$ and $|x| \not\approx \eta$, the sequence $(A_n(x))$ convergesabsolutely weakly. We define the limit to be the continuation of the power series on $\mathcal{R}$.*

4.4 Transcendental Functions

Using Corollary 4.10, we can now extend real functions representable by power series to the new field $\mathcal{R}$.

Definition 4.6 (The Functions Exp, Cos, Sin, Cosh, and Sinh) *By Corollary 4.10, the series*

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},\ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},\ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},\ \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!},\ \textit{and}\ \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

converge absolutely weakly in $\mathcal{R}$ for any $x \in \mathcal{R}$, at most finite in absolute value. For any such x, define

$$\begin{aligned}
\exp(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!}; \\
\cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}; \\
\sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!};
\end{aligned}$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!};$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Remark 4.4 *It follows from Definition 4.6 that for any $x \in \mathcal{R}$, at most finite in absolute value,* $\cos(x)$ *and* $\cosh(x)$ *are even functions, while* $\sin(x)$ *and* $\sinh(x)$ *are odd functions of x.*

Theorem 4.13 (Addition Theorem for the Exponential Function) *Let $x_1, x_2 \in \mathcal{R}$ be at most finite in absolute value. Then*

$$\exp(x_1)\exp(x_2) = \exp(x_1 + x_2).$$

Proof. Since $\sum_{n=0}^{\infty} \frac{x_1^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{x_2^n}{n!}$ both converge absolutely weakly in $\mathcal{R}$ for any x_1 and x_2, at most finite in absolute value, we have by Theorem 4.9 that $\sum_{n=0}^{\infty} c_n$ converges weakly in $\mathcal{R}$ to $\left(\sum_{n=0}^{\infty} \frac{x_1^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{x_2^n}{n!}\right)$, where

$$c_n = \sum_{j=0}^{n} \frac{x_1^j}{j!} \frac{x_2^{(n-j)}}{(n-j)!}.$$

Hence,

$$\begin{aligned}
\exp(x_1)\exp(x_2) &= \left(\sum_{n=0}^{\infty} \frac{x_1^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{x_2^n}{n!}\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} \frac{x_1^j}{j!} \frac{x_2^{(n-j)}}{(n-j)!}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x_1^j x_2^{(n-j)}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}(x_1 + x_2)^n \\
&= \exp(x_1 + x_2).
\end{aligned}$$

Corollary 4.11 *Let $x \in \mathcal{R}$ be at most finite in absolute value. Then*

$$\exp(x) \cdot \exp(-x) = 1.$$

Thus for any such x,

$$\exp(x) \neq 0 \text{ and } \frac{1}{\exp(x)} = \exp(-x).$$

Corollary 4.12 *Let $x \in \mathcal{R}$ be at most finite in absolute value. Then*

$$\exp(x) \approx e^X, \text{ where } X = \Re(x).$$

Proof. Write $x = X + s$. Then by Theorem 4.13,

$$\exp(x) = \exp(X)\exp(s) = \mathrm{e}^X \exp(s).$$

Since $|s|$ is infinitely small, $\exp(s) \approx 1$. Thus $\exp(x) \approx \mathrm{e}^X$ since $\mathrm{e}^X \neq 0$.

Theorem 4.14 (Addition Theorems for Cosine and Sine) *Let $x_1, x_2 \in \mathcal{R}$ be at most finite in absolute value. Then*

$$\cos(x_1 \pm x_2) = \cos(x_1)\cos(x_2) \mp \sin(x_1)\sin(x_2), \text{ and} \tag{4.18}$$

$$\sin(x_1 \pm x_2) = \sin(x_1)\cos(x_2) \pm \cos(x_1)\sin(x_2). \tag{4.19}$$

Proof. Using the definitions of the sine and cosine functions in Definition 4.6, we have that

$$\begin{aligned}
& \cos(x_1)\cos(x_2) \mp \sin(x_1)\sin(x_2) \\
= & \left(\sum_{n=0}^{\infty}(-1)^n \frac{x_1^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty}(-1)^n \frac{x_2^{2n}}{(2n)!}\right) \mp \\
& \left(\sum_{n=0}^{\infty}(-1)^n \frac{x_1^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty}(-1)^n \frac{x_2^{2n+1}}{(2n+1)!}\right) \\
= & \sum_{n=0}^{\infty}(-1)^n \sum_{j=0}^{n} \frac{x_1^{2j}}{(2j)!}\frac{x_2^{2n-2j}}{(2n-2j)!} \mp \sum_{n=0}^{\infty}(-1)^n \sum_{j=0}^{n} \frac{x_1^{2j+1}}{(2j+1)!}\frac{x_2^{2n-2j+1}}{(2n-2j+1)!} \\
= & \sum_{n=0}^{\infty}(-1)^n \sum_{j=0}^{n} \frac{x_1^{2j}}{(2j)!}\frac{x_2^{2n-2j}}{(2n-2j)!} \mp \sum_{n=1}^{\infty}(-1)^{n-1} \sum_{j=0}^{n-1} \frac{x_1^{2j+1}}{(2j+1)!}\frac{x_2^{2n-2j-1}}{(2n-2j-1)!}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}(-1)^n \sum_{j=0}^{n} \frac{x_1^{2j}}{(2j)!}\frac{x_2^{2n-2j}}{(2n-2j)!} \pm \sum_{n=1}^{\infty}(-1)^n \sum_{j=0}^{n-1} \frac{x_1^{2j+1}}{(2j+1)!}\frac{x_2^{2n-2j-1}}{(2n-2j-1)!} \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n \left(\sum_{j=0}^{n} \frac{x_1^{2j}}{(2j)!}\frac{x_2^{2n-2j}}{(2n-2j)!} \pm \sum_{j=0}^{n-1} \frac{x_1^{2j+1}}{(2j+1)!}\frac{x_2^{2n-2j-1}}{(2n-2j-1)!}\right) \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n \left(\sum_{j=0}^{n} \frac{x_1^{2j}}{(2j)!}\frac{(\pm x_2)^{2n-2j}}{(2n-2j)!} + \sum_{j=0}^{n-1} \frac{x_1^{2j+1}}{(2j+1)!}\frac{(\pm x_2)^{2n-2j-1}}{(2n-2j-1)!}\right) \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n \sum_{k=0}^{2n} \frac{x_1^k}{k!}\frac{(\pm x_2)^{2n-k}}{(2n-k)!} \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n \frac{(x_1 \pm x_2)^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{(x_1 \pm x_2)^{2n}}{(2n)!} \\
&= \cos(x_1 \pm x_2),
\end{aligned}$$

which proves Equation (4.18). Similarly, we show that Equation (4.19) holds.

Corollary 4.13 *Let x in $\mathcal{R}$ be at most finite in absolute value, let $X = \Re(x)$, and let $s = x - X$. Then*

$$\cos(x) \approx \begin{cases} \cos(X) & \text{if } \cos(X) \neq 0 \\ -s \cdot sign(\sin(X)) & \text{if } \cos(X) = 0 \end{cases},$$

where

$$sign(Y) = \begin{cases} 1 & \text{if } Y > 0 \\ -1 & \text{if } Y < 0 \end{cases}.$$

Proof. By Theorem 4.14, we have that

$$\cos(x) = \cos(X)\cos(s) - \sin(X)\sin(s).$$

Suppose $\cos(X) \neq 0$; then

$$\cos(x) = \cos(X)\left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{s^{2n}}{(2n)!}\right) - \sin(X)\left(\sum_{n=0}^{\infty}(-1)^n \frac{s^{2n+1}}{(2n+1)!}\right) \approx \cos(X).$$

Now suppose that $\cos(X) = 0$; then $\sin(X) = \mathrm{sign}(\sin(X))$, and hence

$$\begin{aligned} \cos(x) &= -\sin(s)\mathrm{sign}(\sin(X)) = -\left(\sum_{n=0}^{\infty}(-1)^n \frac{s^{2n+1}}{(2n+1)!}\right)\mathrm{sign}(\sin(X)) \\ &\approx -s \cdot \mathrm{sign}(\sin(X)). \end{aligned}$$

Corollary 4.14 *Let x, X, and s be as in Corollary 4.13. Then*

$$\sin(x) \approx \begin{cases} \sin(X) & \text{if } \sin(X) \neq 0 \\ s \cdot sign(\cos(X)) & \text{if } \sin(X) = 0 \end{cases}.$$

Proof. By Theorem 4.14, we have that

$$\sin(x) = \sin(X)\cos(s) + \cos(X)\sin(s).$$

Suppose $\sin(X) \neq 0$; then

$$\sin(x) = \sin(X)\left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{s^{2n}}{(2n)!}\right) + \cos(X)\left(\sum_{n=0}^{\infty}(-1)^n \frac{s^{2n+1}}{(2n+1)!}\right) \approx \sin(X).$$

Now suppose that $\sin(X) = 0$; then $\cos(X) = \mathrm{sign}(\cos(X))$, and hence

$$\sin(x) = \sin(s)\mathrm{sign}(\cos(X)) \approx s \cdot \mathrm{sign}(\cos(X)).$$

Corollary 4.15 *Let $x \in \mathcal{R}$ be at most finite in absolute value. Then*

$$|\sin(x)| \leq |x|.$$

Moreover, equality holds only if $x = 0$.

Proof. Let $X = \Re(x)$, and let $s = x - X$. It suffices to show that

$$\sin(x) \leq x \text{ for } 0 \leq X \leq \pi/2,$$

and that equality holds only if $x = 0$. Suppose $X = 0$. Then

$$\sin(x) = \sin(s) \approx s - \frac{s^3}{3!} \leq s; \text{ thus } \sin(x) = \sin(s) \leq s = x. \tag{4.20}$$

The equality holds in Equation (4.20) only if $x = s = 0$. Now suppose that $0 < X \leq \pi/2$. Then by Corollary 4.14,

$$\sin(x) \approx \sin(X) < X \approx x. \tag{4.21}$$

Since $X - \sin(X)$ is finite, Equation (4.21) entails that $\sin(x) < x$.

Corollary 4.16 *Let $x \in \mathcal{R}$ be at most finite in absolute value. Then*

$$\cos^2(x) + \sin^2(x) = 1.$$

Proof. Using Theorem 4.14, we have that

$$\begin{aligned}
\cos^2(x) + \sin^2(x) &= \cos(x)\cos(x) + \sin(x)\sin(x) \\
&= \cos(x - x) = \cos(0) \\
&= 1.
\end{aligned}$$

Using the results of Theorem 4.14 and Corollary 4.16, we readily obtain the following two corollaries.

Corollary 4.17 *Let $x \in \mathcal{R}$ be at most finite in absolute value. Then*

$$\begin{aligned}
\cos(2x) &= \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x), \text{ and} \\
\sin(2x) &= 2\sin(x)\cos(x).
\end{aligned}$$

Corollary 4.18 *Let $x \in \mathcal{R}$ be at most finite in absolute value. Then*

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}, \text{ and } \sin^2(x) = \frac{1 - \cos(2x)}{2}.$$

Definition 4.7 *For any x in $\mathcal{R}$, at most finite in absolute value and satisfying $\cos(x) \neq 0$, we define*

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

Definition 4.8 *For any x in $\mathcal{R}$, at most finite in absolute value and satisfying* $\sin(x) \neq 0$, *we define*

$$\cot(x) = \frac{\cos(x)}{\sin(x)}.$$

Corollary 4.19 *Let $x \in \mathcal{R}$ be at most finite in absolute value and satisfy* $\sin(x)\cos(x) \neq 0$. *Then*

$$\cot(x) = \frac{1}{\tan(x)}.$$

Corollary 4.20 $\tan(x)$ *and* $\cot(x)$ *are both odd functions of* x.

Corollary 4.21 *Let x_1 and x_2 in $\mathcal{R}$ be such that* $\tan(x_1)$, $\tan(x_2)$, *and* $\tan(x_1+x_2)$ *all exist in $\mathcal{R}$ (i.e. $|x_1|$ and $|x_2|$ are both at most finite, and* $\cos(x_1)\cos(x_2)\cos(x_1+x_2) \neq 0$*). Then*

$$\tan(x_1 + x_2) = \frac{\tan(x_1) + \tan(x_2)}{1 - \tan(x_1)\tan(x_2)}.$$

Corollary 4.22 *Let x_1 and x_2 in $\mathcal{R}$ be such that* $\cot(x_1)$, $\cot(x_2)$, *and* $\cot(x_1+x_2)$ *all exist in $\mathcal{R}$ (i.e. $|x_1|$ and $|x_2|$ are both at most finite, and* $\sin(x_1)\sin(x_2)\sin(x_1+x_2) \neq 0$*). Then*

$$\cot(x_1 + x_2) = \frac{\cot(x_1)\cot(x_2) - 1}{\cot(x_1) + \cot(x_2)}.$$

Lemma 4.19 *Let $x \in \mathcal{R}$ be at most finite in absolute value. Then*

$$\cosh(x) = \frac{\exp(x) + \exp(-x)}{2} \text{ and } \sinh(x) = \frac{\exp(x) - \exp(-x)}{2}.$$

Proof. Using Definition 4.6, we have that

$$\begin{aligned} \frac{\exp(x) + \exp(-x)}{2} &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n + (-x)^n}{n!} \end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$
$$= \cosh(x).$$

Similarly, we can show that the second equality holds.

Corollary 4.23 *Let $x_1, x_2 \in \mathcal{R}$ be at most finite in absolute value. Then*

$$\begin{aligned}
\cosh(x_1 \pm x_2) &= \cosh(x_1)\cosh(x_2) \pm \sinh(x_1)\sinh(x_2), \text{ and} \\
\sinh(x_1 \pm x_2) &= \sinh(x_1)\cosh(x_2) \pm \cosh(x_1)\sinh(x_2).
\end{aligned}$$

Proof. Using the result of Lemma 4.23, we have that

$$\begin{aligned}
& \cosh(x_1)\cosh(x_2) + \sinh(x_1)\sinh(x_2) \\
= & \frac{\exp(x_1) + \exp(-x_1)}{2} \frac{\exp(x_2) + \exp(-x_2)}{2} + \\
& \frac{\exp(x_1) - \exp(-x_1)}{2} \frac{\exp(x_2) - \exp(-x_2)}{2} \\
= & \frac{\exp(x_1 + x_2) + \exp(x_1 - x_2) + \exp(-x_1 + x_2) + \exp(-x_1 - x_2)}{4} + \\
& \frac{\exp(x_1 + x_2) - \exp(x_1 - x_2) - \exp(-x_1 + x_2) + \exp(-x_1 - x_2)}{4} \\
= & \frac{\exp(x_1 + x_2) + \exp(-x_1 - x_2)}{2} \\
= & \cosh(x_1 + x_2).
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
\cosh(x_1 - x_2) &= \cosh(x_1)\cosh(x_2) - \sinh(x_1)\sinh(x_2), \text{ and} \\
\sinh(x_1 \pm x_2) &= \sinh(x_1)\cosh(x_2) \pm \cosh(x_1)\sinh(x_2).
\end{aligned}$$

Corollary 4.24 *Let x in $\mathcal{R}$ be at most finite in absolute value, let $X = \Re(x)$, and let $s = x - X$. Then the following are true*

$\cosh(x) \approx \cosh(X)$.

$\sinh(x) \approx \begin{cases} \sinh(X) & \textit{if } X \neq 0 \\ s & \textit{if } X = 0 \end{cases}$.

$\cosh^2(x) - \sinh^2(x) = 1$.

$\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 2\sinh^2(x) + 1,$ *and*

$\sinh(2x) = 2\sinh(x)\cosh(x)$.

$\cosh^2(x) = \frac{\cosh(2x)+1}{2}$, *and* $\sinh^2(x) = \frac{\cosh(2x)-1}{2}$.

Proof. The proofs of the statements above are similar to those of the corresponding results about the nonhyperbolic functions; and we will omit the details here.

Chapter 5

Calculus on $\mathcal{R}$

In this chapter, we begin with a review of topological continuity and differentiability. We show that, like in R, the family of topologically continuous or differentiable functions at a point or on a domain is closed under addition, multiplication and composition. We also show that if the derivative exists, it must vanish at a local maximum or minimum. However, we show with examples that, unlike in R, a topologically continuous or differentiable function on a closed interval need not be bounded or satisfy any of the common theorems of real calculus. We then review continuity and differentiability, based on the derivate concept [10]. We show that the class of continuous or differentiable functions on a given interval of $\mathcal{R}$ is again closed under operations on functions. We develop a tool for easily checking the differentiability of functions and we finally use the new smoothness criteria to study a large class of functions for which we generalize the intermediate value theorem in [5] and prove an inverse function theorem. We study infinitely often differentiable functions, convergence of their Taylor series and show that power series can be reexpanded around any point of their domain of convergence.

5.1 Topological Continuity and Topological Differentiability

Notation 5.1 *Let $a < b$ be given in $\mathcal{R}$. By $I(a,b)$, we will denote any one of the intervals $[a,b]$, $(a,b]$, $[a,b)$ or (a,b).*

Definition 5.1 *Let $D \subset \mathcal{R}$, and let $f : D \to \mathcal{R}$. Then we say that f is topologically continuous at $x_0 \in D$ if and only if for all $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that*

$$x \in D \text{ and } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

Definition 5.2 *Let $D \subset \mathcal{R}$, and let $f : D \to \mathcal{R}$. Then we say that f is topologically continuous on D if and only if f is topologically continuous at x for all $x \in D$.*

The following example shows that, contrary to the real case, a function topologically continuous on a closed interval $[a,b]$ of $\mathcal{R}$ need not be bounded on $[a,b]$.

Example 5.1 *Let $f : [0,1] \to \mathcal{R}$ be given by*

$$f(x) = \begin{cases} d^{-1} & \text{if } 0 \le x < d \\ d^{-1/\lambda(x)} & \text{if } d \le x \ll 1 \\ 1 & \text{if } x \sim 1 \end{cases}.$$

Then f is topologically continuous on $[0,1]$: Let $x \in [0,1]$ and let $\epsilon > 0$ in $\mathcal{R}$ be given. First assume that $0 \le x < d$. Let $\delta = (d - x)/2$. Then $\delta > 0$ and for all $y \in [0,1]$ satisfying $|y - x| < \delta$, we have that $0 \le y < x < d$ or $0 \le x < y < x + \delta = (d + x)/2 < d$. Thus $f(y) = d^{-1}$; and hence $|f(y) - f(x)| = 0 < \epsilon$.

Now assume that $d \le x \ll 1$. Let $\delta = d \cdot x$. Then for all $y \in [0,1]$ satisfying $|y - x| < \delta$, we have that $\lambda(y) = \lambda(x)$. Thus, either

$$0 < y < d \le x \text{ and } \lambda(x) = \lambda(y) = 1,$$

in which case

$$|f(y) - f(x)| = \left|d^{-1} - d^{-1}\right| = 0 < \epsilon;$$

or

$$d \leq x, y \ll 1 \text{ and } \lambda(x) = \lambda(y),$$

in which case

$$|f(y) - f(x)| = \left|d^{-1/\lambda(y)} - d^{-1/\lambda(x)}\right| = 0 < \epsilon.$$

Finally, assume that $x \sim 1$. Let $\delta = d \cdot x$. Then for all $y \in [0, 1]$ satisfying $|y - x| < \delta$, we have that $y \sim 1$. Thus $f(y) = 1 = f(x)$; and hence $|f(y) - f(x)| = 0 < \epsilon$.

Next we show that f is not bounded on $[0, 1]$: Let $M > 0$ be given in $\mathcal{R}$. Let

$$x = \begin{cases} d^{1/2} & \text{if } \lambda(M) \geq -1 \\ d^{\frac{1}{1-\lambda(M)}} & \text{if } \lambda(M) < -1 \end{cases}.$$

Thus

$$\lambda(x) = \begin{cases} \frac{1}{2} & \text{if } \lambda(M) \geq -1 \\ \frac{1}{1-\lambda(M)} \in \left(0, \frac{1}{2}\right) \cap Q & \text{if } \lambda(M) < -1 \end{cases}.$$

Hence

$$\begin{aligned} |f(x)| &= \begin{cases} d^{-2} & \text{if } \lambda(M) \geq -1 \\ d^{\lambda(M)-1} & \text{if } \lambda(M) < -1 \end{cases} \\ &\gg M. \end{aligned}$$

Thus, for all $M > 0$ in $\mathcal{R}$, there exists $x \in [0, 1]$ such that $|f(x)| > M$. So f is not bounded on $[0, 1]$.

Lemma 5.1 *Let $D \subset \mathcal{R}$ and let $f : D \to \mathcal{R}$. Then f is topologically continuous at $x_0 \in D$ if and only if for any sequence (x_n) in D that converges strongly to x_0, the sequence $(f(x_n))$ converges strongly to $f(x_0)$.*

Proof. Suppose f is topologically continuous at x_0, and let (x_n) be a sequence in D that converges strongly to x_0. Let $\epsilon > 0$ be given in $\mathcal{R}$. There exists $\delta > 0$ in $\mathcal{R}$ such that

$$x \in D \text{ and } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

Since (x_n) converges strongly to x_0, there exists $N \in Z^+$ such that

$$|x_n - x_0| < \delta \text{ for all } n \geq N.$$

Thus,

$$|f(x_n) - f(x_0)| < \epsilon \text{ for all } n \geq N.$$

Hence the sequence $(f(x_n))$ converges strongly to $f(x_0)$.

Now suppose f is not topologically continuous at x_0. Then there exists $\epsilon_0 > 0$ in $\mathcal{R}$ such that for all $\delta > 0$ in $\mathcal{R}$ there exists $x \in D$ such that $|x - x_0| < \delta$ but $|f(x) - f(x_0)| > \epsilon_0$. In particular, for all $n \in Z^+$, there exists $x_n \in D$ such that

$$|x_n - x_0| < d^n \text{ and } |f(x_n) - f(x_0)| > \epsilon_0.$$

Hence (x_n) is a sequence in D that converges strongly to x_0; but the sequence $(f(x_n))$ does not converge strongly to $f(x_0)$.

Theorem 5.1 *Let $D \subset \mathcal{R}$, let $f, g : D \to \mathcal{R}$ be topologically continuous at $x_0 \in D$, and let $\alpha \in \mathcal{R}$ be given. Then $(f + \alpha g)$ and $(f \cdot g)$ are topologically continuous at x_0.*

Proof. Let (x_n) be a sequence in D that converges strongly to x_0. By Lemma 5.1, we have that the sequences $(f(x_n))$ and $(g(x_n))$ converge strongly to $f(x_0)$ and $g(x_0)$, respectively. For all $n \geq 1$, we have that

$$(f + \alpha g)(x_n) = f(x_n) + \alpha g(x_n).$$

Using the results of Lemma 4.5 and Lemma 4.6, the sequence $((f+\alpha g)(x_n))$ converges strongly to $f(x_0)+\alpha g(x_0) = (f+\alpha g)(x_0)$. By Lemma 5.1, $(f+\alpha g)$ is topologically continuous at x_0.

Also, for all $n \geq 1$, we have that

$$\begin{aligned}
(f \cdot g)(x_n) &= f(x_n)\, g(x_n) \\
&= (f(x_n) - f(x_0))(g(x_n) - g(x_0)) \\
&\quad + f(x_0)(g(x_n) - g(x_0)) \\
&\quad + g(x_0)(f(x_n) - f(x_0)) \\
&\quad + f(x_0)\, g(x_0).
\end{aligned}$$

Thus, the sequence $((f \cdot g)(x_n))$ converges strongly to $f(x_0)\,g(x_0) = (f \cdot g)(x_0)$. Again, by Lemma 5.1, $(f \cdot g)$ is topologically continuous at x_0.

Corollary 5.1 *Let $D \subset \mathcal{R}$, let $f, g : D \to \mathcal{R}$ be topologically continuous on D, and let $\alpha \in \mathcal{R}$ be given. Then $(f+\alpha g)$ and $(f \cdot g)$ are topologically continuous on D.*

Theorem 5.2 *Let $D_f, D_g \subset \mathcal{R}$ and let $f : D_f \to \mathcal{R}$ and $g : D_g \to \mathcal{R}$ be such that $f(D_f) \subset D_g$, f is topologically continuous at $x_0 \in D_f$ and g topologically continuous at $f(x_0)$. Then $g \circ f : D_f \to \mathcal{R}$, given by $(g \circ f)(x) = g(f(x))$, is topologically continuous at x_0.*

Proof. Let (x_n) be a sequence in D_f that converges strongly to x_0. Since f is topologically continuous at x_0, the sequence $(f(x_n))$ converges strongly to $f(x_0)$. Since g is topologically continuous at $f(x_0)$, the sequence $(g(f(x_n)))$ converges strongly to $g(f(x_0)) = (g \circ f)(x_0)$. But for all $n \geq 1$, we have that $g(f(x_n)) = (g \circ f)(x_n)$. Thus, the sequence $((g \circ f)(x_n))$ converges strongly to $(g \circ f)(x_0)$. This is true for any sequence (x_n) in D_f that converges strongly to x_0. By Lemma 5.1, $(g \circ f)$ is topologically continuous at x_0.

Corollary 5.2 *Let $D_f, D_g \subset \mathcal{R}$ and let $f : D_f \to \mathcal{R}$ and $g : D_g \to \mathcal{R}$ be such that $f(D_f) \subset D_g$, f is topologically continuous on D_f and g topologically continuous on D_g. Then $g \circ f$ is topologically continuous on D_f.*

Definition 5.3 *Let $D \subset \mathcal{R}$ and let $f : D \to \mathcal{R}$. Then we say that f is topologically uniformly continuous on D if and only if for all $x \in D$ and for all $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that*

$$y \in D \text{ and } |y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon.$$

Lemma 5.2 *Let $D \subset \mathcal{R}$ and let $f : D \to \mathcal{R}$. Then f is topologically uniformly continuous on D if and only if for all $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that*

$$x, y \in D \text{ and } 0 < y - x < \delta \Rightarrow |f(y) - f(x)| < \epsilon.$$

Proof. First assume that f is topologically uniformly continuous on D, and let $\epsilon > 0$ be given in $\mathcal{R}$. Then by Definition 5.3, there exists $\delta > 0$ in $\mathcal{R}$ such that for all $x \in D$, we have that

$$y \in D \text{ and } |y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon.$$

Hence

$$x, y \in D \text{ and } |y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon;$$

in particular,

$$x, y \in D \text{ and } 0 < y - x < \delta \Rightarrow |f(y) - f(x)| < \epsilon.$$

Now assume that for all $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that

$$x, y \in D \text{ and } 0 < y - x < \delta \Rightarrow |f(y) - f(x)| < \epsilon. \tag{5.1}$$

We show that f is topologically uniformly continuous on D. So let $z \in D$ and let $\epsilon > 0$ in $\mathcal{R}$ be given. Let $\delta > 0$ in $\mathcal{R}$ be as in Equation (5.1), and let $w \in D$ be such that

$|w - z| < \delta$. If $z < w$, let $x = z$ and $y = w$ in Equation (5.1) to get $|f(w) - f(z)| < \epsilon$; if $w < z$, let $x = w$ and $y = z$ in Equation (5.1) to get $|f(z) - f(w)| < \epsilon$. Hence for all $z \in D$ and for all $\epsilon > 0$ in $\mathcal{R}$, there exists $\delta > 0$ in $\mathcal{R}$ such that

$$w \in D \text{ and } |w - z| < \delta \Rightarrow |f(w) - f(z)| < \epsilon.$$

Hence f is topologically uniformly continuous on D.

Theorem 5.3 *Let $a < b$ be given in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$ be topologically uniformly continuous on $I(a,b)$. Then there exists a unique function $g : [a,b] \to \mathcal{R}$, topologically uniformly continuous on $[a,b]$, such that*

$$g|_{I(a,b)} = f.$$

Proof. We may assume that $I(a,b) \neq [a,b]$. First assume that $I(a,b) = (a,b]$. For all $n \in Z^+$, let $x_n = a + d^n(b-a)$. Then $x_n \in I(a,b)$ for all $n \geq 1$; and the sequence (x_n) converges strongly to a. We show that the sequence $(f(x_n))$ converges strongly in $\mathcal{R}$. So let $\epsilon > 0$ be given in $\mathcal{R}$. Then there exists $\delta > 0$ in $\mathcal{R}$ such that

$$x, y \in (a,b] \text{ and } |y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon.$$

There exists $N \in Z^+$ such that

$$d^N(b-a) < \delta.$$

Now let $m, n \geq N$ be given. Then

$$|x_m - x_n| = |d^m - d^n|(b-a) \leq d^N(b-a) < \delta.$$

Thus,

$$|f(x_m) - f(x_n)| < \epsilon \text{ for all } m, n \geq N.$$

Hence the sequence $(f(x_n))$ is strongly Cauchy. Since $\mathcal{R}$ is Cauchy complete with respect to the order topology, $(f(x_n))$ converges strongly in $\mathcal{R}$. Define $g:[a,b]\to\mathcal{R}$ by

$$g(x)=\begin{cases} f(x) & \text{if } x\in(a,b] \\ \lim_{n\to\infty} f(x_n) & \text{if } x=a \end{cases}.$$

We show now that g is topologically uniformly continuous on $[a,b]$. Let $\epsilon>0$ be given in $\mathcal{R}$. There exists $\delta>0$ in $\mathcal{R}$ such that

$$x,y\in(a,b] \text{ and } |y-x|<\delta \Rightarrow |f(y)-f(x)|<\frac{\epsilon}{2}.$$

There exists $N\in Z^+$ such that

$$d^N(b-a)<\delta \text{ and } |f(x_N)-g(a)|<\frac{\epsilon}{2}.$$

Now let $x,y\in[a,b]$ be such that $0<y-x<\delta$. Then $y\in(a,b]$. If $x\in(a,b]$, then

$$|g(y)-g(x)|=|f(y)-f(x)|<\frac{\epsilon}{2}<\epsilon.$$

If $x=a$, then

$$0<y-a=y-x<\delta \text{ and } |y-x_N|=|y-a-d^N(b-a)|.$$

Since

$$0<y-a<\delta \text{ and } 0<d^N(b-a)<\delta,$$

we obtain that

$$\begin{aligned} |y-x_N| &= \max\left\{y-a,d^N(b-a)\right\}-\min\left\{y-a,d^N(b-a)\right\} \\ &< \max\left\{y-a,d^N(b-a)\right\}<\delta. \end{aligned}$$

and hence

$$|f(y)-f(x_N)|<\frac{\epsilon}{2}.$$

Thus,

$$\begin{aligned}
|g(y)-g(x)| &= |f(y)-g(a)| \\
&\leq |f(y)-f(x_N)|+|f(x_N)-g(a)| \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

Hence g is topologically uniformly continuous on $[a,b]$ and $g|_{I(a,b)}=f$.

Let $g_1:[a,b]\to\mathcal{R}$ be topologically uniformly continuous on $[a,b]$ with $g_1|_{I(a,b)}=f$. To show that $g_1=g$, it suffices to show that $g_1(a)=g(a)$. Since g_1 is topologically continuous at a and since (x_n) converges strongly to a, we obtain by Lemma 5.1 that the sequence $(g_1(x_n))$ converges strongly to $g_1(a)$. Thus,

$$g_1(a)=\lim_{n\to\infty} g_1(x_n)=\lim_{n\to\infty} f(x_n)=g(a).$$

So $g_1=g$; and hence g is unique.

Similarly we can show that the result is true for the cases $I(a,b)=[a,b)$ and $I(a,b)=(a,b)$.

Example 5.2 *Let $f:(0,1]\to\mathcal{R}$ be given by $g(x)=1/x$.*

We show that f is topologically continuous on $(0,1]$; but there is no function $g:[0,1]\to\mathcal{R}$ such that g is topologically continuous on $[0,1]$ and $g|_{(0,1]}=f$. Let $x\in(0,1]$ and let $\epsilon>0$ in $\mathcal{R}$ be given. Let

$$\delta=\min\left\{\frac{x}{2},\frac{\epsilon x^2}{2}\right\},$$

and let $y\in(0,1]$ be such that $0<|y-x|<\delta$. First assume that that $y<x$. Then $0<x-y<\delta$, and

$$\begin{aligned}
|f(y)-f(x)| &= \frac{1}{y}-\frac{1}{x}<\frac{1}{x-\delta}-\frac{1}{x}=\frac{\delta}{x(x-\delta)} \\
&\leq \frac{\delta}{x\left(x-\frac{x}{2}\right)}=\frac{2\delta}{x^2}\leq\epsilon.
\end{aligned}$$

Now assume that $x < y$. Then $0 < y - x < \delta$, and

$$\begin{aligned} |f(y) - f(x)| &= \frac{1}{x} - \frac{1}{y} < \frac{1}{x} - \frac{1}{x+\delta} = \frac{\delta}{x(x+\delta)} \\ &< \frac{\delta}{x^2} \leq \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Hence f is topologically continuous at x for all $x \in (0,1]$, and hence f is topologically continuous on $(0,1]$.

Suppose there exists $g : [0,1] \to \mathcal{R}$ such that g is topologically continuous on $[0,1]$ and $g|_{(0,1]} = f$. Then, by Lemma 5.1, we have that the sequence $(g(d^n))$ converges strongly to $g(0)$; i.e. the sequence (d^{-n}) converges strongly to $g(0)$, which contradicts the fact that (d^{-n}) is unbounded. So there can be no such g.

Definition 5.4 *Let $D \subset \mathcal{R}$ be open and let $f : D \to \mathcal{R}$. Then we say that f is topologically differentiable at $x_0 \in D$ if and only if there exists a number $f'(x_0) \in \mathcal{R}$ such that the function $F_{1,x_0} : D \to \mathcal{R}$, given by*

$$F_{1,x_0}(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \neq x_0 \\ f'(x_0) & \text{if } x = x_0 \end{cases},$$

is topologically continuous at x_0. If this is the case, we call F_{1,x_0} the derivate function of f at x_0 [10], and $f'(x_0)$ the derivative of f at x_0.

Definition 5.5 *Let $D \subset \mathcal{R}$ be open and let $f : D \to \mathcal{R}$. Then we say that f is topologically differentiable on D if and only if f is topologically differentiable at x for all $x \in D$.*

The following lemma follows directly from Definition 5.4.

Lemma 5.3 *Let D, f, x_0, F_{1,x_0} be as in Definition 5.4. Then we have that*

$$f(x) = f(x_0) + F_{1,x_0}(x)(x - x_0) \text{ for all } x \in D.$$

The following lemma is a direct consequence of Definitions 5.4 and 5.1.

Lemma 5.4 *Let $D \subset \mathcal{R}$ be open and let $f : D \to \mathcal{R}$. Then f is topologically differentiable at $x_0 \in D$ if and only if there exists a number $f'(x_0) \in \mathcal{R}$ such that for all $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that*

$$x \in D \text{ and } 0 < |x - x_0| < \delta \Rightarrow \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon.$$

Theorem 5.4 *Let $D \subset \mathcal{R}$ be open and let $f : D \to \mathcal{R}$ be topologically differentiable at $x_0 \in D$. Then f is topologically continuous at x_0.*

Proof. Let $\epsilon > 0$ be given in $\mathcal{R}$, and let $\epsilon_1 = \epsilon/2$. Since f is topologically differentiable at x_0, there exists $\delta_1 > 0$ in $\mathcal{R}$ such that

$$x \in D \text{ and } 0 < |x - x_0| < \delta_1 \Rightarrow \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon_1.$$

Let

$$\delta = \begin{cases} \min\left\{\delta_1, \frac{\epsilon}{2|f'(x_0)|}, 1\right\} & \text{if } f'(x_0) \neq 0 \\ \min\{\delta_1, 1\} & \text{if } f'(x_0) = 0 \end{cases}.$$

Then $\delta > 0$, and for all $x \in D$ satisfying $0 < |x - x_0| < \delta$, we have that

$$\begin{aligned} |f(x) - f(x_0)| &< \epsilon_1 |x - x_0| + |f'(x_0)| |x - x_0| \\ &< \frac{\epsilon}{2}\delta + |f'(x_0)| \delta \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence f is topologically continuous at x_0.

Corollary 5.3 *Let $D \subset \mathcal{R}$ be open and let $f : D \to \mathcal{R}$ be topologically differentiable on D. Then f is topologically continuous on D.*

Theorem 5.5 *Let $D \subset \mathcal{R}$ be open, let $f, g : D \to \mathcal{R}$ be topologically differentiable at $x_0 \in D$, and let $\alpha \in \mathcal{R}$ be given. Then $(f + \alpha g)$ and $(f \cdot g)$ are topologically differentiable at x_0, with derivatives*

$$\begin{aligned} (f + \alpha g)'(x_0) &= f'(x_0) + \alpha g'(x_0) \quad \textit{and} \\ (f \cdot g)'(x_0) &= f'(x_0) g(x_0) + f(x_0) g'(x_0). \end{aligned}$$

Proof. Let F_{1,x_0} and G_{1,x_0} denote the derivate functions of f and g at x_0, respectively. Then F_{1,x_0} and G_{1,x_0} are topologically continuous at x_0. By Theorem 5.1, we have that the function $F_{1,x_0} + \alpha G_{1,x_0} : D \to \mathcal{R}$, given by

$$\begin{aligned} (F_{1,x_0} + \alpha G_{1,x_0})(x) &= F_{1,x_0}(x) + \alpha G_{1,x_0}(x) \\ &= \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} + \alpha \frac{g(x) - g(x_0)}{x - x_0} & \text{if } x \neq x_0 \\ f'(x_0) + \alpha g'(x_0) & \text{if } x = x_0 \end{cases} \\ &= \begin{cases} \frac{(f + \alpha g)(x) - (f + \alpha g)(x_0)}{x - x_0} & \text{if } x \neq x_0 \\ f'(x_0) + \alpha g'(x_0) & \text{if } x = x_0 \end{cases}, \end{aligned}$$

is topologically continuous at x_0. Thus, $(f + \alpha g)$ is topologically differentiable at x_0, with derivative

$$(f + \alpha g)'(x_0) = f'(x_0) + \alpha g'(x_0).$$

Now let $H : D \to \mathcal{R}$ be given by

$$H(x) = \begin{cases} \frac{(f \cdot g)(x) - (f \cdot g)(x_0)}{x - x_0} & \text{if } x \neq x_0 \\ f'(x_0) g(x_0) + f(x_0) g'(x_0) & \text{if } x = x_0 \end{cases}.$$

To show that $(f \cdot g)$ is topologically differentiable at x_0, we need to show that H is topologically continuous at x_0. Note that

$$\begin{aligned} H(x) &= \begin{cases} \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} & \text{if } x \neq x_0 \\ f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0) & \text{if } x = x_0 \end{cases} \\ &= \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} g(x_0) + f(x)\,\frac{g(x)-g(x_0)}{x-x_0} & \text{if } x \neq x_0 \\ f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0) & \text{if } x = x_0 \end{cases} \\ &= F_{1,x_0}(x)\,g(x_0) + f(x)\,G_{1,x_0}(x). \end{aligned}$$

Hence $H = g(x_0)F_{1,x_0} + f \cdot G_{1,x_0}$. Using Theorem 5.1, we obtain that H is topologically continuous at x_0. Thus, $(f \cdot g)$ is topologically differentiable at x_0, with derivative

$$(f \cdot g)'(x_0) = H(x_0) = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0).$$

Corollary 5.4 *Let $D \subset \mathcal{R}$ be open, let $f, g : D \to \mathcal{R}$ be topologically differentiable on D, and let $\alpha \in \mathcal{R}$ be given. Then $(f + \alpha g)$ and $(f \cdot g)$ are topologically differentiable on D, with derivatives*

$$(f + \alpha g)' = f' + \alpha g' \text{ and } (f \cdot g)' = f' \cdot g + f \cdot g'.$$

Theorem 5.6 (Chain Rule) *Let $D_f, D_g \subset \mathcal{R}$ be open, and let $f : D_f \to \mathcal{R}$ and $g : D_g \to \mathcal{R}$ be such that $f(D_f) \subset D_g$, f is topologically differentiable at $x_0 \in D_f$ and g topologically differentiable at $f(x_0)$. Then $g \circ f$ is topologically differentiable at x_0, with derivative*

$$(g \circ f)'(x_0) = g'(f(x_0))\,f'(x_0).$$

Proof. Let F_{1,x_0} and $G_{1,f(x_0)}$ denote the derivate functions of f at x_0 and of g at $f(x_0)$, respectively. Let $H : D_f \to \mathcal{R}$ be given by

$$H(x) = \begin{cases} \frac{(g\circ f)(x)-(g\circ f)(x_0)}{x-x_0} & \text{if } x \neq x_0 \\ g'(f(x_0))\, f'(x_0) & \text{if } x = x_0 \end{cases}.$$

Then

$$\begin{aligned} H(x) &= \begin{cases} \frac{g(f(x))-g(f(x_0))}{x-x_0} & \text{if } x \neq x_0 \\ g'(f(x_0))\, f'(x_0) & \text{if } x = x_0 \end{cases} \\ &= \begin{cases} \frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \neq x_0 \text{ and } f(x) \neq f(x_0) \\ 0 & \text{if } x \neq x_0 \text{ and } f(x) = f(x_0) \\ g'(f(x_0))\, f'(x_0) & \text{if } x = x_0 \end{cases} \\ &= G_{1,f(x_0)}(f(x))\, F_{1,x_0}(x). \end{aligned}$$

Hence

$$H = \left(G_{1,f(x_0)} \circ f\right) \cdot F_{1,x_0}.$$

Since f is topologically continuous at x_0 and since $G_{1,f(x_0)}$ is topologically continuous at $f(x_0)$, we have by Theorem 5.2 that $\left(G_{1,f(x_0)} \circ f\right)$ is topologically continuous at x_0. Since F_{1,x_0} is topologically continuous at x_0, so is $\left(G_{1,f(x_0)} \circ f\right) \cdot F_{1,x_0} = H$ by Theorem 5.1. Hence $(g \circ f)$ is topologically differentiable at x_0, with derivative

$$(g \circ f)'(x_0) = H(x_0) = g'(f(x_0))\, f'(x_0).$$

Corollary 5.5 *Let $D_f, D_g \subset \mathcal{R}$ be open, and let $f : D_f \to \mathcal{R}$ and $g : D_g \to \mathcal{R}$ be such that $f(D_f) \subset D_g$, f is topologically differentiable on D_f and g topologically differentiable on D_g. Then $g \circ f$ is topologically differentiable on D_f, with derivative*

$$(g \circ f)' = (g' \circ f) \cdot f'.$$

Theorem 5.7 *Let $D \subset \mathcal{R}$ be open, and let $f : D \to \mathcal{R}$ be such that f is topologically differentiable and has a local maximum at $x_0 \in D$. Then*

$$f'(x_0) = 0.$$

Proof. Suppose not; then $|f'(x_0)| > 0$. Since D is open and since f is topologically differentiable at x_0, there exists $\delta > 0$ in $\mathcal{R}$ such that $(x_0 - \delta, x_0 + \delta) \subset D$ and

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < d\,|f'(x_0)| \text{ for all } x \neq x_0 \text{ in } (x_0 - \delta, x_0 + \delta);$$

which entails that

$$\frac{f(x) - f(x_0)}{x - x_0} \approx f'(x_0) \text{ for all } x \neq x_0 \text{ in } (x_0 - \delta, x_0 + \delta).$$

In particular, $(f(x) - f(x_0))/(x - x_0)$ has the same sign (that of $f'(x_0)$) for all $x \neq x_0$ in $(x_0 - \delta, x_0 + \delta)$; which contradicts the fact that f has a local maximum at x_0. Thus, $f'(x_0) = 0$.

Corollary 5.6 *Let $D \subset \mathcal{R}$ be open, and let $f : D \to \mathcal{R}$ be such that f is topologically differentiable and has a local minimum at $x_0 \in D$. Then*

$$f'(x_0) = 0.$$

Proof. Let $g = -f$. Then g is topologically differentiable and has a local maximum at x_0. By Theorem 5.7, we obtain that $g'(x_0) = 0$. Using Theorem 5.5, we finally obtain that

$$f'(x_0) = -g'(x_0) = 0.$$

The following examples show that, contrary to the real case, topological continuity or even topological differentiability of a function on a closed interval of $\mathcal{R}$ are not always sufficient for the function to assume all intermediate values, a maximum, a minimum, or a unique primitive function on the interval.

Example 5.3 *Let* $f:[0,1]\to\mathcal{R}$ *be given by*

$$f(x)=\begin{cases}1 & \textit{if } x\sim 1\\ 0 & \textit{if } 0\le x\ll 1\end{cases}.$$

Then f is topologically continuous on $[0,1]$ and topologically differentiable on $(0,1)$, with derivative $f'(x)=0$ for all $x\in(0,1)$. We have that

$$f(0)=0<d<1=f(1)\,;\text{ but } f(x)\neq d \text{ for all } x\in[0,1]\,.$$

So f does not satisfy the intermediate value theorem on $[0,1]$. Moreover, although $f'(x)=0$ for all $x\in(0,1)$, f is not constant on $[0,1]$.

Example 5.4 *Let* $t>0$ *be given in* Q*, and let* $f:[-1,1]\to\mathcal{R}$ *be given by*

$$f(x)=\begin{cases}d^t\exp(x)+sign(x)\exp\left(-1/x^2\right) & \textit{if } x\sim 1\\ d^t\exp(x) & \textit{if } 0\le x\ll 1\end{cases},$$

where

$$sign(x)=\begin{cases}1 & \textit{if } x>0\\ -1 & \textit{if } x<0\end{cases}.$$

Then f is topologically continuous on $[-1,1]$ and topologically differentiable on $(-1,1)$ with derivative

$$\begin{aligned}f'(x) &= \begin{cases}d^t\exp(x)+\frac{2}{x^3}\text{sign}(x)\exp\left(-1/x^2\right) & \text{if } x\sim 1\\ d^t\exp(x) & \text{if } 0\le x\ll 1\end{cases}\\ &= \begin{cases}d^t\exp(x)+\frac{2}{|x|^3}\exp\left(-1/x^2\right) & \text{if } x\sim 1\\ d^t\exp(x) & \text{if } 0\le x\ll 1\end{cases}\\ &> 0 \text{ for all } x\in(-1,1)\,.\end{aligned}$$

Moreover, f is strictly increasing on $[-1,1]$. We have that

$$f(-1)=d^t\exp(-1)-\exp(-1)<d<d^t\exp(1)+\exp(-1)=f(1)\,;$$

but $f(x)\neq d$ for all $x\in[-1,1]$.

Example 5.5 *Let* $f : [-1,1] \to \mathcal{R}$ *be given by*

$$f(x) = x - \Re(x).$$

Then f is topologically continuous on $[-1,1]$. However, f assumes neither a maximum nor a minimum on $[-1,1]$. The set $f([-1,1])$ is bounded above by any positive real number and below by any negative real number; but it has neither a least upper bound nor a greatest lower bound.

Example 5.6 *Let* $f : [-1,1] \to \mathcal{R}$ *be given by*

$$f(x) = \sum_{\nu=1}^{\infty} x_\nu d^{3q_\nu} \text{ when } x = \Re(x) + \sum_{\nu=1}^{\infty} x_\nu d^{q_\nu}.$$

Then f is topologically continuous on $[-1,1]$ and topologically differentiable on $(-1,1)$, with derivative

$$f'(x) = 0 \text{ for all } x \in (-1,1).$$

f has neither a maximum nor a minimum on $[-1,1]$. Moreover, f is not constant on $[-1,1]$ even though $f'(x) = 0$ for all $x \in (-1,1)$.

Example 5.7 *Let* $f, g : [-1,1] \to \mathcal{R}$ *be given by*

$$f(x) = x \text{ and } g(x) = x + d^{3\lambda(x)+1}.$$

Then f and g are both topologically continuous on $[-1,1]$ and topologically differentiable on $(-1,1)$, with derivatives

$$f'(x) = 1 = g'(x) \text{ for all } x \in (-1,1).$$

So f and g are two primitive functions of 1 on $[-1,1]$ that do not differ by a constant.

In the following section, we introduce stronger smoothness criteria on $\mathcal{R}$ and use them to try and extend the common theorems of real calculus to $\mathcal{R}$.

5.2 Continuity and Differentiability

Definition 5.6 *Let $a<b$ be given in $\mathcal{R}$ and let $f: I(a,b) \to \mathcal{R}$. Then we say that f is continuous on $I(a,b)$ if and only if there exists $M \in \mathcal{R}$, called a Lipschitz constant of f on $I(a,b)$, such that [10]*

$$\left| \frac{f(y)-f(x)}{y-x} \right| \leq M \textit{ for all } x \neq y \textit{ in } I(a,b).$$

Lemma 5.5 *Let $a<b$ be given in $\mathcal{R}$ and let $f: I(a,b) \to \mathcal{R}$ be continuous on $I(a,b)$. Then f is topologically uniformly continuous on $I(a,b)$.*

Proof. Let M be a Lipschitz constant of f on $I(a,b)$, and let $\epsilon>0$ be given in $\mathcal{R}$. Let $\delta=\epsilon/M$. Then $\delta>0$, and for all $x,y \in D$ satisfying $0<y-x<\delta$, we have that

$$|f(y)-f(x)| \leq M(y-x) < M\delta = \epsilon.$$

Hence, using Lemma 5.2, we obtain that f is topologically uniformly continuous on D.

Lemma 5.6 *Let $a<b$ be given in $\mathcal{R}$ and let $f: I(a,b) \to \mathcal{R}$ be continuous on $I(a,b)$. Then f is bounded on $I(a,b)$.*

Proof. Since f is continuous on $I(a,b)$, there exists $M \in \mathcal{R}$ such that

$$\left| \frac{f(y)-f(x)}{y-x} \right| \leq M \text{ for all } x \neq y \text{ in } I(a,b).$$

Thus,

$$\left|\frac{f(x)-f\left(\frac{b-a}{2}\right)}{x-\frac{b-a}{2}}\right| \leq M \text{ for all } x \neq \frac{b-a}{2} \text{ in } I(a,b);$$

and hence

$$|f(x)| \leq \left|f\left(\frac{b-a}{2}\right)\right| + M\left|x-\frac{b-a}{2}\right| \leq \left|f\left(\frac{b-a}{2}\right)\right| + M|b-a|$$

for all $x \in I(a,b)$.

Lemma 5.7 *Let $a<b$ be given in $\mathcal{R}$ and let $f: I(a,b) \to \mathcal{R}$ be continuous on $I(a,b)$ with Lipschitz constant M. Let $x \in I(a,b)$ be given, let $r \in Q$, and let $h \in \mathcal{R}$ be such that $|h| \ll d^r$ and $x+h \in I(a,b)$. Then*

$$f(x+h) =_{r+\lambda(M)} f(x).$$

Proof. If $h=0$, we are done. So we may assume that $h \neq 0$. Thus,

$$\left|\frac{f(x+h)-f(x)}{h}\right| \leq M; \text{ and hence } |f(x+h)-f(x)| \leq M|h|.$$

Thus,

$$\lambda(f(x+h)-f(x)) \geq \lambda(M|h|) = \lambda(M)+\lambda(h) > \lambda(M)+r,$$

which entails that $f(x+h) =_{r+\lambda(M)} f(x)$.

Lemma 5.8 (Remainder Formula 0) *Let $a<b$ in $\mathcal{R}$ and let $f: I(a,b) \to \mathcal{R}$ be continuous on $I(a,b)$ with Lipschitz constant M. Then for all $x, y \in I(a,b)$, we have that*

$$f(y) = f(x) + r_0(x,y)(y-x), \text{ with } \lambda(r_0(x,y)) \geq \lambda(M).$$

Proof. Let $x, y \in I(a,b)$ be given. Let

$$r_0(x,y) = \begin{cases} \frac{f(y)-f(x)}{y-x} & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases}.$$

Then $f(y) = f(x) + r_0(x,y)(y-x)$. Moreover, since f is continuous on $I(a,b)$, we have that

$$|r_0(x,y)| \leq M; \text{ and hence } \lambda(r_0(x,y)) \geq \lambda(M).$$

Theorem 5.8 *Let $a < b$ be given in $\mathcal{R}$, let $f, g : I(a,b) \to \mathcal{R}$ be continuous on $I(a,b)$, and let $\alpha \in \mathcal{R}$. Then $f + \alpha g$ and $f \cdot g$ are continuous on $I(a,b)$.*

Proof. Since f and g are continuous on $I(a,b)$, there exist $M_1, M_2 \in \mathcal{R}$ such that

$$\left|\frac{f(y) - f(x)}{y - x}\right| \leq M_1 \text{ and } \left|\frac{g(y) - g(x)}{y - x}\right| \leq M_2 \text{ for all } x \neq y \text{ in } I(a,b).$$

Let

$$M = \max\{M_1, M_2\}.$$

Then

$$\left|\frac{f(y) - f(x)}{y - x}\right| \leq M \text{ and } \left|\frac{g(y) - g(x)}{y - x}\right| \leq M \text{ for all } x \neq y \text{ in } I(a,b).$$

Now let $x \neq y$ in $I(a,b)$ be given. Then

$$\begin{aligned}
\left|\frac{(f+\alpha g)(y) - (f+\alpha g)(x)}{y - x}\right| &= \left|\frac{f(y) + \alpha g(y) - f(x) - \alpha g(x)}{y - x}\right| \\
&\leq \left|\frac{f(y) - f(x)}{y - x}\right| + |\alpha| \left|\frac{g(y) - g(x)}{y - x}\right| \\
&\leq (1 + |\alpha|) M.
\end{aligned}$$

Hence $f + \alpha g$ is continuous on $I(a,b)$ with Lipschitz constant $(1 + |\alpha|) M$.

Since f and g are continuous on $I(a,b)$, we have by Lemma 5.6 that f and g are bounded on $I(a,b)$. Hence there exists $M_0 \in \mathcal{R}$ such that

$$|f(x)| \leq M_0 \text{ and } |g(x)| \leq M_0 \text{ for all } x \in I(a,b).$$

Now for all $x \neq y$ in $I(a,b)$, we have that

$$\begin{aligned}
\left|\frac{(f \cdot g)(y) - (f \cdot g)(x)}{y-x}\right| &= \left|\frac{f(y)g(y) - f(x)g(x)}{y-x}\right| \\
&= \left|\frac{f(y)(g(y) - g(x)) + (f(y) - f(x))g(x)}{y-x}\right| \\
&\leq |f(y)|\left|\frac{g(y)-g(x)}{y-x}\right| + |g(x)|\left|\frac{f(y)-f(x)}{y-x}\right| \\
&\leq M_0M_2 + M_0M_1 \\
&\leq 2M_0M.
\end{aligned}$$

Hence $f \cdot g$ is continuous on $I(a,b)$ with Lipschitz constant $2M_0M$.

Theorem 5.9 *Let $a < b$ and $c < e$ in $\mathcal{R}$ be given, and let $f : I_1(a,b) \to \mathcal{R}$ and $g : I_2(c,e) \to \mathcal{R}$ be such that $f(I_1(a,b)) \subset I_2(c,e)$, f is continuous on $I_1(a,b)$ and g continuous on $I_2(c,e)$. Then $g \circ f$ is continuous on $I_1(a,b)$.*

Proof. Let M_f and M_g be Lipschitz constants of f on $I_1(a,b)$ and of g on $I_2(c,e)$, respectively. Let $x \neq y$ be given in $I_1(a,b)$. First assume that $f(y) = f(x)$. Then

$$\left|\frac{(g \circ f)(y) - (g \circ f)(x)}{y-x}\right| = \left|\frac{g(f(y)) - g(f(x))}{y-x}\right| = 0 \leq M_gM_f.$$

Now assume that $f(y) \neq f(x)$. Then

$$\begin{aligned}
\left|\frac{(g \circ f)(y) - (g \circ f)(x)}{y-x}\right| &= \left|\frac{g(f(y)) - g(f(x))}{y-x}\right| \\
&= \left|\frac{g(f(y)) - g(f(x))}{f(y)-f(x)}\frac{f(y)-f(x)}{y-x}\right| \\
&= \left|\frac{g(f(y)) - g(f(x))}{f(y)-f(x)}\right|\left|\frac{f(y)-f(x)}{y-x}\right| \\
&\leq M_gM_f.
\end{aligned}$$

Thus, for all $y \neq x$ in $I_1(a,b)$, we have that

$$\left|\frac{(g\circ f)(y)-(g\circ f)(x)}{y-x}\right| \leq M_g M_f;$$

and hence $(g\circ f)$ is continuous on $I_1(a,b)$, with Lipschitz constant $M_g M_f$.

Theorem 5.10 *Let $a<b$ be given in $\mathcal{R}$ and let $f: I(a,b)\to\mathcal{R}$ be continuous on $I(a,b)$. Then there exists a unique function $g:[a,b]\to\mathcal{R}$, continuous on $[a,b]$, such that*

$$g|_{I(a,b)} = f.$$

Proof. We may assume that $I(a,b)\neq[a,b]$. First assume that $I(a,b)=(a,b]$. Let

$$f_0 = \lim_{n\to\infty} f\left(a+d^n(b-a)\right),$$

which exists by the proof of Theorem 5.3 since f is topologically uniformly continuous on $(a,b]$ by Lemma 5.5. Define $g:[a,b]\to\mathcal{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x\in(a,b] \\ f_0 & \text{if } x=a \end{cases}.$$

It remains to show that g is continuous on $[a,b]$. Let M be a Lipschitz constant of f on $(a,b]$ and let $x\neq y$ in $[a,b]$ be given. Without loss of generality, we may assume that $x<y$. Assume that $a<x$, then $x,y\in(a,b]$, and hence

$$\left|\frac{g(y)-g(x)}{y-x}\right| = \left|\frac{f(y)-f(x)}{y-x}\right| \leq M \leq 2M.$$

Now assume that $x=a$. There exists $N\in Z^+$ such that

$$d^N(b-a) < y-a \text{ and } \left|f\left(a+d^N(b-a)\right)-g(a)\right| \leq M(y-a).$$

Then it follows that

$$0 < y-\left(a+d^N(b-a)\right) < y-a;$$

and hence

$$\begin{aligned}
\left|\frac{g(y)-g(x)}{y-x}\right| &= \left|\frac{f(y)-g(a)}{y-a}\right| \\
&= \left|\frac{f(y)-f\left(a+d^N(b-a)\right)+f\left(a+d^N(b-a)\right)-g(a)}{y-a}\right| \\
&\le \left|\frac{f(y)-f\left(a+d^N(b-a)\right)}{y-a}\right| + \left|\frac{f\left(a+d^N(b-a)\right)-g(a)}{y-a}\right| \\
&\le \left|\frac{f(y)-f\left(a+d^N(b-a)\right)}{y-(a+d^N(b-a))}\right| + M \\
&\le M+M=2M.
\end{aligned}$$

Thus, for all $x \neq y$ in $[a,b]$, we have that

$$\left|\frac{g(y)-g(x)}{y-x}\right| \le 2M.$$

Hence g is continuous on $[a,b]$ with Lipschitz constant $2M$, and $g|_{(a,b]} = f$.

Similarly, we can show that the result is true for the cases when $I(a,b)=[a,b)$ and $I(a,b)=(a,b)$.

Definition 5.7 *Let $a<b$ be given in $\mathcal{R}$ and let $f: I(a,b) \to \mathcal{R}$ be continuous on $I(a,b)$. Then we say that f is differentiable on $I(a,b)$ if and only if there exists a function $f': I(a,b) \to \mathcal{R}$, called the derivative of f on $I(a,b)$, such that for all $x \in I(a,b)$, the derivate function $F_{1,x}: I(a,b) \to \mathcal{R}$ [10], given by*

$$F_{1,x}(y) = \begin{cases} \frac{f(y)-f(x)}{y-x} & \text{if } y \neq x \\ f'(x) & \text{if } y = x \end{cases},$$

is continuous on $I(a,b)$.

Lemma 5.9 *Let $a<b$ be given in $\mathcal{R}$ and let $f: I(a,b) \to \mathcal{R}$ be differentiable on $I(a,b)$. Then f is topologically differentiable on (a,b).*

Proof. Let $x \in (a,b)$ be given. Then the derivate function $F_{1,x}$ is continuous on $I(a,b)$. By Lemma 5.5, $F_{1,x}$ is topologically continuous at x. Hence f is topologically differentiable at x. This is true for all $x \in (a,b)$; hence f is topologically differentiable on (a,b).

The following theorem is a generalization of a similar result in [5] and is a central theorem because it reduces computing derivatives to mere arithemtic operations and thus allows rigorous study of differentiation [38, 39, 40, 43].

Theorem 5.11 (Derivatives are Differential Quotients) *Let $a < b$ be given in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$ be differentiable on $I(a,b)$. Let $x \in I(a,b)$ be given, let $F_{1,x}$ be the derivate function of f at x, and let $M_{1,x}$ be a Lipschitz constant of $F_{1,x}$. Let $r \in Q$ be given and let $h \in \mathcal{R}$ be such that $0 < |h| \ll d^r$ and $x + h \in I(a,b)$. Then*

$$f'(x) =_{r+\lambda(M_{1,x})} \frac{f(x+h) - f(x)}{h};$$

which means that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \ll M_{1,x} d^r.$$

Proof. Since $F_{1,x}$ is continuous on $I(a,b)$, we have using Lemma 5.7 that

$$F_{1,x}(x) =_{r+\lambda(M_{1,x})} F_{1,x}(x+h),$$

where

$$F_{1,x}(x) = f'(x) \text{ and } F_{1,x}(x+h) = \frac{f(x+h) - f(x)}{h}.$$

This finishes the proof of the theorem.

Theorem 5.12 (Remainder Formula 1) *Let $a < b$ be given in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$ be differentiable on $I(a,b)$. Let $x \in I(a,b)$ be given, let $F_{1,x}$ be the* *derivate function of f at x, and let $M_{1,x}$ be a Lipschitz constant of $F_{1,x}$. Then for all $y \in I(a,b)$, we have that*

$$f(y) = f(x) + f'(x)(y-x) + r_1(x,y)(y-x)^2, \text{ with } \lambda(r_1(x,y)) \geq \lambda(M_{1,x}).$$

Proof. Since $F_{1,x}$ is continuous on $I(a,b)$, we have by Lemma 5.8 that

$$F_{1,x}(y) = F_{1,x}(x) + r_1(x,y)(y-x), \text{ with } \lambda(r_1(x,y)) \geq \lambda(M_{1,x}).$$

Thus,

$$\frac{f(y)-f(x)}{y-x} = f'(x) + r_1(x,y)(y-x);$$

and hence

$$f(y) = f(x) + f'(x)(y-x) + r_1(x,y)(y-x)^2.$$

Theorem 5.13 *Let $a < b$ be given in $\mathcal{R}$, let $f, g : I(a,b) \to \mathcal{R}$ be differentiable on $I(a,b)$, and let $\alpha \in \mathcal{R}$. Then $f + \alpha g$ and $f \cdot g$ are differentiable on $I(a,b)$, with derivatives*

$$(f+\alpha g)' = f' + \alpha g' \text{ and } (f \cdot g)' = f' \cdot g + f \cdot g'.$$

Proof. Let $x \in I(a,b)$ be given, and let $F_{1,x}$ and $G_{1,x}$ denote the derivate functions of f and g at x, respectively. Then $F_{1,x}$ and $G_{1,x}$ are continuous on $I(a,b)$. It follows by Theorem 5.8 that the function $F_{1,x} + \alpha G_{1,x} : I(a,b) \to \mathcal{R}$, given by

$$\begin{aligned}
(F_{1,x} + \alpha G_{1,x})(y) &= F_{1,x}(y) + \alpha G_{1,x}(y) \\
&= \begin{cases} \frac{f(y)+\alpha g(y)-f(x)-\alpha g(x)}{y-x} & \text{if } y \neq x \\ f'(x) + \alpha g'(x) & \text{if } y = x \end{cases} \\
&= \begin{cases} \frac{(f+\alpha g)(y)-(f+\alpha g)(x)}{y-x} & \text{if } y \neq x \\ (f'+\alpha g')(x) & \text{if } y = x \end{cases},
\end{aligned}$$

is continuous on $I(a,b)$. This is true for all $x \in I(a,b)$. Hence $f+\alpha g$ is differentiable on $I(a,b)$ with derivative $(f+\alpha g)' = f' + \alpha g'$.

Now let $x \in I(a,b)$ be given, and let $H_x : D \to \mathcal{R}$ be given by

$$H_x(y) = \begin{cases} \frac{(f\cdot g)(y)-(f\cdot g)(x)}{y-x} & \text{if } y \neq x \\ f'(x)\,g(x) + f(x)\,g'(x) & \text{if } y = x \end{cases}.$$

We show that H_x is continuous on $I(a,b)$ for all $x \in I(a,b)$. Note that

$$\begin{aligned} H_x(y) &= \begin{cases} \frac{f(y)g(y)-f(x)g(x)}{y-x} & \text{if } y \neq x \\ f'(x)\,g(x) + f(x)\,g'(x) & \text{if } y = x \end{cases} \\ &= \begin{cases} \frac{f(y)-f(x)}{y-x} g(x) + f(y)\,\frac{g(y)-g(x)}{y-x} & \text{if } y \neq x \\ f'(x)\,g(x) + f(x)\,g'(x) & \text{if } y = x \end{cases} \\ &= F_{1,x}(y)\,g(x) + f(y)\,G_{1,x}(y). \end{aligned}$$

Hence

$$H_x = g(x)\,F_{1,x} + f \cdot G_{1,x}.$$

Since f, $F_{1,x}$ and $G_{1,x}$ are continuous on $I(a,b)$, so is H_x by Theorem 5.8. This is true for all $x \in I(a,b)$, and hence $(f \cdot g)$ is differentiable on $I(a,b)$, with derivative

$$(f\cdot g)'(x) = f'(x)\,g(x) + f(x)\,g'(x) \text{ for all } x \in I(a,b).$$

Theorem 5.14 (Chain Rule) *Let $a < b$ and $c < e$ in $\mathcal{R}$, and let $f : I_1(a,b) \to \mathcal{R}$ and $g : I_2(c,e) \to \mathcal{R}$ be such that $f(I_1(a,b)) \subset I_2(c,e)$, f is differentiable on $I_1(a,b)$ and g differentiable on $I_2(c,e)$. Then $g \circ f$ is differentiable on $I_1(a,b)$, with derivative*

$$(g \circ f)' = (g' \circ f) \cdot f'.$$

Let $x \in I(a,b)$ be given, and let $H_x : D \to \mathcal{R}$ be given by

$$H_x(y) = \begin{cases} \frac{(g\circ f)(y)-(g\circ f)(x)}{y-x} & \text{if } y \neq x \\ \\ g'(f(x))\, f'(x) & \text{if } x = x_0 \end{cases}.$$

Then

$$\begin{aligned} H_x(y) &= \begin{cases} \frac{g(f(y))-g(f(x))}{y-x} & \text{if } y \neq x \\ \\ g'(f(x))\, f'(x) & \text{if } y = x \end{cases} \\ &= \begin{cases} \frac{g(f(y))-g(f(x))}{f(y)-f(x)} \frac{f(y)-f(x)}{y-x} & \text{if } y \neq x \text{ and } f(y) \neq f(x) \\ 0 & \text{if } y \neq x \text{ and } f(y) = f(x) \\ g'(f(x))\, f'(x) & \text{if } y = x \end{cases} \\ &= G_{1,f(x)}(f(y))\, F_{1,x}(y), \end{aligned}$$

where $F_{1,x}$ is the derivate function of f at x, and $G_{1,f(x)}$ the derivate function of g at $f(x)$. Hence

$$H_x = \left(G_{1,f(x)} \circ f\right) \cdot F_{1,x}.$$

Since f is continuous on $I_1(a,b)$ and since $G_{1,f(x)}$ is continuous on $I_2(c,e)$, we have by Theorem 5.9 that $\left(G_{1,f(x)} \circ f\right)$ is continuous on $I_1(a,b)$. Since $F_{1,x}$ is continuous on $I_1(a,b)$, so is $\left(G_{1,f(x)} \circ f\right) \cdot F_{1,x} = H_x$ by Theorem 5.8. Hence $(g \circ f)$ is differentiable on $I_1(a,b)$, with derivative

$$\begin{aligned} (g \circ f)'(x) &= H_x(x) = g'(f(x))\, f'(x) \\ &= ((g' \circ f) \cdot f')(x) \text{ for all } x \in I_1(a,b). \end{aligned}$$

The following result provides a useful tool for checking the differentiability of functions and will be used frequently later, as in the proofs of Theorem 5.20 and Theorem 5.23 and in Example 5.10.

Theorem 5.15 *Let $a < b$ be given in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$ be continuous on $I(a,b)$. Suppose there exists $M \in \mathcal{R}$ and there exists a function $g : I(a,b) \to \mathcal{R}$ such that*

$$\left| \frac{f(y) - f(x)}{y - x} - g(x) \right| \leq M |y - x| \text{ for all } y \neq x \text{ in } I(a,b).$$

Then f is differentiable on $I(a,b)$, with derivative $f' = g$.

Proof. We need to show that for all $x \in I(a,b)$, the function $F_{1,x} : I(a,b) \to \mathcal{R}$, given by

$$F_{1,x}(y) = \begin{cases} \frac{f(y)-f(x)}{y-x} & \text{if } y \neq x \\ g(x) & \text{if } y = x \end{cases},$$

is continuous on $I(a,b)$. It is sufficient to show that for all $x \in I(a,b)$, we have that

$$\left| \frac{F_{1,x}(y) - F_{1,x}(z)}{y - z} \right| \leq d^{-1} M \text{ for all } y \neq z \text{ in } I(a,b).$$

So let $x \in I(a,b)$ be given; and let $y \neq z$ be given in $I(a,b)$. Four cases are to be considered.

As a first case, assume that $y = x$. Then

$$\begin{aligned} \left| \frac{F_{1,x}(y) - F_{1,x}(z)}{y - z} \right| &= \left| \frac{F_{1,x}(x) - F_{1,x}(z)}{x - z} \right| = \left| \frac{g(x) - \frac{f(z)-f(x)}{z-x}}{x - z} \right| \\ &= \frac{\left| \frac{f(z)-f(x)}{z-x} - g(x) \right|}{|z - x|} \\ &\leq \frac{M|z - x|}{|z - x|} = M \\ &\leq d^{-1} M. \end{aligned}$$

As a second case, assume that $z = x$. Then

$$\left| \frac{F_{1,x}(y) - F_{1,x}(z)}{y - z} \right| = \left| \frac{F_{1,x}(y) - F_{1,x}(x)}{y - x} \right| = \left| \frac{\frac{f(y)-f(x)}{y-x} - g(x)}{y - x} \right|$$

$$\begin{aligned}
&= \frac{\left|\frac{f(y)-f(x)}{y-x} - g(x)\right|}{|y-x|} \\
&\le \frac{M|y-x|}{|y-x|} = M \\
&\le d^{-1}M.
\end{aligned}$$

As a third case, assume that $y \neq x \neq z$ and $|y-z|$ is not infinitely smaller than $|y-x|$. Then $|y-z|$ is not infinitely smaller than $|z-x|$; for if $|y-z| \ll |z-x|$, then $|y-x| = |y-z+(z-x)| \approx |z-x| \gg |y-z|$, a contradiction. Thus,

$$\begin{aligned}
|F_{1,x}(y) - F_{1,x}(z)| &= \left|\frac{f(y)-f(x)}{y-x} - \frac{f(z)-f(x)}{z-x}\right| \\
&= \left|\left(\frac{f(y)-f(x)}{y-x} - g(x)\right) - \left(\frac{f(z)-f(x)}{z-x} - g(x)\right)\right| \\
&\le \left|\frac{f(y)-f(x)}{y-x} - g(x)\right| + \left|\frac{f(z)-f(x)}{z-x} - g(x)\right| \\
&\le M|y-x| + M|z-x| \\
&\le d^{-1}M|y-z| \text{ since } d^{-1}|y-z| \gg |y-x| + |z-x|.
\end{aligned}$$

Hence

$$\left|\frac{F_{1,x}(y) - F_{1,x}(z)}{y-z}\right| \le d^{-1}M.$$

Finally, assume that $y \neq x \neq z$ and $|y-z| \ll |y-x|$. Then

$$z - x = z - y + (y - x) \approx y - x.$$

Thus

$$\begin{aligned}
|F_{1,x}(y) - F_{1,x}(z)| &= \left|\frac{f(y)-f(x)}{y-x} - \frac{f(z)-f(x)}{z-x}\right| \\
&= \left|\frac{f(y)-f(x)}{y-x} - \frac{f(z)-f(x)}{y-x}\frac{y-x}{z-x}\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left|\frac{f(y)-f(x)}{y-x} - \frac{f(z)-f(x)}{y-x}\left(1+\frac{y-z}{z-x}\right)\right| \\
&= \left|\frac{f(y)-f(z)}{y-x} - \frac{f(z)-f(x)}{y-x}\frac{y-z}{z-x}\right| \\
&= \left|\frac{f(y)-f(z)}{y-z}\frac{y-z}{y-x} - \frac{f(z)-f(x)}{z-x}\frac{y-z}{y-x}\right| \\
&= \left|\frac{y-z}{y-x}\right|\left|\frac{f(y)-f(z)}{y-z} - \frac{f(z)-f(x)}{z-x}\right|.
\end{aligned}
$$

By hypothesis, we have that

$$\frac{f(y)-f(z)}{y-z} = g(z) + r_1 \text{ and } \frac{f(z)-f(x)}{z-x} = g(z) + r_2;$$

where

$$|r_1| \leq M|y-z| \text{ and } |r_2| \leq M|z-x|.$$

Thus,

$$
\begin{aligned}
|F_{1,x}(y) - F_{1,x}(z)| &= \left|\frac{y-z}{y-x}\right| |r_1 - r_2| \\
&\leq \left|\frac{y-z}{y-x}\right| (|r_1| + |r_2|) \\
&\leq \left|\frac{y-z}{y-x}\right| M(|y-z| + |z-x|).
\end{aligned}
$$

Since $|z-x| \approx |y-x|$, we obtain that $|z-x| < d^{-1}|y-x|/2$. Also, since $|y-z| \ll |y-x|$, we obtain that $|y-z| < |y-x| < d^{-1}|y-x|/2$. Therefore,

$$|F_{1,x}(y) - F_{1,x}(z)| \leq \left|\frac{y-z}{y-x}\right| M d^{-1}|y-x| = d^{-1}M|y-z|;$$

and hence

$$\left|\frac{F_{1,x}(y) - F_{1,x}(z)}{y-z}\right| \leq d^{-1}M.$$

This finishes the proof of the theorem.

Remark 5.1 *The proof of Theorem 5.15 is yet another example of how the non-Archimedean properties of $\mathcal{R}$ allow us to obtain results that would not hold in R or in any other Archimedean structure. In the third and fourth cases above, the existence of an infinitely large number (d^{-1} in our proof) was essential to get the final inequality. Indeed we could replace d^{-1} in the proof of Theorem 5.15 by any positive infinitely large number without having to change anything else in the proof.*

5.3 n-times Differentiability

Definition 5.8 *Let $a < b$ be given in $\mathcal{R}$, and let $f : I(a,b) \to \mathcal{R}$. Let $n \geq 2$ be given in Z^+. Then we define n-times differentiability of f on $I(a,b)$ inductively as follows: Having defined $(n-1)$-times differentiability, we say that f is n-times differentiable on $I(a,b)$ if and only if f is $(n-1)$-times differentiable on $I(a,b)$ and for all $x \in I(a,b)$, the $(n-1)$st derivate function $F_{n-1,x}$ is differentiable on $I(a,b)$. For all $x \in I(a,b)$, the number*

$$f^{(n)}(x) = n! F'_{n-1,x}(x)$$

will be called the nth derivative of f at x and the derivate function $F_{n,x}$ of $F_{n-1,x}$ at x will be called the nth derivate function of f at x [10].

In connection with the derivate functions, we introduce the secants of different orders.

Definition 5.9 *Let $a < b$ be given in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$. Then for all $x \in I(a,b)$, the function $S_{1,x} : I(a,b) \setminus \{x\} \to \mathcal{R}$, given by*

$$S_{1,x}(y) = \frac{f(y) - f(x)}{y - x},$$

will be called the first secant of f at x.

Definition 5.10 *Let $a < b$ be given in $\mathcal{R}$, let $n \in Z^+$ be given and let $f : I(a,b) \to \mathcal{R}$ be n-times differentiable on $I(a,b)$. Let $x \in I(a,b)$ be given, and let $F_{1,x}, \ldots, F_{n,x}$ denote the first,…, the nth derivate functions of f at x. For all $l \in \{2, \ldots, n+1\}$, define $S_{l,x} : I(a,b) \setminus \{x\} \to \mathcal{R}$ to be the first secant of $F_{l-1,x}$ at x. Then $S_{l,x}$ will be called the lth secant of f at x.*

Lemma 5.10 *Let $n \in Z^+$ be given, let $a < b$ be given in $\mathcal{R}$, and let $f : I(a,b) \to \mathcal{R}$ be n-times differentiable on $I(a,b)$. Let $f', \ldots, f^{(n)}$ denote the first,…,nth derivative functions of f on $I(a,b)$, and for all $x \in I(a,b)$, let $F_{1,x}, \ldots, F_{n,x}$ denote the first,…,nth derivate functions of f at x. Then for all $x, y \in I(a,b)$, we have [10] that*

$$\begin{aligned} f(y) &= f(x) + F_{1,x}(y)(y-x) \\ &= f(x) + f'(x)(y-x) + F_{2,x}(y)(y-x)^2 \\ &\vdots \\ &= f(x) + \sum_{j=1}^{n-1} \frac{f^{(j)}(x)}{j!}(y-x)^j + F_{n,x}(y)(y-x)^n . \end{aligned}$$

Proof. By induction on n. The assertion is true for $n = 1$. Suppose it is true for $n = l$ and show it is true for $n = l+1$. So let f be $(l+1)$-times differentiable on $I(a,b)$. Since f is l-times differentiable on $I(a,b)$, we have by the induction hypothesis that

$$f(y) = f(x) + \sum_{j=1}^{l-1} \frac{f^{(j)}(x)}{j!}(y-x)^j + F_{l,x}(y)(y-x)^l \text{ for all } x, y \in I(a,b). \quad (5.2)$$

Since f is $(l+1)$-times differentiable on $I(a,b)$, we have that $F_{l,x}$ is differentiable on $I(a,b)$, with derivative $F'_{l,x}(x) = f^{(l+1)}(x)/(l+1)!$ and with derivate function $F_{l+1,x}$, the $(l+1)$st derivate function of f at x. Thus,

$$\begin{aligned} F_{l,x}(y) &= F_{l,x}(x) + F_{l+1,x}(y)(y-x) \\ &= F'_{l-1,x}(x) + F_{l+1,x}(y)(y-x) \\ &= \frac{f^{(l)}(x)}{l!} + F_{l+1,x}(y)(y-x) \text{ for all } x, y \in I(a,b). \end{aligned} \quad (5.3)$$

Substituting Equation (5.3) into Equation (5.2) yields

$$f(y) = f(x) + \sum_{j=1}^{l} \frac{f^{(j)}(x)}{j!}(y-x)^j + F_{l+1,x}(y)(y-x)^{l+1} \text{ for all } x, y \in I(a,b).$$

So the assertion is true for $n = l+1$; and hence it is true for all $n \in Z^+$.

Corollary 5.7 *Let $a < b$ be given in $\mathcal{R}$, let $n \in Z^+$ be given and let $f : I(a,b) \to \mathcal{R}$ be n-times differentiable on $I(a,b)$. Then for all $l \in \{2, \ldots, n+1\}$ and for all $y \neq x$ in $I(a,b)$, we have that*

$$S_{l,x}(y) = \frac{f(y) - \sum_{j=0}^{l-1} \frac{f^{(j)}(x)}{j!}(y-x)^j}{(y-x)^l}.$$

Proof. Let $y \neq x$ in $I(a,b)$ and $l \in \{2, \ldots, n+1\}$ be given. Then, using Lemma 5.10, we obtain that

$$\begin{aligned} S_{l,x}(y) &= \frac{F_{l-1,x}(y) - F_{l-1,x}(x)}{y-x} \\ &= \frac{1}{y-x}\left(\frac{f(y) - \sum_{j=0}^{l-2} \frac{f^{(j)}(x)}{j!}(y-x)^j}{(y-x)^{l-1}} - \frac{f^{(l-1)}(x)}{(l-1)!}\right) \\ &= \frac{f(y) - \sum_{j=0}^{l-1} \frac{f^{(j)}(x)}{j!}(y-x)^j}{(y-x)^l}. \end{aligned}$$

Corollary 5.8 (Remainder Formula n) *Let $a < b$ be given in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$ be n-times differentiable on $I(a,b)$. Let $x \in I(a,b)$ be given, let $F_{n,x}$ be the nth order derivate function of f at x, and let $M_{n,x}$ be a Lipschitz constant of $F_{n,x}$. Then for all $y \in I(a,b)$, we have that*

$$f(y) = f(x) + \sum_{j=1}^{n} \frac{f^{(j)}(x)}{j!}(y-x)^j + r_n(x,y)(y-x)^{n+1}, \text{ with } \lambda(r_n(x,y)) \geq \lambda(M_{n,x}).$$

Proof. If $y = x$, there is nothing to prove; so we may assume that $y \neq x$. Since $F_{n,x}$ is continuous on $I(a,b)$, we have by Lemma 5.8 that

$$F_{n,x}(y) = F_{n,x}(x) + r_n(x,y)(y-x), \text{ with } \lambda(r_n(x,y)) \geq \lambda(M_{n,x}).$$

Using Lemma 5.10, we have that

$$F_{n,x}(y) = \frac{f(y) - f(x) - \sum_{j=1}^{n-1} \frac{f^{(j)}(x)}{j!}(y-x)^j}{(y-x)^n}.$$

Also, from Definition 5.8, we obtain that

$$F_{n,x}(x) = F'_{n-1,x}(x) = \frac{f^{(n)}(x)}{n!}.$$

Thus,

$$\frac{f(y) - f(x) - \sum_{j=1}^{n-1} \frac{f^{(j)}(x)}{j!}(y-x)^j}{(y-x)^n} = \frac{f^{(n)}(x)}{n!} + r_n(x,y)(y-x);$$

and hence

$$f(y) = f(x) + \sum_{j=1}^{n} \frac{f^{(j)}(x)}{j!}(y-x)^j + r_n(x,y)(y-x)^{n+1}.$$

Theorem 5.16 *Let $n \in Z^+$ be given; let $a < b$ be given in $\mathcal{R}$; let $f, g : I(a,b) \to \mathcal{R}$ be n-times differentiable on $I(a,b)$, with derivatives $f', \ldots, f^{(n)}$ and $g', \ldots, g^{(n)}$, respectively; and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ is n-times differentiable on $I(a,b)$, with derivatives*

$$(f + \alpha g)^{(l)} = f^{(l)} + \alpha g^{(l)} \textit{ for all } l \in \{1, \ldots, n\}.$$

Proof. By induction on n. The assertion is true for $n = 1$ by Theorem 5.13. Suppose it is true for $n = m$ and show it is true for $n = m + 1$. So we have that f and g are $(m+1)$-times differentiable on $I(a,b)$. By the induction hypothesis, we have that $(f + \alpha g)$ is m-times equidifferentiable on $I(a,b)$ with derivatives

$$(f + \alpha g)^{(l)} = f^{(l)} + \alpha g^{(l)} \text{ for all } l \in \{1, \ldots, m\}. \tag{5.4}$$

Now let $x \in I(a,b)$ be given. Since f and g are $(m+1)$-times differentiable on $I(a,b)$, we have that the $(m+1)$st derivate functions of f and g at x, $F_{m+1,x}, G_{m+1,x}$:

$I(a,b) \to \mathcal{R}$, given by

$$F_{m+1,x}(y) = \begin{cases} \frac{f(y)-f(x)-\sum_{l=1}^{m} \frac{f^{(l)}(x)}{l!}(y-x)^l}{(y-x)^{m+1}} & \text{if } y \neq x \\ \frac{f^{(m+1)}(x)}{(m+1)!} & \text{if } y = x \end{cases}$$

and

$$G_{m+1,x}(y) = \begin{cases} \frac{g(y)-g(x)-\sum_{l=1}^{m} \frac{g^{(l)}(x)}{l!}(y-x)^l}{(y-x)^{m+1}} & \text{if } y \neq x \\ \frac{g^{(m+1)}(x)}{(m+1)!} & \text{if } y = x \end{cases},$$

are continuous on $I(a,b)$. By Theorem 5.8, we have that $F_{m+1,x}+\alpha G_{m+1,x} : I(a,b) \to \mathcal{R}$, given by

$$(F_{m+1,x} + \alpha G_{m+1,x})(y) = F_{m+1,x}(y) + \alpha G_{m+1,x}(y)$$

$$= \begin{cases} \frac{f(y)+\alpha g(y)-(f(x)+\alpha g(x))-\sum_{l=1}^{m} \frac{f^{(l)}(x)+\alpha g^{(l)}(x)}{l!}(y-x)^l}{(y-x)^{m+1}} & \text{if } y \neq x \\ \frac{f^{(m+1)}(x)+\alpha g^{(m+1)}(x)}{(m+1)!} & \text{if } y = x \end{cases}$$

$$= \begin{cases} \frac{(f+\alpha g)(y)-(f+\alpha g)(x)-\sum_{l=1}^{m} \frac{(f+\alpha g)^{(l)}(x)}{l!}(y-x)^l}{(y-x)^{m+1}} & \text{if } y \neq x \\ \frac{f^{(m+1)}(x)+\alpha g^{(m+1)}(x)}{(m+1)!} & \text{if } y = x \end{cases},$$

is continuous on $I(a,b)$, where use has been made of Equation (5.4). Hence $f + \alpha g$ is $(m+1)$-times differentiable on $I(a,b)$, with $(m+1)$st derivative

$$(f+\alpha g)^{(m+1)}(x) = f^{(m+1)}(x) + \alpha g^{(m+1)}(x) \text{ for all } x \in I(a,b).$$

So the assertion is true for $n = m+1$, and hence it is true for all $n \in Z^+$.

Definition 5.11 *Let $a < b$ be given in $\mathcal{R}$, and let $f : I(a,b) \to \mathcal{R}$. Then we say that f is infinitely often differentiable on $I(a,b)$ if and only if for all $n \in Z^+$, f is n-times differentiable on $I(a,b)$.*

Using Theorem 4.11, we obtain the following result.

Theorem 5.17 *Let $a < b$ be given in $\mathcal{R}$, and let $f : I(a,b) \to \mathcal{R}$ be infinitely often differentiable on $I(a,b)$. Let $x_0 \in I(a,b)$ be given, and let*

$$\lambda_0 = \limsup_{n\to\infty}\left(\frac{-\lambda\left(f^{(n)}(x_0)\right)}{n}\right).$$

Then $\sum_{n=0}^{\infty} f^{(n)}(x_0)/(n!)(x-x_0)^n$ converges strongly if $\lambda(x-x_0) > \lambda_0$, and it is strongly divergent if $\lambda(x-x_0) < \lambda_0$.

Using Theorem 4.12, we obtain:

Theorem 5.18 *Let $a < b$ be given in $\mathcal{R}$, and let $f : I(a,b) \to \mathcal{R}$ be infinitely often differentiable on $I(a,b)$. Let $x_0 \in I(a,b)$ be given, and let*

$$\lambda_0 = \limsup_{n\to\infty}\left(\frac{-\lambda\left(f^{(n)}(x_0)\right)}{n}\right).$$

Let $x \in I(a,b)$ be such that $\lambda(x-x_0) = \lambda_0$. For all $n \geq 0$, let

$$b_n = d^{n\lambda_0}\frac{f^{(n)}(x_0)}{n!}.$$

Suppose that the sequence (b_n) is regular; and write $\cup_{n=0}^{\infty} supp(b_n) = \{q_1, q_2, \ldots\}$ with $q_{j_1} < q_{j_2}$ if $j_1 < j_2$. For all $n \geq 0$, write $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$ where $b_{n_j} = b_n[q_j]$; and let

$$r = \frac{1}{\sup\left\{\limsup_{n\to\infty}|b_{n_j}|^{1/n} : j \geq 1\right\}}.$$

Then $\sum_{n=0}^{\infty} f^{(n)}(x_0)/(n!)(x-x_0)^n$ converges weakly if $|(x-x_0)[\lambda_0]| < r$, and it is weakly divergent if $|(x-x_0)[\lambda_0]| > r$.

The following two examples show that, even when $\sum_{n=0}^{\infty} f^{(n)}(x_0)/(n!)(x-x_0)^n$ converges, the series need not converge to $f(x)$. It converges to $f(x)$ only if the remainder term $r_n(x_0,x)(x-x_0)^{n+1}$ converges to 0; and Theorem 5.19 below provides a criterion for that.

Example 5.8 *Let $f:[-1,1]\to\mathcal{R}$ be given by*

$$f(x)=\begin{cases}\exp\left(-1/x^2\right) & \text{if } x\sim 1\\ 0 & \text{if } 0\le |x|\ll 1\end{cases}.$$

Then f is infinitely often differentiable on $[-1,1]$, and we have that

$$f^{(n)}(0)=0 \text{ for all } n\ge 1.$$

Thus,

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \text{ converges strongly to 0 for all } x\in[-1,1].$$

Hence the limit is equal to $f(x)$ if $0\le|x|\ll 1$ and is different from $f(x)$ if $x\sim 1$.

Example 5.9 *Let $f:[0,1]\to\mathcal{R}$ be given by*

$$f(x)=\begin{cases}0 & \text{if } x=0\\ \sum_{j=1}^{n}\frac{d^{-j}x^j}{j!} & \text{if } n\ge 1 \text{ and } n-1\le\lambda(x)<n\end{cases}.$$

Then f is infinitely often differentiable on $[0,1]$ with derivatives at 0 given by

$$f^{(l)}(0)=d^{-l} \text{ for all } l\ge 1.$$

First we show that f is differentiable on $[0,1]$ with derivative

$$f'(x)=g_1(x)=\begin{cases}d^{-1} & \text{if } x=0\\ \sum_{j=1}^{n}\frac{d^{-j}x^{j-1}}{(j-1)!} & \text{if } n\ge 1 \text{ and } n-1\le\lambda(x)<n\end{cases}.$$

We will show that

$$\left|\frac{f(y)-f(x)}{y-x}-g_1(x)\right|<d^{-11/4}\,|y-x| \text{ for all } x\neq y \text{ in } [0,1]\,.$$

So let $x\neq y$ in $[0,1]$ be given. Five cases are to be considered.

First case: $x=0<y$. In this case, $f(x)=f(0)=0$ and $g_1(x)=g_1(0)=d^{-1}$; so

$$\begin{aligned}
\left|\frac{f(y)-f(x)}{y-x}-g_1(x)\right| &= \left|\frac{f(y)}{y}-d^{-1}\right| \\
&= \begin{cases} 0 & \text{if } 0\leq\lambda(y)<1 \\ \sum_{j=2}^{m}\frac{d^{-j}y^{j-1}}{j!} & \text{if } m\geq 2 \text{ and } m-1\leq\lambda(y)<m \end{cases} \\
&= y\cdot\begin{cases} 0 & \text{if } 0\leq\lambda(y)<1 \\ \sum_{j=2}^{m}\frac{d^{-j}y^{j-2}}{j!} & \text{if } m\geq 2 \text{ and } m-1\leq\lambda(y)<m \end{cases}.
\end{aligned}$$

Note that for all $m\geq 2$ and for $m-1\leq\lambda(y)<m$, we have that

$$\lambda\left(d^{-j}y^{j-2}\right)=-2+(j-2)\,\lambda\left(d^{-1}y\right)\geq -2 \text{ for all } j\in\{2,\ldots,m\}\,.$$

Thus,

$$\lambda\left(\sum_{j=2}^{m}\frac{d^{-j}y^{j-2}}{j!}\right)\geq -2 \text{ for all } m\geq 2;$$

and hence

$$\left|\frac{f(y)-f(x)}{y-x}-g_1(x)\right|<d^{-11/4}y=d^{-11/4}\,|y-x|\,.$$

Second case: $0=y<x$. In this case, we have that

$$\left|\frac{f(y)-f(x)}{y-x}-g_1(x)\right|=\left|\frac{f(x)}{x}-g_1(x)\right|$$

$$\begin{aligned}
&= -\begin{cases} 0 & \text{if } 0\leq\lambda(x)<1 \\ \sum_{j=2}^{n}\frac{d^{-j}x^{j-1}}{j!}-\sum_{j=2}^{n}\frac{d^{-j}x^{j-1}}{(j-1)!} & \text{if } n\geq 2 \text{ and } n-1\leq\lambda(x)<n \end{cases} \\
&= x\cdot\begin{cases} 0 & \text{if } 0\leq\lambda(x)<1 \\ \sum_{j=2}^{n}\left(\frac{1}{(j-1)!}-\frac{1}{j!}\right)d^{-j}x^{j-2} & \text{if } n\geq 2 \text{ and } n-1\leq\lambda(x)<n \end{cases}.
\end{aligned}$$

For all $n \geq 2$ and for $n-1 \leq \lambda(x) < n$, we have that

$$\lambda\left(d^{-j}x^{j-2}\right) = -2 + (j-2)\,\lambda\left(d^{-1}x\right) \geq -2 \text{ for all } j \in \{2,\ldots,n\}.$$

Thus,

$$\lambda\left(\sum_{j=2}^{n}\left(\frac{1}{(j-1)!} - \frac{1}{j!}\right) d^{-j}x^{j-2}\right) \geq -2 \text{ for all } n \geq 2;$$

and hence

$$\left|\frac{f(y)-f(x)}{y-x} - g_1(x)\right| < d^{-11/4}x = d^{-11/4}\,|y-x|\,.$$

Third case: $x > 0$, $y > 0$, and $n-1 \leq \lambda(x), \lambda(y) < n$; for $n \geq 1$. In this case, we have that

$$\begin{aligned}
&\left|\frac{f(y)-f(x)}{y-x} - g_1(x)\right| \\
= &\left|\begin{cases} 0 & \text{if } n = 1 \\ \sum_{j=2}^{n}\left(\frac{d^{-j}}{j!}\frac{y^j-x^j}{y-x}\right) - \sum_{j=2}^{n}\frac{d^{-j}x^{j-1}}{(j-1)!} & \text{if } n \geq 2 \end{cases}\right| \\
= &\ d^{-11/4}\,|y-x| \cdot \left|\begin{cases} 0 & \text{if } n = 1 \\ \frac{d^{11/4}}{y-x}\sum_{j=2}^{n}\frac{d^{-j}}{j!}\left(\frac{y^j-x^j}{y-x} - jx^{j-1}\right) & \text{if } n \geq 2 \end{cases}\right|.
\end{aligned}$$

So it remains to show that

$$\left|\frac{d^{11/4}}{y-x}\sum_{j=2}^{n}\frac{d^{-j}}{j!}\left(\frac{y^j-x^j}{y-x} - jx^{j-1}\right)\right| < 1 \text{ for all } n \geq 2.$$

We have that

$$\begin{aligned}
\frac{y^j-x^j}{y-x} - jx^{j-1} &= y^{j-1} + y^{j-2}x + \cdots + yx^{j-2} + x^{j-1} - jx^{j-1} \\
&= y^{j-1} + y^{j-2}x + \cdots + yx^{j-2} - (j-1)x^{j-1} \\
&= \left(y^{j-1} - x^{j-1}\right) + x\left(y^{j-2} - x^{j-2}\right) + \cdots + x^{j-2}(y-x)\,.
\end{aligned}$$

Thus, for all $j \in \{2,\ldots,n\}$, we have that

$$\lambda\left(\frac{d^{11/4}}{y-x}\frac{d^{-j}}{j!}\left(\frac{y^j-x^j}{y-x} - jx^{j-1}\right)\right)$$

$$
\begin{aligned}
&= \frac{11}{4} - j + \lambda \left(\frac{\left(y^{j-1} - x^{j-1}\right) + x\left(y^{j-2} - x^{j-2}\right) + \cdots + x^{j-2}\left(y - x\right)}{y - x} \right) \\
&= \frac{11}{4} - j + \lambda \left(\left(y^{j-2} + \cdots + x^{j-2} \right) + x \left(y^{j-3} + \cdots + x^{j-3} \right) + \cdots + x^{j-2} \right) \\
&= \frac{3}{4} + \lambda \left(\left(\left(d^{-1} y \right)^{j-2} + \cdots + \left(d^{-1} x \right)^{j-2} \right) + \cdots + \left(d^{-1} x \right)^{j-2} \right) \\
&\geq \frac{3}{4} \text{ since } \lambda(x) \geq 1 \text{ and } \lambda(y) \geq 1.
\end{aligned}
$$

Hence

$$\lambda \left(\frac{d^{11/4}}{y - x} \sum_{j=2}^{n} \frac{d^{-j}}{j!} \left(\frac{y^j - x^j}{y - x} - j x^{j-1} \right) \right) \geq \frac{3}{4} \text{ for all } n \geq 2.$$

Therefore,

$$\left| \frac{d^{11/4}}{y - x} \sum_{j=2}^{n} \frac{d^{-j}}{j!} \left(\frac{y^j - x^j}{y - x} - j x^{j-1} \right) \right| < 1 \text{ for all } n \geq 2;$$

and hence

$$\left| \frac{f(y) - f(x)}{y - x} - g_1(x) \right| < d^{-11/4} \left| y - x \right|.$$

Similarly, we can show that

$$\left| \frac{f(y) - f(x)}{y - x} - g_1(x) \right| < d^{-11/4} \left| y - x \right|$$

for the remaining two cases, namely the case when $x > 0$, $y > 0$, and $n - 1 \leq \lambda(x) < n \leq m - 1 \leq \lambda(y) < m$, for $n \geq 1$; and the case when $x > 0$, $y > 0$, and $m - 1 \leq \lambda(y) < m \leq n - 1 \leq \lambda(x) < n$, for $m \geq 1$. Thus, f is differentiable on $[0, 1]$, with derivative $f'(x) = g_1(x)$ for all $x \in [0, 1]$. In particular, we have that $f'(0) = g_1(0) = d^{-1}$.

Similarly, we can show for all $l \geq 2$ that f is l-times differentiable on $[0, 1]$, with l-th derivative

$$f^{(l)}(x) = \begin{cases} d^{-l} & \text{if } x = 0 \\ 0 & \text{if } 0 \leq \lambda(x) < l - 1 \\ \sum_{j=l}^{n} \frac{d^{-j} x^{j-l}}{(j-l)!} & \text{if } n \geq l \text{ and } n - 1 \leq \lambda(x) < n \end{cases}.$$

Since

$$\limsup_{l\to\infty}\left(\frac{-\lambda\left(f^{(l)}(0)\right)}{l}\right)=\limsup_{l\to\infty}\left(\frac{l}{l}\right)=1,$$

we obtain that

$$\sum_{l=0}^{\infty}\frac{f^{(l)}(0)}{l!}x^{l} \text{ converges strongly for } 0<x\ll d.$$

However, for no such x does $\sum_{l=0}^{\infty} f^{(l)}(0)/(l!)\,x^{l}$ converge to $f(x)$. This is so because, as we will show below, for $0<x\ll d$ the remainder term $r_l(0,x)\,x^{l+1}$ does not converge to 0 as l goes to ∞. So let $x\in[0,1]$ be such that $0<x\ll d$. There exists $m\geq 2$ in Z^{+} such that $m-1\leq\lambda(x)<m$. Then for all $l>\lambda(x)+2$, we have that

$$r_l(0,x)=\sum_{j=m+1}^{l}\frac{d^{-j}}{j!x^{l+1-j}};$$

so

$$r_l(0,x)\,x^{l+1}=\sum_{j=m+1}^{l}\frac{d^{-j}}{j!x^{-j}}=\sum_{j=m+1}^{l}\frac{\left(d^{-1}x\right)^{j}}{j!}\approx\frac{\left(d^{-1}x\right)^{m+1}}{(m+1)!}$$

since $0<d^{-1}x\ll 1$. Thus,

$$\lambda\left(r_l(0,x)\,x^{l+1}\right)=(m+1)\left(\lambda(x)-1\right)<(m+1)\,\lambda(x)<\infty \text{ for all } l>2\lambda(x)+1,$$

which entails that

$$\lim_{l\to\infty} r_l(0,x)\,x^{l+1}\neq 0.$$

As an example, let $x=d^{2}$; then

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{f^{(l)}(0)}{l!}x^{l} &= \sum_{l=1}^{\infty}\frac{d^{-l}}{l!}d^{2l}=\sum_{l=1}^{\infty}\frac{d^{l}}{l!}=\exp(d)-1\\ &\neq f\left(d^{2}\right)=d+\frac{d^{2}}{2!}+\frac{d^{3}}{3!}.\end{aligned}$$

Theorem 5.19 *Let $a<b$ be given in $\mathcal{R}$ and let $f:I(a,b)\to\mathcal{R}$ be infinitely often differentiable on $I(a,b)$. Let $x_0\in I(a,b)$ be given and for each $l\in Z^{+}$, let F_{l,x_0} denote the lth derivate function of f at x_0 [10]. For each $l\in Z^{+}$, let*

$$\alpha_l=\sup_{in\ \mathcal{R}}\{\lambda(M_l): M_l \text{ is a Lipschitz constant of } F_{l,x_0}\},$$

and let

$$\lambda_0 = \limsup_{l\to\infty}\left(\frac{-\alpha_l}{l}\right).$$

Then $\sum_{n=0}^{\infty} f^{(n)}(x_0)/(n!)(x-x_0)^n$ *converges strongly to* $f(x)$ *for all* $x \in I(a,b)$ *satisfying*

$$\lambda(x-x_0) > \lambda_0.$$

Proof. Let $x \in I(a,b)$ be such that $\lambda(x-x_0) > \lambda_0$ and let $l \in Z^+$ be given. By the remainder formula, Corollary 5.8, we have that

$$f(x) = \sum_{n=0}^{l} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + r_l(x_0,x)(x-x_0)^{l+1},$$

where

$$\lambda\left(r_l(x_0,x)\right) \geq \lambda(M_l)$$

for all Lipschitz constant M_l of F_{l,x_0}; and hence

$$\lambda\left(r_l(x_0,x)\right) \geq \alpha_l.$$

We need to show that

$$\lim_{l\to\infty}\left(r_l(x_0,x)(x-x_0)^{l+1}\right) = 0.$$

Since $\lambda(x-x_0) > \lambda_0$, there exists $t \in Q^+$ such that

$$\lambda(x-x_0) - t > \lambda_0.$$

Hence there exists $N \in Z^+$ such that

$$\lambda(x-x_0) - t > \frac{-\alpha_l}{l} \text{ for all } l \geq N.$$

Hence

$$\alpha_l + l\lambda(x-x_0) > lt \text{ for all } l \geq N.$$

Thus,

$$\lambda\left(r_l(x_0,x)(x-x_0)^l\right) > lt \text{ for all } l \geq N,$$

which entails that

$$\lim_{l\to\infty}\left(r_l(x_0,x)(x-x_0)^l\right) = 0.$$

Hence

$$\lim_{l\to\infty}\left(r_l(x_0,x)(x-x_0)^{l+1}\right) = (x-x_0)\lim_{l\to\infty}\left(r_l(x_0,x)(x-x_0)^l\right) = 0.$$

This finishes the proof of the theorem.

The following result is a generalization of the corresponding result about power series with real coefficients, which was proved in [5]; and the arguments in the proof are very similar to those in the proof of the previous result.

Theorem 5.20 *Let $x_0 \in \mathcal{R}$ be given, let (a_n) be a sequence in $\mathcal{R}$, let*

$$\lambda_0 = \limsup_{n\to\infty}\left\{\frac{-\lambda\left(a_n\right)}{n}\right\};$$

and for all $n \geq 0$ let $b_n = d^{n\lambda_0}a_n$. Suppose that the sequence (b_n) is regular; and write $\cup_{n=0}^{\infty} supp(b_n) = \{q_1, q_2, \ldots\}$ with $q_{j_1} < q_{j_2}$ if $j_1 < j_2$. For all $n \geq 0$, write $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$ where $b_{n_j} = b_n\left[q_j\right]$; and let

$$\eta = \frac{1}{\sup\left\{\limsup_{n\to\infty}|b_{n_j}|^{1/n} : j \geq 1\right\}} \text{ in } R. \tag{5.5}$$

Then, for all $\sigma \in R$ satisfying $0 < \sigma < \eta$, the function $f : \left[x_0 - \sigma d^{\lambda_0}, x_0 + \sigma d^{\lambda_0}\right] \to \mathcal{R}$, given by

$$f\left(x\right) = \sum_{n=0}^{\infty} a_n (x-x_0)^n,$$

is infinitely often differentiable on $\left[x_0 - \sigma d^{\lambda_0}, x_0 + \sigma d^{\lambda_0}\right]$, and the derivatives are given by

$$f^{(k)}\left(x\right) = g_k\left(x\right) = \sum_{n=k}^{\infty} n\left(n-1\right)\cdots\left(n-k+1\right) a_n \left(x-x_0\right)^{n-k}$$

for all $k \geq 1$ *and for all* $x \in \left[x_0 - \sigma d^{\lambda_0}, x_0 + \sigma d^{\lambda_0}\right]$. *In particular, we have that*

$$a_k = \frac{f^{(k)}(x_0)}{k!} \text{ for all } k = 0, 1, 2, \ldots;$$

and hence for all $x \in \left[x_0 - \sigma d^{\lambda_0}, x_0 + \sigma d^{\lambda_0}\right]$, *we have that*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n .$$

Proof. By a remark made at the beginning of the proof of Theorem 4.12, we may assume that $\lambda_0 = 0$, $b_n = a_n$ for all $n \geq 0$, and

$$\min\left(\cup_{n=0}^{\infty} \operatorname{supp}(a_n)\right) = 0.$$

Using induction on k, it suffices to show that the result is true for $k = 1$. Since $\lim_{n\to\infty} (n)^{1/n} = 1$ and $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges weakly for $x \in [x_0 - \sigma, x_0 + \sigma]$, we obtain that $\sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$ converges weakly for $x \in [x_0 - \sigma, x_0 + \sigma]$. Next we show that f is differentiable at x with derivative $f'(x) = g_1(x)$ for all $x \in [x_0 - \sigma, x_0 + \sigma]$; by Theorem 5.15, it suffices to show that

$$\left| \frac{f(x+h) - f(x)}{h} - g_1(x) \right| < d^{-1} |h|$$

for all $x \in [x_0 - \sigma, x_0 + \sigma]$ and for all $h \neq 0$ in $\mathcal{R}$ satisfying $x + h \in [x_0 - \sigma, x_0 + \sigma]$. So let $x \in [x_0 - \sigma, x_0 + \sigma]$ be given and let $h \neq 0$ in $\mathcal{R}$ be such that $x + h \in [x_0 - \sigma, x_0 + \sigma]$. First let $|h|$ be finite. Since $f(x)$, $f(x+h)$ and $g_1(x)$ are all at most finite in absolute value, we obtain that

$$\lambda\left(\frac{f(x+h) - f(x)}{h} - g_1(x) \right) \geq 0.$$

On the other hand, we have that

$$\lambda\left(d^{-1}|h|\right) = \lambda\left(d^{-1}\right) + \lambda(h) = -1 + 0 = -1.$$

Hence

$$\left|\frac{f(x+h)-f(x)}{h}-g_1(x)\right| \ll d^{-1}|h|.$$

Now let $|h|$ be infinitely small. Write $h = h_0 d^r (1+h_1)$ with $h_0 \in R$, $0 < r \in Q$ and $0 \leq |h_1| \ll 1$. Let $s \leq 2r$ be given. Since (a_n) is regular, there exist only finitely many elements in $[0,s] \cap \cup_{n=0}^{\infty}\mathrm{supp}(a_n)$; write

$$[0,s] \cap \cup_{n=0}^{\infty}\mathrm{supp}(a_n) = \{q_{1,s}, q_{2,s}, \ldots, q_{j,s}\}.$$

Thus,

$$\begin{aligned}
f(x+h)[s] &= \left(\sum_{n=0}^{\infty} a_n (x+h-x_0)^n\right)[s] \\
&= \sum_{n=0}^{\infty} \left(a_n (x+h-x_0)^n\right)[s] \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=1}^{j} a_n[q_{l,s}] (x+h-x_0)^n [s-q_{l,s}]\right) \\
&= \sum_{l=1}^{j} \left(\sum_{n=0}^{\infty} a_n[q_{l,s}] (x+h-x_0)^n [s-q_{l,s}]\right) \\
&= \sum_{l=1}^{j} \left(\sum_{n=0}^{\infty} a_n[q_{l,s}] \sum_{\nu=0}^{n} \left(\frac{n!}{\nu!(n-\nu)!} h^{\nu} (x-x_0)^{n-\nu}\right)[s-q_{l,s}]\right) \\
&= \sum_{l=1}^{j} \left(\begin{array}{c} \sum_{n=0}^{\infty} a_n[q_{l,s}] (x-x_0)^n [s-q_{l,s}] \\ +\sum_{n=1}^{\infty} n a_n[q_{l,s}] \left(h(x-x_0)^{n-1}\right)[s-q_{l,s}] \\ +\sum_{n=2}^{\infty} \frac{n(n-1)}{2} a_n[q_{l,s}] \left(h^2 (x-x_0)^{n-2}\right)[s-q_{l,s}] \end{array}\right).
\end{aligned}$$

Other terms are not relevant, since the corresponding powers of h are infinitely smaller than d^s in absolute value, and hence infinitely smaller than $d^{s-q_{l,s}}$ for all $l \in \{1, \ldots, j\}$. Thus

$$f(x+h)[s] = \sum_{n=0}^{\infty} \left(\sum_{l=1}^{j} a_n[q_{l,s}] (x-x_0)^n [s-q_{l,s}]\right)$$

$$+\sum_{n=1}^{\infty}\left(\sum_{l=1}^{j} n a_n\left[q_{l,s}\right]\left(h\left(x-x_0\right)^{n-1}\right)\left[s-q_{l,s}\right]\right)$$

$$+\sum_{n=2}^{\infty}\left(\sum_{l=1}^{j} \frac{n\left(n-1\right)}{2} a_n\left[q_{l,s}\right]\left(h^2\left(x-x_0\right)^{n-2}\right)\left[s-q_{l,s}\right]\right)$$

$$= \sum_{n=0}^{\infty}\left(a_n\left(x-x_0\right)^n\right)[s]+\sum_{n=1}^{\infty}\left(n h a_n\left(x-x_0\right)^{n-1}\right)[s]$$

$$+\sum_{n=2}^{\infty}\left(\frac{n\left(n-1\right)}{2} h^2 a_n\left(x-x_0\right)^{n-2}\right)[s].$$

Therefore, we obtain that

$$\frac{f\left(x+h\right)-f\left(x\right)}{h}-g_1\left(x\right)=_r h_0 d^r \sum_{n=2}^{\infty} \frac{n\left(n-1\right)}{2} a_n\left(x-x_0\right)^{n-2}. \tag{5.6}$$

Since $\lambda\left(a_n\right) \geq 0$ for all $n \geq 2$ and since $\lambda\left(x-x_0\right) \geq 0$, we obtain that

$$\lambda\left(\sum_{n=2}^{\infty} \frac{n\left(n-1\right)}{2} a_n\left(x-x_0\right)^{n-2}\right) \geq 0.$$

Thus, Equation (5.6) entails that

$$\lambda\left(\frac{f\left(x+h\right)-f\left(x\right)}{h}-g_1\left(x\right)\right) \geq r=\lambda\left(h\right)>\lambda\left(h\right)-1=\lambda\left(d^{-1}\left|h\right|\right);$$

and hence

$$\left|\frac{f\left(x+h\right)-f\left(x\right)}{h}-g_1\left(x\right)\right| \ll d^{-1}\left|h\right|.$$

This finishes the proof of the theorem.

Theorem 5.21 (Reexpansion of Power Series) *Let $x_0 \in \mathcal{R}$ be given, let $\left(a_n\right)$ be a regular sequence in $\mathcal{R}$, with*

$$\lambda_0=\limsup_{n \rightarrow \infty}\left\{\frac{-\lambda\left(a_n\right)}{n}\right\}=0;$$

and let $\eta \in R$ be the radius of weak convergence of $f\left(x\right)=\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n$, given by Equation (5.5). Let $y_0 \in \mathcal{R}$ be such that $\left|\Re\left(y_0-x_0\right)\right|<\eta$. Then, for *all* $x \in \mathcal{R}$ *satisfying* $|\Re(x-y_0)| < \eta - |\Re(y_0-x_0)|$*, we have that the power series* $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)(x-y_0)^k$ *converges weakly to* $f(x)$*; i.e.*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(y_0)}{k!}(x-y_0)^k = f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n .$$

Moreover, the radius of weak convergence of $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)(x-y_0)^k$ *is exactly* $\eta - |\Re(y_0-x_0)|$.

Proof. Let $x \in \mathcal{R}$ be such that $|\Re(x-y_0)| < \eta - |\Re(y_0-x_0)|$. By Theorem 5.20, we have, since $|\Re(y_0-x_0)| < \eta$, that

$$f^{(k)}(y_0) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n (y_0-x_0)^{n-k} \text{ for all } k \geq 0.$$

Since $|\Re(x-y_0)| < \eta - |\Re(y_0-x_0)|$, we obtain that

$$\begin{aligned} |\Re(x-x_0)| &= |\Re(x-y_0+y_0-x_0)| = |\Re(x-y_0)+\Re(y_0-x_0)| \\ &\leq |\Re(x-y_0)| + |\Re(y_0-x_0)| \\ &< \eta. \end{aligned}$$

Hence $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely weakly in $\mathcal{R}$. Now let $q \in Q$ be given. Then

$$\begin{aligned} & f(x)[q] \\ = & \left(\sum_{n=0}^{\infty} a_n (x-x_0)^n\right)[q] \\ = & \left(\sum_{n=0}^{\infty} a_n (y_0-x_0+x-y_0)^n\right)[q] \\ = & \left(\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (y_0-x_0)^{n-k}(x-y_0)^k\right)[q] \\ = & \left(\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_n (y_0-x_0)^{n-k}(x-y_0)^k\right)[q] \end{aligned}$$

$$= \left(\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n(n-1)\dots(n-k+1)}{k!} a_n (y_0 - x_0)^{n-k} (x - y_0)^k \right) [q]$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{n(n-1)\dots(n-k+1)}{k!} a_n (y_0 - x_0)^{n-k} (x - y_0)^k \right) [q] . \qquad (5.7)$$

Because of absolute convergence in R, we can interchange the order of the sums in Equation (5.7) to get

$$\begin{aligned} f(x)[q] &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \left(\frac{n(n-1)\dots(n-k+1)}{k!} a_n (y_0 - x_0)^{n-k} (x - y_0)^k \right) [q] \\ &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)\, a_n (y_0 - x_0)^{n-k} \right) (x - y_0)^k \right) [q] \\ &= \left(\sum_{k=0}^{\infty} \frac{f^{(k)}(y_0)}{k!} (x - y_0)^k \right) [q] . \end{aligned}$$

Thus, for all $q \in Q$, we have that

$$\left(\sum_{k=0}^{\infty} \frac{f^{(k)}(y_0)}{k!} (x - y_0)^k \right) [q] \text{ converges in } R \text{ to } f(x)[q] .$$

Now consider the sequence $(A_m)_{m \geq 1}$, where for each m,

$$A_m = \sum_{k=0}^{m} \frac{f^{(k)}(y_0)}{k!} (x - y_0)^k .$$

Since (a_n) is regular and since $\lambda(y_0 - x_0) \geq 0$, we obtain that the sequence $\left(f^{(k)}(y_0)\right)$ is regular. Since, in addition, $\lambda(x - y_0) \geq 0$, we obtain that the sequence (A_m) itself is regular. Since (A_m) is regular and since $(A_m[q])$ converges in R to $f(x)[q]$ for all $q \in Q$, we finally obtain that (A_m) converges weakly to $f(x)$; and we can write

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(y_0)}{k!} (x - y_0)^k = f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for all x satisfying $|\Re(x - y_0)| < \eta - |\Re(y_0 - x_0)|$.

Next we show that $\eta - |\Re(y_0 - x_0)|$ is indeed the radius of weak convergence of $\sum_{k=0}^{\infty} f^{(k)}(y_0) / (k!) (x - y_0)^k$. So let $r > \eta - |\Re(y_0 - x_0)|$ be given in R; we show that there exists $x \in \mathcal{R}$ satisfying $|\Re(x-y_0)| < r$ such that $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)(x-y_0)^k$ is weakly divergent in $\mathcal{R}$. Let

$$x = \begin{cases} y_0 + \frac{r+\eta-|\Re(y_0-x_0)|}{2} & \text{if } y_0 \geq x_0 \\ y_0 - \frac{r+\eta-|\Re(y_0-x_0)|}{2} & \text{if } y_0 < x_0 \end{cases}.$$

Then

$$|\Re(x-y_0)| = |x-y_0| = \frac{r+\eta-|\Re(y_0-x_0)|}{2} < r.$$

But

$$\begin{aligned} \Re(x-x_0) &= \begin{cases} \Re(y_0-x_0) + \frac{r+\eta-|\Re(y_0-x_0)|}{2} & \text{if } y_0 \geq x_0 \\ \Re(y_0-x_0) - \frac{r+\eta-|\Re(y_0-x_0)|}{2} & \text{if } y_0 < x_0 \end{cases} \\ &= \begin{cases} |\Re(y_0-x_0)| + \frac{r+\eta-|\Re(y_0-x_0)|}{2} & \text{if } y_0 \geq x_0 \\ -|\Re(y_0-x_0)| - \frac{r+\eta-|\Re(y_0-x_0)|}{2} & \text{if } y_0 < x_0 \end{cases} \\ &= \begin{cases} \frac{r+\eta+|\Re(y_0-x_0)|}{2} & \text{if } y_0 \geq x_0 \\ -\frac{r+\eta+|\Re(y_0-x_0)|}{2} & \text{if } y_0 < x_0 \end{cases}; \end{aligned}$$

and hence

$$|\Re(x-x_0)| = \frac{r+\eta+|\Re(y_0-x_0)|}{2} > \eta.$$

Hence $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is weakly divergent in $\mathcal{R}$. Thus, there exists $t_0 \in Q$ such that $\left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)[t_0]$ diverges in R. Hence $\left(\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)(x-y_0)^k\right)[t_0]$ diverges in R; and it follows that $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)(x-y_0)^k$ is weakly divergent in $\mathcal{R}$. So $\eta - |\Re(y_0-x_0)|$ is the radius of weak convergence of $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)(x-y_0)^k$.

In Chapter 6, we will study a large class of functions that are given locally by power series; and we will prove more results about power series.

5.4 Intermediate Value Theorem and Inverse Function Theorem

Notation 5.2 *Let $D_1, D_2 \subset \mathcal{R}$ and let $f_1 : D_1 \to \mathcal{R}$ and $f_2 : D_2 \to \mathcal{R}$. Then we say that $f_1 \sim f_2$ if and only if there exists $n \in Z^+$ such that*

$$\frac{1}{n}|f_1(x)| \leq |f_2(y)| \leq n|f_1(x)| \text{ for all } x \in D_1 \text{ and for all } y \in D_2.$$

Definition 5.12 *Let $a < b$ be given in $\mathcal{R}$, and let $f : [a, b] \to \mathcal{R}$ be differentiable. Then we say that f is quasi-linear on $[a, b]$ if and only if*

$$S_{1,x} \sim S_{1,\bar{x}} \text{ and } S_{2,x} \sim S_{2,\bar{x}} \text{ for all } x, \bar{x} \in [a, b], \tag{5.8}$$

where $S_{1,x}$ and $S_{2,x}$ denote the first and second secants of f at x, respectively.

Remark 5.2 *It follows directly from Definition 5.12 that if f is quasi-linear on $[a, b]$ and if $a \leq a_1 < b_1 \leq b$, then f is quasi-linear on $[a_1, b_1]$.*

Lemma 5.11 *Let $a < b$ be given in $\mathcal{R}$, and let $f : [a, b] \to \mathcal{R}$ be quasi-linear. Then there exist $n, m \in Z^+$ such that*

$$\frac{1}{n}\frac{|f(b) - f(a)|}{b - a} \leq \left|\frac{f(y) - f(x)}{y - x}\right| \leq n\frac{|f(b) - f(a)|}{b - a}, \tag{5.9}$$

$$\frac{1}{n}\frac{|f(b) - f(a)|}{b - a} \leq |f'(x)| \leq n\frac{|f(b) - f(a)|}{b - a} \text{ and} \tag{5.10}$$

$$\left|\frac{f(y) - f(x)}{y - x} - f'(x)\right| \leq m\frac{|f(b) - f(a)|}{(b - a)^2}|y - x| \tag{5.11}$$

for all x and for all $y \neq x$ in $[a, b]$.

Proof. Since f is quasi-linear on $[a, b]$, there exists n such that

$$\frac{1}{n}\left|\frac{f(\bar{y}) - f(\bar{x})}{\bar{y} - \bar{x}}\right| \leq \left|\frac{f(y) - f(x)}{y - x}\right| \leq n\left|\frac{f(\bar{y}) - f(\bar{x})}{\bar{y} - \bar{x}}\right| \tag{5.12}$$

and

$$\begin{aligned}\frac{1}{n}\frac{|f(\bar{y})-f(\bar{x})-f'(\bar{x})(\bar{y}-\bar{x})|}{(\bar{y}-\bar{x})^2} &\leq \frac{|f(y)-f(x)-f'(x)(y-x)|}{(y-x)^2} \\ &\leq n\frac{|f(\bar{y})-f(\bar{x})-f'(\bar{x})(\bar{y}-\bar{x})|}{(\bar{y}-\bar{x})^2} \qquad (5.13)\end{aligned}$$

for all $y \neq x$ and for all $\bar{y} \neq \bar{x}$ in $[a,b]$. Letting $\bar{x} = a$ and $\bar{y} = b$ in Equation (5.12), we obtain Equation (5.9). Since

$$f'(x) = \lim_{y \to x}\frac{f(y)-f(x)}{y-x}$$

for all $x \in [a,b]$, Equation (5.10) follows readily from Equation (5.9).

Equation (5.13) entails that

$$\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| \leq n\frac{|f(b)-f(a)-f'(a)(b-a)|}{(b-a)^2}|y-x|$$

for all x and for all $y \neq x$ in $[a,b]$. Using Equation (5.9) and Equation (5.10), we have that

$$\begin{aligned}\frac{|f(b)-f(a)-f'(a)(b-a)|}{(b-a)^2} &\leq \frac{1}{b-a}\left(\frac{|f(b)-f(a)|}{b-a} + |f'(a)|\right) \\ &\leq \frac{1}{b-a}(n+1)\frac{|f(b)-f(a)|}{b-a} \\ &= (n+1)\frac{|f(b)-f(a)|}{(b-a)^2}\end{aligned}$$

Thus,

$$\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| \leq n(n+1)\frac{|f(b)-f(a)|}{(b-a)^2}|y-x|$$

for all x and for all $y \neq x$ in $[a,b]$.

Corollary 5.9 *Let $a < b$ be given in $\mathcal{R}$ and let $f : [a,b] \to \mathcal{R}$ be quasi-linear on $[a,b]$. Let n be as in Lemma 5.11. Then*

$$\frac{1}{n^2}|f'(x)| \leq |f'(\bar{x})| \leq n^2|f'(x)| \text{ for all } x, \bar{x} \in [a,b]; \qquad (5.14)$$

and hence $f' \sim f'$.

Proof. Let $x, \bar{x} \in [a,b]$ be given. Using Equation (5.10), we have that

$$\frac{1}{n}\frac{|f(b)-f(a)|}{b-a} \leq |f'(x)| \leq n\frac{|f(b)-f(a)|}{b-a} \text{ and} \tag{5.15}$$

$$\frac{1}{n}\frac{|f(b)-f(a)|}{b-a} \leq |f'(\bar{x})| \leq n\frac{|f(b)-f(a)|}{b-a}. \tag{5.16}$$

Combining the results of Equation (5.15) and Equation (5.16), we obtain Equation (5.14).

Corollary 5.10 *Let $a < b$ be given in $\mathcal{R}$ and let $f : [a,b] \to \mathcal{R}$ be quasi-linear. Then*

$$\frac{f(y)-f(x)}{y-x} \sim \frac{f(b)-f(a)}{b-a} \text{ for all } y \neq x \text{ in } [a,b], \text{ and} \tag{5.17}$$

$$f'(x) \sim \frac{f(b)-f(a)}{b-a} \text{ for all } x \in [a,b]. \tag{5.18}$$

Lemma 5.12 *Let $a < b$ be given in $\mathcal{R}$, and let $f : [a,b] \to \mathcal{R}$ be quasi-linear. Then f is continuously differentiable on $[a,b]$.*

Proof. Let $m \in Z^+$ be as in Lemma 5.11, and let $x \neq y$ in $[a,b]$ be given. Then

$$\begin{aligned}
|f'(y)-f'(x)| &\leq \left|\frac{f(y)-f(x)}{y-x} - f'(y)\right| + \left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| \\
&\leq m\frac{|f(b)-f(a)|}{(b-a)^2}|y-x| + m\frac{|f(b)-f(a)|}{(b-a)^2}|y-x| \\
&= 2m\frac{|f(b)-f(a)|}{(b-a)^2}|y-x|.
\end{aligned}$$

Hence

$$\left|\frac{f'(y)-f'(x)}{y-x}\right| \leq 2m\frac{|f(b)-f(a)|}{(b-a)^2} \text{ for all } y \neq x \text{ in } [a,b].$$

Therefore, f' is continuous on $[a,b]$.

Lemma 5.13 *Let $a < b$ be given in $\mathcal{R}$, and let $f : [a,b] \to \mathcal{R}$ be quasi-linear. Then for all $x, y \in [a,b]$, we have that*

$$\lambda\left(f'(y) - f'(x)\right) \geq \lambda\left(\frac{f(b) - f(a)}{b-a}\right) + \lambda\left(\frac{y-x}{b-a}\right).$$

Proof. Let $x, y \in [a,b]$ be given. Using the proof of Theorem 5.12, we have that

$$\begin{aligned} f(y) &= f(x) + f'(x)(y-x) + r(x,y)(y-x)^2 \text{ and} && (5.19)\\ f(x) &= f(y) + f'(y)(x-y) + r(y,x)(y-x)^2, && (5.20) \end{aligned}$$

where, using Equation (5.11), we have that

$$\begin{aligned} \lambda(r(x,y)) &\geq \lambda\left(\frac{f(b) - f(a)}{(b-a)^2}\right) = \lambda\left(\frac{f(b)-f(a)}{b-a}\right) - \lambda(b-a) \text{ and} \\ \lambda(r(y,x)) &\geq \lambda\left(\frac{f(b)-f(a)}{b-a}\right) - \lambda(b-a). \end{aligned}$$

Adding Equations (5.19) and (5.20), we obtain that

$$\left(f'(y) - f'(x)\right)(y-x) = \left(r(x,y) + r(y,x)\right)(y-x)^2;$$

and hence

$$f'(y) - f'(x) = \left(r(x,y) + r(y,x)\right)(y-x).$$

Thus,

$$\begin{aligned} \lambda\left(f'(y) - f'(x)\right) &= \lambda\left(r(x,y) + r(y,x)\right) + \lambda(y-x) \\ &\geq \lambda\left(\frac{f(b)-f(a)}{b-a}\right) - \lambda(b-a) + \lambda(y-x) \\ &= \lambda\left(\frac{f(b)-f(a)}{b-a}\right) + \lambda\left(\frac{y-x}{b-a}\right). \end{aligned}$$

Lemma 5.14 *Let $a < b$ be given in $\mathcal{R}$, and let $f : [a,b] \to \mathcal{R}$ be quasi-linear. If $f(a) = f(b)$, then f is constant on $[a,b]$.*

Proof. Let $x \in (a, b]$ be given. Then, by Equation (5.17), we have that

$$\frac{f(x) - f(a)}{x - a} \sim \frac{f(b) - f(a)}{b - a} = 0,$$

which entails that $f(x) = f(a)$.

Definition 5.13 *Let $a < b$ be given in $\mathcal{R}$, and let $f : [a, b] \to \mathcal{R}$ be quasi-linear and nonconstant. Define $g : [0, 1] \to \mathcal{R}$ by*

$$g(x) = \frac{f((b-a)x + a) - f(a)}{f(b) - f(a)}.$$

Then g will be called the scaled function of f on $[0, 1]$.

Lemma 5.15 *Let $a < b$ be given in $\mathcal{R}$, let $f : [a, b] \to \mathcal{R}$ be quasi-linear and nonconstant, and let g be the scaled function of f on $[0, 1]$. Then g is quasi-linear on $[0, 1]$, with*

$$\lambda(g(x)) = \lambda(x) \geq 0 \text{ and } \lambda(g'(x)) = 0 \text{ for all } x \in [0, 1].$$

Proof. Using Theorems 5.8 and 5.9, we obtain that g is continuous on $[0, 1]$. Using Theorems 5.13 and 5.14, we obtain that g is differentiable on $[0, 1]$, with derivative

$$g'(x) = \frac{b - a}{f(b) - f(a)} f'((b-a)x + a) \text{ for all } x \in [0, 1].$$

Using Equation (5.18), we have that

$$f'((b-a)x + a) \sim \frac{f(b) - f(a)}{b - a} \text{ for all } x \in [0, 1].$$

Hence

$$g'(x) \sim 1 \text{ for all } x \in [0, 1].$$

Moreover, for all $x \in [0, 1]$, we have that

$$\lambda(g(x)) = \lambda\left(\frac{f((b-a)x + a) - f(a)}{f(b) - f(a)}\right)$$

$$\begin{aligned} &= \lambda\left(\frac{f\left(\left(b-a\right)x+a\right)-f\left(a\right)}{\left(b-a\right)x}\frac{\left(b-a\right)x}{f\left(b\right)-f\left(a\right)}\right) \\ &= \lambda\left(\frac{f\left(\left(b-a\right)x+a\right)-f\left(a\right)}{\left(b-a\right)x}\right)+\lambda\left(\frac{b-a}{f\left(b\right)-f\left(a\right)}\right)+\lambda\left(x\right). \end{aligned}$$

Using Equation (5.17), we have that

$$\lambda\left(\frac{f\left(\left(b-a\right)x+a\right)-f\left(a\right)}{\left(b-a\right)x}\right)=\lambda\left(\frac{f\left(b\right)-f\left(a\right)}{b-a}\right)=-\lambda\left(\frac{b-a}{f\left(b\right)-f\left(a\right)}\right)$$

for all $x\in[0,1]$. Hence

$$\lambda\left(g\left(x\right)\right)=\lambda\left(x\right)\geq 0 \text{ for all } x\in\left[0,1\right].$$

Now let $x,\bar{x}\in[0,1]$ be given, let $S_{1,x}$ and $S_{1,\bar{x}}$ denote the first secants of g at x and $\bar{x}$ and let $T_{1,(b-a)x+a}$ and $T_{1,(b-a)\bar{x}+a}$ denote the first secants of f at $(b-a)x+a$ and $(b-a)\bar{x}+a$. Then for all $y\neq x$ in $[0,1]$, we have that

$$\begin{aligned} S_{1,x}(y) &= \frac{g\left(y\right)-g\left(x\right)}{y-x} \\ &= \frac{1}{f\left(b\right)-f\left(a\right)}\frac{f\left(\left(b-a\right)y+a\right)-f\left(\left(b-a\right)x+a\right)}{y-x} \\ &= \frac{b-a}{f\left(b\right)-f\left(a\right)}\frac{f\left(\left(b-a\right)y+a\right)-f\left(\left(b-a\right)x+a\right)}{\left(\left(b-a\right)y+a\right)-\left(\left(b-a\right)x+a\right)} \\ &= \frac{b-a}{f(b)-f(a)}T_{1,(b-a)x+a}((b-a)y+a). \end{aligned}$$

Similarly, for all $\bar{y}\neq\bar{x}$ in $[0,1]$, we have that

$$S_{1,\bar{x}}(\bar{y})=\frac{b-a}{f(b)-f(a)}T_{1,(b-a)\bar{x}+a}((b-a)\bar{y}+a).$$

Since f is quasi-linear on $[a,b]$, we have that

$$T_{1,(b-a)x+a}\sim T_{1,(b-a)\bar{x}+a}.$$

Hence

$$S_{1,x}\sim S_{1,\bar{x}}.$$

Finally let $S_{2,x}$ and $S_{2,\bar{x}}$ denote the second secants of g at x and $\bar{x}$ and let $T_{2,(b-a)x+a}$ and $T_{2,(b-a)\bar{x}+a}$ denote the second secants of f at $(b-a)x+a$ and $(b-a)\bar{x}+a$. Then for all $y \neq x$ in $[0,1]$, we have that

$$\begin{aligned}
S_{2,x}(y) &= \frac{g(y)-g(x)-g'(x)(y-x)}{(y-x)^2} \\
&= \frac{(b-a)^2}{f(b)-f(a)} \cdot \\
&\quad \frac{f((b-a)y+a)-f((b-a)x+a)-f'((b-a)x+a)(b-a)(y-x)}{((b-a)(y-x))^2} \\
&= \frac{(b-a)^2}{f(b)-f(a)} T_{2,(b-a)x+a}((b-a)y+a).
\end{aligned}$$

Similarly, for all $\bar{y} \neq \bar{x}$ in $[0,1]$, we have that

$$S_{2,\bar{x}}(\bar{y}) = \frac{(b-a)^2}{f(b)-f(a)} T_{2,(b-a)\bar{x}+a}((b-a)\bar{y}+a).$$

Since f is quasi-linear on $[a,b]$, we have that

$$T_{2,(b-a)x+a} \sim T_{2,(b-a)\bar{x}+a}.$$

Hence

$$S_{2,x} \sim S_{2,\bar{x}}.$$

This finishes the proof of the lemma.

The following result follows immediately from Lemma 5.15 and Lemma 5.13.

Corollary 5.11 *Let $a<b$ be given in $\mathcal{R}$, let $f:[a,b]\to\mathcal{R}$ be quasi-linear and nonconstant, and let g be the scaled function of f on $[0,1]$. Then for all $x,y\in[0,1]$, we have that*

$$\lambda\left(g'(y)-g'(x)\right) \geq \lambda\left(y-x\right).$$

Lemma 5.16 *Let $a < b$ be given in $\mathcal{R}$, let $f : [a,b] \to \mathcal{R}$ be quasi-linear and nonconstant, and let g be the scaled function of f on $[0,1]$. Let $g_R : [0,1] \cap R \to R$ be given by*

$$g_R(X) = g(X)[0].$$

Then g_R is continuously differentiable on $[0,1] \cap R$, with derivative

$$(g_R)'(X) = g'(X)[0] \neq 0 \text{ for all } X \in [0,1] \cap R.$$

Proof. Since g is quasi-linear on $[0,1]$ by Lemma 5.15, there exists $n \in Z^+$ such that

$$\left| \frac{g(y) - g(x)}{y - x} - g'(x) \right| \leq n|y - x| \text{ for all } y \neq x \text{ in } [0,1].$$

Now let $X \in [0,1] \cap R$ be given. Then

$$\left| \frac{g(Y) - g(X)}{Y - X} - g'(X) \right| \leq n|Y - X| \text{ for all } Y \neq X \text{ in } [0,1] \cap R.$$

Thus,

$$\begin{aligned} \left| \frac{g_R(Y) - g_R(X)}{Y - X} - g'(X)[0] \right| &= \left| \frac{g(Y)[0] - g(X)[0]}{Y - X} - g'(X)[0] \right| \\ &= \left| \left(\frac{g(Y) - g(X)}{Y - X} - g'(X) \right)[0] \right| \\ &\leq 2n|Y - X| \text{ for all } Y \neq X \text{ in } [0,1] \cap R, \end{aligned}$$

which entails that g_R is differentiable (in the real sense) at X with derivative

$$(g_R)'(X) = g'(X)[0] \neq 0,$$

since $\lambda(g'(X)) = 0$ by Lemma 5.15.

Next we show that $(g_R)'$ is continuous on $[0,1] \cap R$. As in the proof of Lemma 5.12, we have that

$$|g'(y) - g'(x)| \leq 2n|y - x| \text{ for all } x, y \in [0,1].$$

In particular,

$$|g'(Y) - g'(X)| \le 2n|Y - X| \text{ for all } X, Y \in [0,1] \cap R.$$

It follows that

$$\begin{aligned} |(g_R)'(Y) - (g_R)'(X)| &= |g'(Y)[0] - g'(X)[0]| \\ &\le 3n|Y - X| \text{ for all } X, Y \in [0,1] \cap R, \end{aligned}$$

which entails that $(g_R)'$ is (uniformly) continuous on $[0,1] \cap R$. Thus, g_R is continuously differentiable on $[0,1] \cap R$.

Lemma 5.17 *Let $a < b$ be given in $\mathcal{R}$, and let $f : [a,b] \to \mathcal{R}$ be quasi-linear and nonconstant. Then f is strictly monotone on $[a,b]$.*

Proof. Let $g : [0,1] \to \mathcal{R}$ be the scaled function of f on $[0,1]$. We show that g is strictly increasing on $[0,1]$. Let g_R be as in Lemma 5.16. Then g_R is continuously differentiable on $[0,1] \cap R$ and $(g_R)'(X) \neq 0$ for all $X \in [0,1] \cap R$. Thus, g_R is strictly monotone on $[0,1] \cap R$. Since $g_R(0) = 0 < 1 = g_R(1)$, we obtain that g_R is strictly increasing on $[0,1] \cap R$. Now let $x, y \in [0,1]$ be such that $x < y$, and let $X = \Re(x)$ and $Y = \Re(y)$. As a first case, assume that $X < Y$; then $g_R(X) < g_R(Y)$. Hence

$$\begin{aligned} g(y) - g(x) &= g_R(Y) - g_R(X) \\ &\quad + (g(y) - g(Y)) + (g(Y) - g_R(Y)) \\ &\quad + (g_R(X) - g(X)) + (g(X) - g(x)), \end{aligned}$$

where the first term is positive and real. By Theorem 5.12, we have that

$$g(y) - g(Y) = g'(Y)(y - Y) + r(Y, y)(y - Y)^2,$$

where

$$\lambda(g'(Y)) = 0, \ \lambda(y - Y) > 0, \ \text{and } \lambda(r(Y, y)) \ge 0.$$

Hence $|g(y) - g(Y)|$ is infinitely small. Similarly, $|g(X) - g(x)|$ is infinitely small. Since

$$\lambda(g(Y)) \geq 0 \text{ and } g_R(Y) = g(Y)[0],$$

we obtain that $|g(Y) - g_R(Y)|$ is infinitely small. Similarly, $|g_R(X) - g(X)|$ is infinitely small. So

$$g(y) - g(x) \approx g_R(Y) - g_R(X) > 0; \text{ and hence } g(x) < g(y).$$

As a second case, assume that $X = Y$. Then $y - x \ll 1$, and hence

$$\begin{aligned} g(y) - g(x) &= g'(x)(y-x) + r(x,y)(y-x)^2 \\ &\approx g'(x)(y-x), \end{aligned} \tag{5.21}$$

since $|r(x,y)|$ is at most finite and hence

$$\begin{aligned} \lambda\left(r(x,y)(y-x)^2\right) &= \lambda(r(x,y)) + 2\lambda(y-x) \\ &\geq 2\lambda(y-x) \\ &> \lambda(y-x) = \lambda(g'(x)) + \lambda(y-x) \\ &= \lambda(g'(x)(y-x)). \end{aligned}$$

By Corollary 5.11, we have that

$$\lambda(g'(x) - g'(X)) \geq \lambda(x - X) > 0.$$

Since

$$g'(x) \sim 1,\ g'(X) \sim 1 \text{ and } |g'(x) - g'(X)| \ll 1,$$

we obtain that

$$g'(x) \approx g'(X) \approx (g_R)'(X) > 0. \tag{5.22}$$

From Equations (5.21) and (5.22), we obtain that $g(y) - g(x) > 0$. Thus,

$$g(x) < g(y) \text{ for all } x < y \text{ in } [0,1];$$

and hence g is strictly increasing on $[0,1]$. Since

$$f(x) = (f(b) - f(a))\, g\left(\frac{x-a}{b-a}\right) + f(a) \text{ for all } x \in [a,b]$$

and since g is strictly increasing on $[0,1]$, we obtain that f is strictly increasing on $[a,b]$ if $f(a) < f(b)$, and f is strictly decreasing on $[a,b]$ if $f(a) > f(b)$.

The following theorem generalizes the intermediate value theorem which was discussed in [5] and which applied to functions whose domain and range are both finite and whose derivatives are finite everywhere. We offer two proofs; and after scaling, the second proof is similar to that of the previous version of the intermediate value theorem.

Theorem 5.22 (Intermediate Value Theorem) *Let $a < b$ be given in $\mathcal{R}$, and let $f : [a,b] \to \mathcal{R}$ be quasi-linear. Then f assumes every intermediate value between $f(a)$ and $f(b)$.*

Proof. If $f(a) = f(b)$, then f is constant on $[a,b]$ by Lemma 5.14, and there is nothing to prove. So we may assume that $f(a) \neq f(b)$. Let $g : [0,1] \to \mathcal{R}$ be the scaled function of f on $[0,1]$. For all $x \in [a,b]$, we have that

$$\begin{aligned} f(x) &= (f(b) - f(a))\, g\left(\frac{x-a}{b-a}\right) + f(a) \\ &= l_2 \circ g \circ l_1(x), \end{aligned}$$

where l_1 and l_2 are linear functions. Hence it suffices to show that g assumes every intermediate value between $g(0) = 0$ and $g(1) = 1$. We present two proofs.

First proof (by iteration): Let g_R be as in Lemma 5.16, let $S \in (0,1)$ be given, and let $S_R = \Re(S)$. Then $S_R \in [0,1] \cap R$. Since g_R is continuous on $[0,1] \cap R$ by Lemma 5.16, there exists $X \in [0,1] \cap R$ such that $g_R(X) = S_R$. Thus,

$$\begin{aligned} |S - g(X)| &\leq |S - S_R| + |S_R - g_R(X)| + |g_R(X) - g(X)| \\ &= |S - S_R| + |g_R(X) - g(X)| \end{aligned}$$

is infinitely small.

Now let $s = S - g(X)$. If $s = 0$, then $g(X) = S$ and we are done. So we may assume that $s \neq 0$; we try to find $x \in \mathcal{R}$ such that $0 < |x| \ll 1$, $X + x \in [0,1]$ and $g(X + x) = S$. Let $a_1 = 1/g'(X)$, and let

$$x_1 = a_1 s = \frac{s}{g'(X)}.$$

Then, by Theorem 5.12, we have that

$$\begin{aligned} g(X + x_1) &= g\left(X + \frac{s}{g'(X)}\right) = g(X) + s + r(X, X + x_1)\, x_1^2 \\ &= S + r(X, X + x_1)\, x_1^2; \end{aligned}$$

and hence

$$|g(X + x_1) - S| = |r(X, X + x_1)|\, x_1^2,$$

where $|r(X, X + x_1)|$ is at most finite. So

$$|g(X + x_1) - S| = |r(X, X + x_1)| \frac{s^2}{(g'(X))^2} = c_1 s^2,$$

where

$$c_1 = \frac{|r(X, X + x_1)|}{(g'(X))^2}.$$

Let

$$s_1 = S - g(X + x_1) = -r(X, X + x_1)\, x_1^2,$$

let

$$a_2 = -\frac{r(X, X + x_1)\, a_1^2}{g'(X + x_1)}$$

and let

$$x_2 = x_1 + \frac{s_1}{g'(X + x_1)}.$$

Then

$$x_2 = x_1 - \frac{r(X, X + x_1)\, x_1^2}{g'(X + x_1)} = x_1 - \frac{r(X, X + x_1)\, a_1^2}{g'(X + x_1)} s^2 = x_1 + a_2 s^2.$$

Since $g'(z) \sim 1$ for all $z \in [0,1]$ by Lemma 5.15, we obtain that $|a_1|$ and $|a_2|$ are both at most finite, and $x_2 \approx x_1$. Moreover,

$$\begin{aligned} g(X + x_2) &= g\left(X + x_1 + \frac{s_1}{g'(X + x_1)}\right) \\ &= g(X + x_1) + s_1 + r(X + x_1, X + x_2)(x_2 - x_1)^2 \\ &= S + r(X + x_1, X + x_2)\, a_2^2 s^4, \end{aligned}$$

where $|r(X + x_1, X + x_2)|$ is at most finite. Let

$$c_2 = |r(X + x_1, X + x_2)|\, a_2^2.$$

Then c_2 is at most finite and

$$|g(X + x_2) - S| = c_2 s^4.$$

By induction, we obtain a sequence (x_n) such that for all $n \geq 1$, we have that

$$x_n = \sum_{j=1}^{n} a_j s^{2^{j-1}} \approx a_1 s = x_1 \text{ and } |g(X + x_n) - S| = c_n s^{2^n},$$

with

$$\lambda(a_n) \geq 0 \text{ and } \lambda(c_n) \geq 0 \text{ for all } n \geq 1.$$

Since $|s| \ll 1$ and since $\lambda(c_n) \geq 0$ for all $n \geq 1$, we have that

$$\lim_{n \to \infty} c_n s^{2^n} = 0; \text{ and hence } \lim_{n \to \infty} g(X + x_n) = S.$$

Also, the sequence (x_n) converges strongly. Let

$$x = \lim_{n \to \infty} x_n = \sum_{j=1}^{\infty} a_j s^{2^{j-1}}.$$

Then

$$x \approx x_1 = a_1 s = \frac{s}{g'(X)}.$$

We show that $g(X+x) = S$. Since g is continuous on $[0,1]$ and since the sequence (x_n) converges strongly to x, we obtain by Lemma 5.5 and Lemma 5.1 that the sequence $(g(X+x_n))$ converges strongly to $g(X+x)$. Hence

$$g(X+x) = \lim_{n\to\infty} g(X+x_n) = S.$$

Finally we show that $X+x \in (0,1)$. First assume that $X=0$; then $S > 0 = g(X)$ and hence $s = S - g(X) > 0$. Since $g'(0) \approx (g_R)'(0) > 0$, we obtain that

$$X + x = x \approx \frac{s}{g'(0)} > 0.$$

Moreover, $x \ll 1$; hence $X + x = x \in (0,1)$. Now assume that $X = 1$; then $S < 1 = g(1)$ and hence $s < 0$. It follows that

$$x \approx \frac{s}{g'(1)} < 0 \text{ and hence } X + x = 1 + x < 1.$$

Since $|x| \ll 1$, we obtain that $X+x = 1+x \in (0,1)$. Finally assume that $0 < X < 1$; then X is finitely away from 0 and 1. Since $|x| \ll 1$, we obtain that $X + x \in (0,1)$.

Second proof (using the fixed point theorem): Having found $X \in [0,1] \cap R$ such that $g_R(X) = S_R$, we proceed to find x such that $0 < |x| \ll 1$, $X + x \in [0,1]$ and $g(X+x) = S$. Since g is differentiable on $[0,1]$, we have, by Theorem 5.12, that

$$S = g(X+x) = g(X) + g'(X)\,x + r(X, X+x)\,x^2, \tag{5.23}$$

where $|r(X, X+x)|$ is at most finite.

Transforming Equation (5.23) into a fixed point problem yields

$$\begin{aligned} x &= \frac{s}{g'(X)} - \frac{r(X, X+x)}{g'(X)}x^2 \\ &= h(x), \end{aligned} \tag{5.24}$$

where $s = S - g(X)$ is infinitely small in absolute value. Let

$$M = \{z \in \mathcal{R} : \lambda(z) \geq \lambda(s)\}.$$

Let $x \in M$ be given. Since $|r(X, X+x)|$ is at most finite and since $g'(X) \sim 1$, we have that

$$\lambda\left(\frac{r(X, X+x)}{g'(X)}x^2\right) \geq 2\lambda(x) > \lambda(x) \geq \lambda(s) = \lambda\left(\frac{s}{g'(X)}\right).$$

Hence $s/g'(X)$ is the leading term on the right hand side of Equation (5.24). Thus,

$$h(x) \approx \frac{s}{g'(X)}; \text{ and hence } \lambda(h(x)) = \lambda(s) \text{ for all } x \in M.$$

Hence

$$h(M) \subset M.$$

Now let $x_1 \neq x_2$ be given in M. Then

$$\begin{aligned} |h(x_1) - h(x_2)| &= \left|\frac{r(X, X+x_2)x_2^2 - r(X, X+x_1)x_1^2}{g'(X)}\right| \\ &= \left|\frac{g(X+x_2) - g(X+x_1)}{g'(X)} + x_1 - x_2\right|. \end{aligned}$$

But

$$g(X+x_2) = g(X+x_1) + g'(X+x_1)(x_2 - x_1) + r(X+x_1, X+x_2)(x_2 - x_1)^2,$$

where $|r(X+x_1, X+x_2)|$ is at most finite. Thus,

$$\begin{aligned} & |h(x_1) - h(x_2)| \\ = & \left|\frac{g'(X+x_1)(x_2-x_1) + r(X+x_1, X+x_2)(x_2-x_1)^2}{g'(X)} + x_1 - x_2\right| \\ = & \left|\frac{g'(X+x_1) - g'(X)}{g'(X)}(x_2 - x_1) + \frac{r(X+x_1, X+x_2)}{g'(X)}(x_2-x_1)^2\right| \\ \leq & |x_1 - x_2|\left(\frac{|g'(X+x_1) - g'(X)|}{g'(X)} + \frac{|r(X+x_1, X+x_2)|}{g'(X)}|x_1 - x_2|\right). \end{aligned}$$

Using Corollary 5.11 and the fact that $g'(X) \sim 1$, we have that

$$\begin{aligned} \lambda\left(\frac{|g'(X+x_1) - g'(X)|}{g'(X)}\right) &= \lambda(g'(X+x_1) - g'(X)) \\ &\geq \lambda(x_1) \geq \lambda(s) \\ &> \frac{\lambda(s)}{2}. \end{aligned}$$

Also

$$\begin{aligned}\lambda\left(\frac{|r(X+x_1,X+x_2)|}{g'(X)}|x_1-x_2|\right) &\geq \lambda(x_1-x_2)\\ &\geq \min\{\lambda(x_1),\lambda(x_2)\}\geq\lambda(s)\\ &> \frac{\lambda(s)}{2}.\end{aligned}$$

Hence

$$|h(x_1)-h(x_2)| \ll d^{\lambda(s)/2}|x_1-x_2|,$$

where $\lambda(s)>0$. So h is contracting on M, and hence h has a fixed point x in M. The same argument as at the end of the first proof shows that $X+x\in(0,1)$.

The following two examples show that the conditions in Equation (5.8) are necessary to obtain Theorem 5.22.

Example 5.10 *Let $f:[-1,1]\to\mathcal{R}$ be given by*

$$f(x)=\begin{cases} d & \text{if } 0\leq|x|\ll 1\\ x^3 & \text{if } x\sim 1\end{cases}.$$

Then f is continuous on $[-1,1]$ since

$$\left|\frac{f(y)-f(x)}{y-x}\right|\leq 3 \text{ for all } x\neq y \text{ in } [-1,1].$$

Next we show that f is differentiable on $[-1,1]$ with derivative

$$f'(x)=g(x)=\begin{cases} 0 & \text{if } 0\leq|x|\ll 1\\ 3x^2 & \text{if } x\sim 1\end{cases}.$$

Let $x\neq y$ be given in $[-1,1]$. We show that

$$\left|\frac{f(y)-f(x)}{y-x}-g(x)\right|<3|y-x|.$$

First assume that $0\leq|x|\ll 1$. Then

$$\left|\frac{f(y)-f(x)}{y-x}-g(x)\right|=\left|\frac{f(y)-f(x)}{y-x}\right|=\begin{cases} 0 & \text{if } 0\leq|y|\ll 1\\ \left|\frac{y^3-d}{y-x}\right| & \text{if } y\sim 1\end{cases}.$$

But for $y \sim 1$, we have that

$$\left|\frac{y^3-d}{y-x}\right| \approx y^2 \leq |y| \approx |y-x| \, ; \text{ and hence } \left|\frac{y^3-d}{y-x}\right| < 2\,|y-x| \, .$$

Thus, for $0 \leq |x| \ll 1$ and for all $y \neq x$ in $[-1,1]$, we obtain that

$$\left|\frac{f(y)-f(x)}{y-x} - g(x)\right| < 2\,|y-x| \, .$$

Now assume that $x \sim 1$ and $0 \leq |y| \ll 1$. Then

$$\begin{aligned}\left|\frac{f(y)-f(x)}{y-x} - g(x)\right| &= \left|\frac{d-x^3}{y-x} - 3x^2\right| \\ &\approx 2x^2 \leq 2\,|x| \approx 2\,|y-x| \, ;\end{aligned}$$

and hence

$$\left|\frac{f(y)-f(x)}{y-x} - g(x)\right| < 3\,|y-x| \, .$$

Finally, assume that $x \sim 1$ and $y \sim 1$. Then

$$\begin{aligned}\left|\frac{f(y)-f(x)}{y-x} - g(x)\right| &= \left|\frac{y^3-x^3}{y-x} - 3x^2\right| = \left|y^2+xy-2x^2\right| \\ &= |(y+2x)(y-x)| = |y+2x|\,|y-x| \\ &< 3\,|y-x| \, .\end{aligned}$$

Using Theorem 5.15, we have that f is differentiable on $[-1,1]$, with derivative

$$f'(x) = g(x) = \begin{cases} 0 & \text{if } 0 \leq |x| \ll 1 \\ 3x^2 & \text{if } x \sim 1 \end{cases} \, .$$

But f is not quasi-linear on $[-1,1]$ since f is neither constant nor strictly monotone on $[-1,1]$. f does not satisfy the intermediate value theorem on $[-1,1]$ since

$$f(-1) = -1 < 0 < 1 = f(1) \, ; \text{ but } f(x) \neq 0 \text{ for all } x \in [-1,1] \, .$$

Example 5.11 *Let $f : [0,1] \to \mathcal{R}$ be given by*

$$f(x) = \begin{cases} \pi x & \text{if } x \sim 1 \text{ and } x[0] \in Q \\ 2x & \text{if } x \sim 1 \text{ and } x[0] \notin Q \\ x & \text{if } 0 \leq x \ll 1 \end{cases} \, .$$

Then f is differentiable on $[0,1]$, with derivative

$$f'(x)=\begin{cases}\pi & \text{if } x\sim 1 \text{ and } x[0]\in Q\\ 2 & \text{if } x\sim 1 \text{ and } x[0]\notin Q\\ 1 & \text{if } 0\leq x\ll 1\end{cases}.$$

Thus

$$f'(x)\sim 1=\frac{f(1)-f(0)}{1-0} \text{ for all } x\in[0,1].$$

But f is not quasi-linear on $[0,1]$ since f is neither constant nor strictly monotone on $[0,1]$. We have that

$$f(0)=0<1<\pi=f(1);\text{ but } f(x)\neq 1 \text{ for all } x\in[0,1].$$

In this example, the first condition in Equation (5.8) is satisfied, but the second condition is not. We also note that even though $f'(x)$ is positive and finite for all $x\in[0,1]$, f is not strictly increasing on $[0,1]$, which is another manifestation of the differences between R and $\mathcal{R}$.

Using Lemma 5.17 and Theorem 5.22, we readily obtain the following result.

Corollary 5.12 *Let $a<b$ be given in $\mathcal{R}$, and let $f:[a,b]\to\mathcal{R}$ be quasi-linear. Let $\alpha=\min\{f(a),f(b)\}$ and $\beta=\max\{f(a),f(b)\}$. Then*

$$f([a,b])=[\alpha,\beta].$$

Theorem 5.23 (Inverse Function Theorem) *Let $a<b$ be given in $\mathcal{R}$, and let $f:[a,b]\to\mathcal{R}$ be quasi-linear and nonconstant. Let $\alpha=\min\{f(a),f(b)\}$ and $\beta=\max\{f(a),f(b)\}$. Then f has an inverse function $f^{-1}:[\alpha,\beta]\to[a,b]$ that is quasi-linear on $[\alpha,\beta]$, with derivative*

$$\left(f^{-1}\right)'(\rho)=\frac{1}{f'\left(f^{-1}(\rho)\right)} \textit{ for all } \rho\in[\alpha,\beta].$$

Proof. That f^{-1} exists follows immediately from the fact that f is strictly monotone on $[a,b]$, by Lemma 5.17. Let $m,n \in Z^+$ be as in Lemma 5.11. First we show that f^{-1} is continuous on $[\alpha,\beta]$. Let $\rho \neq \upsilon$ in $[\alpha,\beta]$ be given and let $x = f^{-1}(\rho)$ and $y = f^{-1}(\upsilon)$. Then, using Equation (5.9), we obtain that

$$\begin{aligned}\left|\frac{f^{-1}(\upsilon)-f^{-1}(\rho)}{\upsilon-\rho}\right| &= \left|\frac{y-x}{f(y)-f(x)}\right| \\ &\leq n\frac{b-a}{|f(b)-f(a)|}.\end{aligned}$$

Thus

$$\left|\frac{f^{-1}(\upsilon)-f^{-1}(\rho)}{\upsilon-\rho}\right| \leq n\frac{b-a}{|f(b)-f(a)|} \text{ for all } \rho \neq \upsilon \text{ in } [\alpha,\beta];$$

and hence f^{-1} is continuous on $[\alpha,\beta]$.

Next we show that f^{-1} is differentiable on $[\alpha,\beta]$. Let $\rho \neq \upsilon$ in $[\alpha,\beta]$ be given, and let $x = f^{-1}(\rho)$ and $y = f^{-1}(\upsilon)$. Then, using Equations (5.9), (5.10) and (5.11), we obtain that

$$\begin{aligned}&\left|\frac{f^{-1}(\upsilon)-f^{-1}(\rho)}{\upsilon-\rho}-\frac{1}{f'(f^{-1}(\rho))}\right| \\ &= \left|\frac{y-x}{f(y)-f(x)}-\frac{1}{f'(x)}\right| \\ &= \left|\frac{y-x}{f(y)-f(x)}\right|\frac{1}{|f'(x)|}\left|\frac{f(y)-f(x)}{y-x}-f'(x)\right| \\ &\leq \left|\frac{y-x}{f(y)-f(x)}\right|\frac{1}{|f'(x)|}m\frac{|f(b)-f(a)|}{(b-a)^2}|y-x| \\ &= m\frac{|f(b)-f(a)|}{(b-a)^2}\left(\frac{y-x}{f(y)-f(x)}\right)^2\frac{1}{|f'(x)|}|f(y)-f(x)| \\ &= m\frac{|f(b)-f(a)|}{(b-a)^2}\left(\frac{y-x}{f(y)-f(x)}\right)^2\frac{1}{|f'(x)|}|\upsilon-\rho| \\ &\leq m\frac{|f(b)-f(a)|}{(b-a)^2}n^2\frac{(b-a)^2}{(f(b)-f(a))^2}n\frac{b-a}{|f(b)-f(a)|}|\upsilon-\rho|\end{aligned}$$

$$= \quad mn^3 \frac{b-a}{\left(f(b)-f(a)\right)^2} |v-\rho|.$$

Using Theorem 5.15, we obtain that f^{-1} is differentiable on $[\alpha, \beta]$ with derivative

$$\left(f^{-1}\right)'(\rho) = \frac{1}{f'\left(f^{-1}(\rho)\right)} \text{ for all } \rho \in [\alpha, \beta].$$

Now let $\rho, \bar{\rho} \in [\alpha, \beta]$ be given, let $S_{1,\rho}$ and $S_{1,\bar{\rho}}$ denote the first secants of f^{-1} at ρ and $\bar{\rho}$, let $x = f^{-1}(\rho)$ and $\bar{x} = f^{-1}(\bar{\rho})$, and let $T_{1,x}$ and $T_{1,\bar{x}}$ denote the first secants of f at x and $\bar{x}$. For all $v \neq \rho$ and for all $\bar{v} \neq \bar{\rho}$, let $y = f^{-1}(v)$ and $\bar{y} = f^{-1}(\bar{v})$. Then we have that

$$S_{1,\rho}(v) \quad = \quad \frac{f^{-1}(v) - f^{-1}(\rho)}{v - \rho} = \frac{y - x}{f(y) - f(x)} = \frac{1}{T_{1,x}(y)} \text{ and} \tag{5.25}$$

$$S_{1,\bar{\rho}}(\bar{v}) \quad = \quad \frac{f^{-1}(\bar{v}) - f^{-1}(\bar{\rho})}{\bar{v} - \bar{\rho}} = \frac{\bar{y} - \bar{x}}{f(\bar{y}) - f(\bar{x})} = \frac{1}{T_{1,\bar{x}}(\bar{y})}. \tag{5.26}$$

Since f is quasi-linear on $[a, b]$, we have that $T_{1,x} \sim T_{1,\bar{x}}$. Hence Equations (5.25) and (5.26) entail that

$$S_{1,\rho} \sim S_{1,\bar{\rho}} \text{ for all } \rho, \bar{\rho} \in [\alpha, \beta].$$

Finally, using Corollary 5.9 and the fact that $T_{1,x} \sim T_{1,\bar{x}}$ and $T_{2,x} \sim T_{2,\bar{x}}$ for all $x, \bar{x} \in [a, b]$, we can easily verify that

$$S_{2,\rho} \sim S_{2,\bar{\rho}} \text{ for all } \rho, \bar{\rho} \in [\alpha, \beta],$$

where $S_{2,\rho}$ denotes the second secant of f^{-1} at ρ and $T_{2,x}$ the second secant of f at x. Hence f^{-1} is quasi-linear on $[\alpha, \beta]$.

Chapter 6

Expandable Functions

In this chapter, we will present a detailed study of a large class of functions on $\mathcal{R}$, which are given locally by power series with $\mathcal{R}$ coefficients [41, 42]. We show that these functions have all the desired smoothness properties; in particular they satisfy all the common theorems of real calculus.

6.1 Definition and Algebraic Properties

Definition 6.1 *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$, let $f : [a, b] \to \mathcal{R}$ and let $x_0 \in [a, b]$. Then we say that f is expandable at x_0 if and only if there exists $\delta > 0$, finite in $\mathcal{R}$, and there exists a regular sequence $(a_n(x_0))$ in $\mathcal{R}$ such that, under weak convergence, $f(x) = \sum_{n=0}^{\infty} a_n(x_0)(x - x_0)^n$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$.*

Definition 6.2 *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$ and let $f : [a, b] \to \mathcal{R}$. Then we say that f is expandable on $[a, b]$ if and only if f is expandable at each $x \in [a, b]$.*

Definition 6.3 *Let $a < b$ in $\mathcal{R}$ be such that $t = \lambda(b - a) \neq 0$ and let $f : [a, b] \to \mathcal{R}$. Then we say that f is expandable on $[a, b]$ if and only if the function $g : [d^{-t}a, d^{-t}b] \to \mathcal{R}$, given by*

$$g(x) = f(d^t x),$$

is expandable on $[d^{-t}a, d^{-t}b]$.

Lemma 6.1 *Let* $a, b \in \mathcal{R}$ *be such that* $0 < b - a \sim 1$, *let* $f, g : [a, b] \to \mathcal{R}$ *be expandable on* $[a, b]$ *and let* $\alpha \in \mathcal{R}$ *be given. Then* $f + \alpha g$ *and* $f \cdot g$ *are expandable on* $[a, b]$.

Proof. Let $x \in [a, b]$ be given. Then there exist finite $\delta_1 > 0$ and $\delta_2 > 0$, and there exist regular sequences (a_n) and (b_n) in $\mathcal{R}$ such that

$$0 \le |h| < \delta_1 \quad \Rightarrow \quad f(x + h) = \sum_{n=0}^{\infty} a_n h^n \text{ and}$$

$$0 \le |h| < \delta_2 \quad \Rightarrow \quad g(x + h) = \sum_{n=0}^{\infty} b_n h^n.$$

Let $\delta = \min\{\delta_1/2, \delta_2/2\}$. Then $0 < \delta \sim 1$. Moreover, for $0 \le |h| < \delta$, we have that

$$\begin{aligned} (f + \alpha g)(x + h) &= f(x + h) + \alpha g(x + h) \\ &= \sum_{n=0}^{\infty} a_n h^n + \alpha \sum_{n=0}^{\infty} b_n h^n \\ &= \sum_{n=0}^{\infty} a_n h^n + \sum_{n=0}^{\infty} (\alpha b_n) h^n \\ &= \sum_{n=0}^{\infty} (a_n + \alpha b_n) h^n, \end{aligned}$$

where $\sum_{n=0}^{\infty} (a_n + \alpha b_n) h^n$ converges weakly and where the sequence $(a_n + \alpha b_n)$ is regular by Lemma 4.1. Thus $(f + \alpha g)$ is expandable at x. This is true for all $x \in [a, b]$; hence $(f + \alpha g)$ is expandable on $[a, b]$.

Now for each n, let

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}.$$

Then the sequence (c_n) is regular by Lemma 4.1. Since $\sum_{n=0}^{\infty} a_n h^n$ converges weakly for all h satisfying $x + h \in [a, b]$ and $0 \le |h| < \delta_1$, so does $\sum_{n=0}^{\infty} a_n [t] h^n$ for all

$t \in \cup_n \text{supp}(a_n)$. Hence $\sum_{n=0}^{\infty} |(a_n [t] h^n)[q]|$ converges in R for all $q \in Q$, for all $t \in \cup_n \text{supp}(a_n)$ and for all h satisfying

$$x + h \in [a, b], \ 0 \leq |h| < \frac{3}{2}\delta \leq \frac{3}{4}\delta_1 \text{ and } |h| \not\approx \frac{3}{2}\delta.$$

Now let $h \in \mathcal{R}$ be such that $x + h \in [a, b]$ and $0 \leq |h| < \delta$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} |(a_n h^n)[q]| &= \sum_{n=0}^{\infty} \left| \sum_{\substack{q_1 \in \text{supp}(a_n),\, q_2 \in \text{supp}(h^n) \\ q_1 + q_2 = q}} a_n [q_1] h^n [q_2] \right| \\ &\leq \sum_{\substack{q_1 \in \cup_{n=0}^{\infty}\text{supp}(a_n),\, q_2 \in \cup_{n=0}^{\infty}\text{supp}(h^n) \\ q_1 + q_2 = q}} \sum_{n=0}^{\infty} |a_n [q_1]| \, |h^n [q_2]| \, . \end{aligned}$$

Since $\sum_{n=0}^{\infty} |a_n [q_1]| \, |h^n [q_2]|$ converges in R and since only finitely many terms contribute to the sum $\sum_{\substack{q_1 \in \cup_{n=0}^{\infty}\text{supp}(a_n),\, q_2 \in \cup_{n=0}^{\infty}\text{supp}(h^n) \\ q_1 + q_2 = q}}$ by regularity, we obtain that $\sum_{n=0}^{\infty} |(a_n h^n)[q]|$ converges for each $q \in Q$. Since $\sum_{n=0}^{\infty} a_n h^n$ converges absolutely weakly, since $\sum_{n=0}^{\infty} b_n h^n$ converges weakly and since the sequences $\left(\sum_{m=0}^{n} a_m h^m\right)$ and $\left(\sum_{m=0}^{n} b_m h^m\right)$ are both regular, we obtain by Theorem 4.9 that

$$\sum_{n=0}^{\infty} a_n h^n \cdot \sum_{n=0}^{\infty} b_n h^n = \sum_{n=0}^{\infty} c_n h^n; \text{ hence}$$

$$(f \cdot g)(x + h) = \sum_{n=0}^{\infty} c_n h^n.$$

Thus $(f \cdot g)$ is expandable at x. This is true for all $x \in [a, b]$; hence $(f \cdot g)$ is expandable on $[a, b]$.

Corollary 6.1 *Let $a < b$ in $\mathcal{R}$ be given, let $f, g : [a, b] \to \mathcal{R}$ be expandable on $[a, b]$ and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are expandable on $[a, b]$.*

Proof. Let $t = \lambda(b - a)$, and let $F, G : [d^{-t}a, d^{-t}b]$ be given by

$$F(x) = f(d^t x) \text{ and } G(x) = g(d^t x).$$

Then, by definition, F and G are both expandable on $[d^{-t}a, d^{-t}b]$; and hence so is $F + \alpha G$ by Lemma 6.1. For all $x \in [d^{-t}a, d^{-t}b]$, we have that

$$\begin{aligned}(F+\alpha G)(x) &= F(x) + \alpha G(x)\\ &= f(d^t x) + \alpha g(d^t x)\\ &= (f+\alpha g)(d^t x).\end{aligned}$$

Since $(F + \alpha G)$ is expandable on $[d^{-t}a, d^{-t}b]$, so is $(f + \alpha g)$ on $[a, b]$.

Also by Lemma 6.1, we have that $(F \cdot G)$ is expandable on $[d^{-t}a, d^{-t}b]$, where for all $x \in [d^{-t}a, d^{-t}b]$,

$$\begin{aligned}(F\cdot G)(x) &= F(x)\cdot G(x)\\ &= f(d^t x)\cdot g(d^t x)\\ &= (f\cdot g)(d^t x).\end{aligned}$$

Hence $(f \cdot g)$ is expandable on $[a, b]$.

Lemma 6.2 *Let $a < b$ and $c < e$ in $\mathcal{R}$ be such that $b - a$ and $e - c$ are both finite. Let $f : [a, b] \to \mathcal{R}$ be expandable on $[a, b]$, let $g : [c, e] \to \mathcal{R}$ be expandable on $[c, e]$, and let $f([a, b]) \subset [c, e]$. Then $g \circ f$ is expandable on $[a, b]$.*

Proof. Let $x \in [a, b]$ be given. There exist finite $\delta_1 > 0$ and $\delta_2 > 0$, and there exist regular sequences (a_n) and (b_n) in $\mathcal{R}$ such that

$$|h| < \delta_1 \text{ and } x + h \in [a, b] \;\Rightarrow\; f(x+h) = f(x) + \sum_{n=1}^{\infty} a_n h^n \text{ ; and}$$

$$|y| < \delta_2 \text{ and } f(x) + y \in [c, e] \;\Rightarrow\; g(f(x) + y) = g(f(x)) + \sum_{n=1}^{\infty} b_n y^n.$$

Since $F(h) = \left(\sum_{n=1}^{\infty} a_n h^n\right)[0]$ is continuous on R, we can choose $\delta \in (0, \delta_1/2]$ such that

$$|h| < \delta \text{ and } x + h \in [a, b] \Rightarrow \left|\sum_{n=1}^{\infty} a_n h^n\right| < \frac{\delta_2}{2}.$$

Thus, for $|h| < \delta$ and $x+h \in [a,b]$, we have that

$$\begin{aligned}
(g \circ f)(x+h) &= g(f(x+h)) \\
&= g\left(f(x) + \sum_{n=1}^{\infty} a_n h^n\right) \\
&= g(f(x)) + \sum_{k=1}^{\infty} b_k \left(\sum_{n=1}^{\infty} a_n h^n\right)^k \\
&= (g \circ f)(x) + \sum_{k=1}^{\infty} b_k \left(\sum_{n=1}^{\infty} a_n h^n\right)^k . \qquad (6.1)
\end{aligned}$$

For each k, let $V_k(h) = b_k \left(\sum_{n=1}^{\infty} a_n h^n\right)^k$. Then $V_k(h)$ is a power series in h

$$V_k(h) = \sum_{j=1}^{\infty} a_{kj} h^j,$$

where the sequence $(a_{kj})_{j \geq 1}$ is regular in $\mathcal{R}$ for each k. By our choice of δ, we have that for all $q \in Q$, $\sum_{j=1}^{\infty} |(a_{kj} h^j)[q]|$ converges in R; so we can rearrange the terms in $V_k(h)[q] = \sum_{j=1}^{\infty} (a_{kj} h^j)[q]$. Moreover, the double sum $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (a_{kj} h^j)[q]$ converges; so (see for example [31] pp 205-208) we obtain that

$$\begin{aligned}
((g \circ f)(x+h))[q] &= ((g \circ f)(x))[q] + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left(a_{kj} h^j\right)[q] \\
&= ((g \circ f)(x))[q] + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(a_{kj} h^j\right)[q]
\end{aligned}$$

for all $q \in Q$. Therefore,

$$\begin{aligned}
(g \circ f)(x+h) &= (g \circ f)(x) + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj} h^j \\
&= (g \circ f)(x) + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{kj} h^j .
\end{aligned}$$

Thus rearranging and regrouping the terms in Equation (6.1), we obtain that

$$(g \circ f)(x+h) = (g \circ f)(x) + \sum_{l=1}^{\infty} c_l h^l,$$

where the sequence (c_l) is regular.

Corollary 6.2 *Let $a < b$ and $c < e$ in $\mathcal{R}$ be given. Let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$, let $g : [c,e] \to \mathcal{R}$ be expandable on $[c,e]$, and let $f([a,b]) \subset [c,e]$. Then $g \circ f$ is expandable on $[a,b]$.*

Lemma 6.3 *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$, and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Then f is bounded on $[a,b]$.*

Proof. Let $(a_n(a))$ and $(a_n(b))$ be the expansion coefficients of f to the right of a and to the left of b, respectively. Let $a_R = \Re(a)$ and $b_R = \Re(b)$, and define $\overline{f} : [a,b] \cup [a_R, b_R] \to \mathcal{R}$ by

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in [a,b] \\ \sum_{n=0}^{\infty} a_n(a)(x-a)^n & \text{if } x \in [a_R, a) \\ \sum_{n=0}^{\infty} a_n(b)(x-b)^n & \text{if } x \in (b, b_R] \end{cases}.$$

Then $\overline{f}$ is expandable on $[a,b] \cup [a_R, b_R]$. For all $X \in [a_R, b_R] \cap R$ there exists $\delta > 0$ in R and there exists a regular sequence $(a_n(X))$ in $\mathcal{R}$ such that $f(x) = \sum_{n=0}^{\infty} a_n(X)(x-X)^n$ for all $x \in (X - \delta(X), X + \delta(X)) \cap [a,b]$. We have that $\{(X - \delta(X)/2, X + \delta(X)/2) \cap R : X \in [a_R, b_R] \cap R\}$ is a real open cover of the compact real set $[a_R, b_R] \cap R$. There exists $m \in Z^+$ and there exist $X_1, \ldots, X_m \in [a_R, b_R] \cap R$ such that

$$[a_R, b_R] \cap R \subset \cup_{j=1}^{m} \left(\left(X_j - \frac{\delta(X_j)}{2}, X_j + \frac{\delta(X_j)}{2} \right) \cap R \right).$$

It follows that

$$[a,b] \cup [a_R, b_R] \subset \cup_{j=1}^{m} (X_j - \delta(X_j), X_j + \delta(X_j)).$$

Let

$$l = \min_{1 \le j \le m} \{ \min \{ \cup_{n=0}^{\infty} \text{supp}(a_n(X_j)) \} \}.$$

Then

$$\left|\overline{f}(x)\right| < d^{l-1} \text{ for all } x \in [a,b] \cup [a_R, b_R],$$

and hence $\overline{f}$ is bounded on $[a,b] \cup [a_R, b_R]$. In particular, f is bounded on $[a,b]$.

Remark 6.1 *In the proof of Lemma 6.3, l is independent of the choice of the cover. It depends only on a, b, and f; we will call it the index of f on $[a,b]$ and we will denote it by $i(f)$. Moreover, for all $\Delta > 0$ in R and for all $X \in (a_R, b_R) \cap R$ there exists $Y_1 \in [a,b] \cap R \cap (X - \Delta, X)$ and there exists $Y_2 \in [a,b] \cap R \cap (X, X + \Delta)$ such that $\lambda(f(Y_1)) = i(f) = \lambda(f(Y_2))$.*

Proof. Let $X_1, \ldots, X_m$ and l be as in the proof of Lemma 6.3. Let $Z_1, \ldots, Z_k \in [a_R, b_R] \cap R$, let $\{(Z_j - \delta(Z_j), Z_j + \delta(Z_j)) \cap R : j \in \{1, \ldots, k\}\}$ be an open cover of $[a_R, b_R]$, with $\delta(Z_j) > 0$ and real for all $j \in \{1, \ldots, k\}$; and let

$$l_1 = \min_{1 \le j \le k} \left\{ \min \left\{ \cup_{n=0}^{\infty} \operatorname{supp}(a_n(Z_j)) \right\} \right\}.$$

Suppose $l_1 \neq l$. Without loss of generality, we may assume that $l < l_1$. In particular, $l < \infty$. Define $f_R : [a_R, b_R] \cap R \to R$ by

$$f_R(Y) = \overline{f}(Y)[l].$$

Then for $Y \in (X_j - \delta(X_j), X_j + \delta(X_j)) \cap [a_R, b_R] \cap R$, we have that

$$\begin{aligned} f_R(Y) &= \left(\sum_{n=0}^{\infty} a_n(X_j)(Y - X_j)^n \right)[l] \\ &= \sum_{n=0}^{\infty} a_n(X_j)[l](Y - X_j)^n. \end{aligned} \tag{6.2}$$

Thus f_R is analytic on $[a_R, b_R] \cap R$. Moreover,

$$f_R(Y) = \overline{f}(Y)[l] = 0$$

for all $Y \in \left(Z_1 - \frac{\delta(Z_1)}{2}, Z_1 + \frac{\delta(Z_1)}{2}\right) \cap [a_R, b_R] \cap R$. Using the Identity Theorem for analytic real functions, we obtain that $f_R(Y) = 0$ for all $Y \in [a_R, b_R] \cap R$. Using Equation (6.2), we obtain that

$$a_n(X_j)[l] = 0 \text{ for all } n \geq 0 \text{ and for all } j \in \{1, \ldots, m\},$$

which contradicts the definition of l. Thus $l_1 = l$.

Corollary 6.3 *Let a, b and f be as in Lemma 6.3 and let $i(f)$ be the index of f on $[a, b]$. Then*

$$i(f) = \min\{supp(f(x)) : x \in [a, b]\}.$$

Corollary 6.4 *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be expandable on $[a, b]$. Then f is bounded on $[a, b]$.*

Definition 6.4 *Let a, b, f and $\overline{f}$ be as in Lemma 6.3 and let $i(f)$ be the index of f on $[a, b]$. The function $f_R : [a_R, b_R] \cap R \to R$, defined by $f_R(X) = \overline{f}(X)[i(f)]$, will be called the underlying real function of f on $[a_R, b_R] \cap R$.*

Remark 6.2 *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$, let $f : [a, b] \to \mathcal{R}$ be expandable on $[a, b]$, and let f_R be the underlying real function of f on $[a_R, b_R] \cap R$. Then, by the proof of Remark 6.1, f_R is analytic on $[a_R, b_R] \cap R$; in particular, f_R is continuous on $[a_R, b_R] \cap R$.*

6.2 Calculus on the Expandable Functions

In this section we show that like in the case of continuous real functions over closed and bounded real intervals, the expandable functions over closed intervals satisfy an intermediate value theorem, a maximum theorem, and a mean value theorem. Consequently, the expandable functions are integrable.

6.2.1 Intermediate Value Theorem

In this section, we state and prove the intermediate value theorem for the expandable functions, which is a generalization of the corresponding result for normal functions [5]. We first find a real intermediate point where the real part of intermediate value is assumed by the real part of the function; then we look for a solution in the infinitely small neighborhood of this real point where the function is given by an infinite power series with $\mathcal{R}$ coefficients. Thus, finding the point where the intermediate value is assumed requires finding a root of a power series. We find this root by first finding a root x_0 of the leading polynomial in the power series and then applying the fixed point theorem in the second iteration, as the proof below will illustrate in details.

Theorem 6.1 (Intermediate Value Theorem) *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$, and let $f : [a, b] \to \mathcal{R}$ be expandable on $[a, b]$. Then f assumes every intermediate value between $f(a)$ and $f(b)$.*

Proof. Let $\overline{f}$ be as in the proof of Lemma 6.3 and let f_R be the underlying real function of f on $[a_R, b_R] \cap R$. Without loss of generality, we may assume that $i(f) = 0$. Now let S be between $f(a)$ and $f(b)$. Without loss of generality, we may assume that $f(a) < 0 = S < f(b)$. Since f_R is continuous on $[a_R, b_R] \cap R$, there exists $X \in [a_R, b_R] \cap R$ such that $f_R(X) = 0$. Let $Z_{f_R} = \{X \in [a_R, b_R] \cap R : f_R(X) = 0\}$, let

$$A = \begin{cases} \emptyset & \text{if } \{a_R, b_R\} \cap Z_{f_R} = \emptyset \\ \{a\} & \text{if } \{a_R, b_R\} \cap Z_{f_R} = \{a_R\} \\ \{b\} & \text{if } \{a_R, b_R\} \cap Z_{f_R} = \{b_R\} \\ \{a, b\} & \text{if } \{a_R, b_R\} \cap Z_{f_R} = \{a_R, b_R\} \end{cases},$$

and let $B = (Z_{f_R} \setminus \{a_R, b_R\}) \cup A$. If there exists $X \in B$ such that $f(X) = 0$, then we are done. So we may assume that $f(X) \neq 0$ for all $X \in B$.

First Claim: There exists $X_0 \in B$ such that for all finite $\Delta > 0$ there exists $x \in (X_0 - \Delta, X_0 + \Delta) \cap [a, b]$ with $\lambda(x - X_0) = 0$ such that $f(x)/f(X_0) < 0$.

Proof of the first claim: Suppose not. Then for all $X \in B$ there exists $\Delta(X) > 0$, finite in $\mathcal{R}$, such that

$$\frac{f(x)}{f(X)} \geq 0 \text{ for all } x \in (X - \Delta(X), X + \Delta(X)) \cap [a, b] \text{ with } \lambda(x - X) = 0. \quad (6.3)$$

Since f_R is continuous on $[a_R, b_R] \cap R$, we have that for all $Y \in ([a_R, b_R] \cap R) \setminus Z_{f_R}$ there exists a real $\Delta(Y) > 0$ such that $f_R(X)/f_R(Y) > 0$ for all $X \in [a_R, b_R] \cap R \cap (Y - 2\Delta(Y), Y + 2\Delta(Y))$. It follows that, for all $Y \in ([a_R, b_R] \cap R) \setminus Z_{f_R}$,

$$\frac{f(x)}{f(Y)} > 0 \text{ for all } x \in (Y - \Delta(Y), Y + \Delta(Y)) \cap [a, b].$$

In particular,

$$\frac{f(x)}{f(Y)} > 0 \text{ for all } x \in (Y - \Delta(Y), Y + \Delta(Y)) \cap [a, b] \text{ with } \lambda(x - Y) = 0. \quad (6.4)$$

Combining Equation (6.3) and Equation (6.4), we obtain that for all $X \in [a_R, b_R] \cap R$ there exists a finite $\delta(X) > 0$ such that

$$\begin{cases} \frac{f(x)}{f(X)} \geq 0 & \begin{array}{l}\text{for all } x \in (X - \delta(X), X + \delta(X)) \cap [a, b] \\ \text{with } \lambda(x - X) = 0, \text{ if } X \in (a_R, b_R)\end{array} \\ \frac{f(x)}{f(a)} \geq 0 & \begin{array}{l}\text{for all } x \in [a, a + \delta(X)) \cap [a, b] \\ \text{with } \lambda(x - a) = 0, \text{ if } X = a_R\end{array} \\ \frac{f(x)}{f(b)} \geq 0 & \begin{array}{l}\text{for all } x \in (b - \delta(X), b] \cap [a, b] \\ \text{with } \lambda(b - x) = 0, \text{ if } X = b_R\end{array} \end{cases}. \quad (6.5)$$

$\{(X - \Re(\delta(X))/2, X + \Re(\delta(X))/2) \cap R : X \in [a_R, b_R] \cap R\}$ is a real open cover of the compact real set $[a_R, b_R] \cap R$. Hence there exists $m \in Z^+$ and there exist $X_1, \ldots, X_m \in [a_R, b_R] \cap R$ such that

$$[a_R, b_R] \cap R \subset \cup_{j=1}^{m} \left(\left(X_j - \frac{\Re(\delta(X_j))}{2}, X_j + \frac{\Re(\delta(X_j))}{2} \right) \cap R \right).$$

Thus

$$[a, b] \subset \cup_{j=1}^{m} (X_j - \delta(X_j), X_j + \delta(X_j)). \quad (6.6)$$

By Equation (6.5), we have for $j \in \{1, \ldots, m\}$ that

$$\begin{cases} \frac{f(x)}{f(X_j)} \geq 0 & \text{for all } x \in (X_j - \delta(X_j), X_j + \delta(X_j)) \cap [a,b] \\ & \text{with } \lambda(x - X_j) = 0, \text{ if } X_j \in (a_R, b_R) \\ \frac{f(x)}{f(a)} \geq 0 & \text{for all } x \in [a, a + \delta(X_j)) \cap [a,b] \\ & \text{with } \lambda(x - X_j) = 0, \text{ if } X_j = a_R \\ \frac{f(x)}{f(b)} \geq 0 & \text{for all } x \in (b - \delta(X_j), b] \cap [a,b] \\ & \text{with } \lambda(x - X_j) = 0, \text{ if } X_j = b_R \end{cases} . \tag{6.7}$$

Using Equation (6.6) and Equation (6.7), we obtain that $f(b)/f(a) \geq 0$, a contradiction to the fact that $f(a) < 0 < f(b)$. This finishes the proof of the first claim.

Since f is expandable at X_0, there exists a real $\delta(X_0) > 0$ and there exists a regular sequence $(a_n(X_0))_{n \geq 1}$ in $\mathcal{R}$ such that

$$0 \leq |h| < \delta(X_0) \Rightarrow f(X_0 + h) = f(X_0) + \sum_{n=1}^{\infty} a_n(X_0) h^n.$$

Now we look for x such that $0 < |x| \ll 1$ and $f(X_0 + x) = S = 0$. That is we look for a root of the equation

$$f(X_0) + \sum_{n=1}^{\infty} a_n(X_0) x^n = 0.$$

Since $f_R(X_0) = 0$, we have that $0 < |f(X_0)| \ll 1$. Let

$$m = \min\{n \geq 1 : \lambda(a_n(X_0)) = 0\}.$$

Such an m exists by virtue of Remark 6.1. Consider the polynomial

$$P(x) = f(X_0) + a_1(X_0)x + \cdots + a_{m-1}(X_0)x^{m-1} + a_m(X_0)x^m.$$

Second Claim: $P(x)$ has a root $x_0 \in \mathcal{R}$.

Proof of the second claim: Suppose not. Then by the Fundamental Theorem of Algebra, we have that m is even and $\frac{a_m(X_0)}{f(X_0)} > 0$. Thus

$$\frac{P(x)}{f(X_0)} > 0 \text{ for all } x \in (-\delta(X_0), \delta(X_0)) \text{ with } \lambda(x) = 0. \tag{6.8}$$

There exist $M_1 > 0$ and $M_2 > 0$ in R such that

$$|P(x)| > M_1 \text{ for all } x \in \left[-\frac{\delta(X_0)}{2}, \frac{\delta(X_0)}{2}\right] \text{ with } \lambda(x) = 0$$

and

$$\left|\sum_{n>m} a_n(X_0)\, x^n\right| < M_2 |x|^{m+1} \text{ for all } x \in \left[-\frac{\delta(X_0)}{2}, \frac{\delta(X_0)}{2}\right].$$

Let

$$\delta_1 = \min\left\{\left(\frac{M_1}{2M_2}\right)^{\frac{1}{m+1}}, \frac{\delta(X_0)}{2}\right\}.$$

Then $\delta_1 > 0$, δ_1 is finite, and

$$\begin{aligned}\left|\sum_{n>m} a_n(X_0)\, x^n\right| &< M_2 |x|^{m+1} \\ &< \frac{M_1}{2} \\ &< \frac{|P(x)|}{2} \text{ for all } x \in (-\delta_1, \delta_1) \text{ with } \lambda(x) = 0.\end{aligned}$$

Thus $f(X_0 + x) = P(x) + \sum_{n>m} a_n(X_0)\, x^n$ has the same sign as $P(x)$ for all $x \in (-\delta_1, \delta_1)$ with $\lambda(x) = 0$. Using Equation (6.8) and the fact that $\delta_1 < \delta(X_0)$, we obtain that $\frac{f(X_0+x)}{f(X_0)} > 0$ for all $x \in (-\delta_1, \delta_1)$ with $\lambda(x) = 0$, or

$$\frac{f(x)}{f(X_0)} > 0 \text{ for all } x \in (X_0 - \delta_1, X_0 + \delta_1) \text{ with } \lambda(x - X_0) = 0,$$

which contradicts the result of the first claim. This finishes the proof of the second claim.

So $P(x)$ has a root $x_0 \in \mathcal{R}$. Since $\lambda(a_k(X_0)) > 0$ for all $k < m$, we obtain that $\lambda(x_0) > 0$. Let $j \in \{1, \ldots, m\}$ be such that

$$\begin{aligned}\lambda(a_n(X_0)\, x_0^n) &> \lambda\left(a_j(X_0)\, x_0^j\right) \text{ for all } n < j \text{ and} \\ \lambda(a_n(X_0)\, x_0^n) &\geq \lambda\left(a_j(X_0)\, x_0^j\right) \text{ for all } n \geq j.\end{aligned} \tag{6.9}$$

Such a j exists by our choice of m and the fact that $\lambda(x_0) > 0$. We look for $x \approx x_0$ such that $0 = g(x) = f(X_0 + x) = P(x) + \sum_{n>m} a_n(X_0) x^n$. Write $x = x_0 + y$ with $\lambda(y) > \lambda(x_0)$. Then, using the results of Theorems 5.20 and 5.21, we have that

$$g(x_0 + y) = g(x_0) + \sum_{k=1}^{\infty} \beta_k(X_0, x_0) y^k = 0, \tag{6.10}$$

where for $k = 1, 2, \ldots$

$$\beta_k(X_0, x_0) = \sum_{n=k}^{\infty} \frac{n \ldots (n-k+1)}{k!} a_n(X_0) x_0^{n-k}.$$

Using Equation (6.9), we obtain that $\lambda(\beta_1(X_0, x_0)) = \lambda\left(a_j(X_0) x_0^{j-1}\right)$ and $\lambda(\beta_k(X_0, x_0)) \geq \lambda\left(a_j(X_0) x_0^{j-k}\right)$ for all $k \geq 2$.

We write Equation (6.10) as a fixed point problem

$$y = \frac{-g(x_0)}{\beta_1(X_0, x_0)} - \sum_{k=2}^{\infty} \frac{\beta_k(X_0, x_0)}{\beta_1(X_0, x_0)} y^k = h(y), \tag{6.11}$$

where $\lambda(g(x_0)) = \lambda\left(\sum_{n>m} a_n(X_0) x_0^n\right) \geq (m+1)\lambda(x_0)$. Note that, by our choice of j,

$$\lambda(\beta_1(X_0, x_0)) = \lambda\left(a_j(X_0) x_0^{j-1}\right) \leq \lambda\left(a_m(X_0) x_0^{m-1}\right) = (m-1)\lambda(x_0).$$

Thus $\lambda(g(x_0)/\beta_1(X_0, x_0)) = \lambda(g(x_0)) - \lambda(\beta_1(X_0, x_0)) \geq 2\lambda(x_0)$.

Let

$$M = \left\{ z \in \mathcal{R} : \lambda(z) \geq \lambda\left(\frac{g(x_0)}{\beta_1(X_0, x_0)}\right) \right\}$$

and let $y \in M$ be given. For all $k \geq 2$, we have that

$$\begin{aligned}
\lambda\left(\frac{\beta_k(X_0, x_0)}{\beta_1(X_0, x_0)} y^{k-1}\right) &> \lambda\left(\frac{\beta_k(X_0, x_0)}{\beta_1(X_0, x_0)} x_0^{k-1}\right) \\
&= \lambda\left(\beta_k(X_0, x_0) x_0^{k-1}\right) - \lambda(\beta_1(X_0, x_0)) \\
&\geq \lambda\left(a_j(X_0) x_0^{j-k} x_0^{k-1}\right) - \lambda\left(a_j(X_0) x_0^{j-1}\right) = 0.
\end{aligned}$$

Thus

$$\lambda\left(\frac{\beta_k(X_0, x_0)}{\beta_1(X_0, x_0)} y^k\right) > \lambda(y) \geq \lambda\left(\frac{g(x_0)}{\beta_1(X_0, x_0)}\right)$$

for all $k \geq 2$. So $-g(x_0)/\beta_1(X_0, x_0)$ is the leading term on the right hand side of Equation (6.11), and hence $h(y) \approx -g(x_0)/\beta_1(X_0, x_0)$. Thus $h(M) \subset M$. Now let $y_1, y_2 \in M$ be given. Then

$$h(y_1) - h(y_2) = (y_2 - y_1)\sum_{k=2}^{\infty}\left(\frac{\beta_k(X_0, x_0)}{\beta_1(X_0, x_0)}\left(\sum_{l=0}^{k-1} y_2^l y_1^{k-1-l}\right)\right).$$

Since $\lambda(y_1) \geq 2\lambda(x_0)$ and since $\lambda(y_2) \geq 2\lambda(x_0)$, we have for all $k \geq 2$ that

$$\begin{aligned}
& \lambda\left(\frac{\beta_k(X_0, x_0)}{\beta_1(X_0, x_0)}\left(\sum_{l=0}^{k-1} y_2^l y_1^{k-1-l}\right)\right) \geq \lambda\left(\frac{\beta_k(X_0, x_0)}{\beta_1(X_0, x_0)} x_0^{2(k-1)}\right) \\
= \;& \lambda(\beta_k(X_0, x_0)) - \lambda(\beta_1(X_0, x_0)) + 2(k-1)\lambda(x_0) \\
\geq \;& \lambda\left(a_j(X_0)x_0^{j-k}\right) - \lambda\left(a_j(X_0)x_0^{j-1}\right) + (2k-2)\lambda(x_0) \\
= \;& (k-1)\lambda(x_0) \\
\geq \;& \lambda(x_0) \\
> \;& \frac{\lambda(x_0)}{2} > 0.
\end{aligned}$$

Hence

$$|h(y_1) - h(y_2)| \ll d^{\frac{\lambda(x_0)}{2}} |y_1 - y_2| \text{ for all } y_1, y_2 \in M,$$

where

$$d^{\frac{\lambda(x_0)}{2}} \ll 1.$$

So h is contracting on M; and hence, using Theorem 3.3, h has a (unique) fixed point y_0 in M. Thus

$$f(X_0 + x_0 + y_0) = 0.$$

Remark 6.3 *In the proof of Theorem 6.1,* $g(x) = f(X_0) + \sum_{n=1}^{\infty} a_n(X_0)x^n$ *has at most* m *roots in* $\mathcal{R}_s = \{x \in \mathcal{R} : 0 < |x| \ll 1\}$.

Proof. Let $x_1, \ldots, x_{m_0}$ be the roots of

$$P(x) = f(X_0) + a_1(X_0)x + \cdots + a_{m-1}(X_0)x^{m-1} + a_m(X_0)x^m$$

in $\mathcal{R}_s$. Then $m_0 \leq m$. By the proof of Theorem 6.1, we have that for each $l \in \{1, \ldots, m_0\}$ there exists a unique $\eta_l \approx x_l$ in $\mathcal{R}_s$ such that $g(\eta_l) = 0$. We show that $\eta_1, \ldots, \eta_{m_0}$ are the only roots of $g(x)$ in $\mathcal{R}_s$. So let $\eta \in \mathcal{R}_s$ be such that $g(\eta) = 0$. Thus

$$f(X_0) + a_1(X_0)\eta + \cdots + a_m(X_0)\eta^m + \sum_{n>m} a_n(X_0)\eta^n = 0,$$

and hence

$$P(\eta) = f(X_0) + a_1(X_0)\eta + \cdots + a_m(X_0)\eta^m = -\sum_{n>m} a_n(X_0)\eta^n. \tag{6.12}$$

We look for $y \in \mathcal{R}_s$ such that $\lambda(y) > \lambda(\eta)$ and $P(\eta + y) = 0$. Thus,

$$\begin{aligned} 0 &= P(\eta + y) = f(X_0) + a_1(X_0)(\eta + y) + \cdots + a_m(X_0)(\eta + y)^m \\ &= P(\eta) + \sum_{k=1}^{m} \alpha_k(X_0, \eta) y^k, \end{aligned} \tag{6.13}$$

where for $k = 1, 2, \ldots, m$

$$\alpha_k(X_0, \eta) = \sum_{n=k}^{m} \frac{n \ldots (n-k+1)}{k!} a_n(X_0)\eta^{n-k}.$$

Let $j \in \{1, \ldots, m\}$ be such that

$$\begin{aligned} \lambda(a_n(X_0)\eta^n) &> \lambda\left(a_j(X_0)\eta^j\right) \text{ for all } n < j, \text{ and} \\ \lambda(a_n(X_0)\eta^n) &\geq \lambda\left(a_j(X_0)\eta^j\right) \text{ for all } n \in \{j, \ldots, m\}. \end{aligned} \tag{6.14}$$

Thus

$$\begin{aligned} \lambda(\alpha_1(X_0, \eta)) &= \lambda\left(a_j(X_0)\eta^{j-1}\right) \text{ and} \\ \lambda(\alpha_k(X_0, \eta)) &\geq \lambda\left(a_j(X_0)\eta^{j-k}\right) \text{ for all } k \in \{2, \ldots, m\}. \end{aligned} \tag{6.15}$$

We write Equation (6.13) as a fixed point problem

$$\begin{aligned} y &= \frac{-P(\eta)}{\alpha_1(X_0, \eta)} - \sum_{k=2}^{m} \frac{\alpha_k(X_0, \eta)}{\alpha_1(X_0, \eta)} y^k \\ &= h(y), \end{aligned} \tag{6.16}$$

where

$$\begin{aligned}
\lambda\left(-P\left(\eta\right)\right) &= \lambda\left(\sum_{n>m} a_n\left(X_0\right)\eta^n\right) \text{ using Equation (6.12)} \\
&\geq (m+1)\,\lambda\left(\eta\right).
\end{aligned}$$

Note that, by our choice of j,

$$\lambda\left(\alpha_1\left(X_0,\eta\right)\right) = \lambda\left(a_j\left(X_0\right)\eta^{j-1}\right) \leq \lambda\left(a_m\left(X_0\right)\eta^{m-1}\right) = (m-1)\,\lambda\left(\eta\right).$$

Thus

$$\begin{aligned}
\lambda\left(\frac{-P\left(\eta\right)}{\alpha_1\left(X_0,\eta\right)}\right) &= \lambda\left(-P\left(\eta\right)\right) - \lambda\left(\alpha_1\left(X_0,\eta\right)\right) \\
&\geq (m+1)\,\lambda\left(\eta\right) - (m-1)\,\lambda\left(\eta\right) \\
&= 2\lambda\left(\eta\right).
\end{aligned}$$

Let

$$M = \left\{z \in \mathcal{R} : \lambda(z) \geq \lambda\left(\frac{P\left(\eta\right)}{\alpha_1\left(X_0,\eta\right)}\right)\right\}.$$

Let $y \in M$ be given. For all $k \in \{2,\ldots,m\}$, we have that

$$\begin{aligned}
\lambda\left(\frac{\alpha_k\left(X_0,\eta\right)}{\alpha_1\left(X_0,\eta\right)}y^k\right) &= \lambda\left(\frac{\alpha_k\left(X_0,\eta\right)}{\alpha_1\left(X_0,\eta\right)}y^{k-1}\right) + \lambda\left(y\right) \\
&> \lambda\left(\frac{\alpha_k\left(X_0,\eta\right)}{\alpha_1\left(X_0,\eta\right)}\eta^{k-1}\right) + \lambda\left(\frac{P\left(\eta\right)}{\alpha_1\left(X_0,\eta\right)}\right).
\end{aligned}$$

But

$$\begin{aligned}
\lambda\left(\frac{\alpha_k\left(X_0,\eta\right)}{\alpha_1\left(X_0,\eta\right)}\eta^{k-1}\right) &= \lambda\left(\alpha_k\left(X_0,\eta\right)\eta^{k-1}\right) - \lambda\left(\alpha_1\left(X_0,\eta\right)\right) \\
&\geq \lambda\left(a_j\left(X_0\right)\eta^{j-k}\eta^{k-1}\right) - \lambda\left(a_j\left(X_0\right)\eta^{j-1}\right) \\
&= 0.
\end{aligned}$$

Thus

$$\lambda\left(\frac{\alpha_k\left(X_0,\eta\right)}{\alpha_1\left(X_0,\eta\right)}y^k\right) > \lambda\left(\frac{P\left(\eta\right)}{\alpha_1\left(X_0,\eta\right)}\right) \text{ for all } k \in \{2,\ldots,m\}.$$

So $-P(\eta)/\alpha_1(X_0,\eta)$ is the leading term on the right hand side of Equation (6.16), and hence $h(y) \approx -P(\eta)/\alpha_1(X_0,\eta)$. Thus

$$h(M) \subset M.$$

Now let $y_1, y_2 \in M$ be given. Then

$$\begin{aligned} h(y_1) - h(y_2) &= \sum_{k=2}^{m} \frac{\alpha_k(X_0,\eta)}{\alpha_1(X_0,\eta)} \left(y_2^k - y_1^k\right) \\ &= (y_2 - y_1) \sum_{k=2}^{m} \left(\frac{\alpha_k(X_0,\eta)}{\alpha_1(X_0,\eta)} \left(\sum_{l=0}^{k-1} y_2^l y_1^{k-1-l} \right) \right). \end{aligned}$$

Since $\lambda(y_1) \geq 2\lambda(\eta)$ and since $\lambda(y_2) \geq 2\lambda(\eta)$, we have for all $k \in \{2, \ldots, m\}$ that

$$\begin{aligned} & \lambda\left(\frac{\alpha_k(X_0,\eta)}{\alpha_1(X_0,\eta)} \left(\sum_{l=0}^{k-1} y_2^l y_1^{k-1-l} \right) \right) \geq \lambda\left(\frac{\alpha_k(X_0,\eta)}{\alpha_1(X_0,\eta)} \eta^{2(k-1)} \right) \\ = \;& \lambda(\alpha_k(X_0,\eta)) - \lambda(\alpha_1(X_0,\eta)) + 2(k-1)\lambda(\eta) \\ \geq \;& \lambda\left(a_j(X_0)\eta^{j-k}\right) - \lambda\left(a_j(X_0)\eta^{j-1}\right) + (2k-2)\lambda(\eta) \\ = \;& (k-1)\lambda(\eta) \geq \lambda(\eta) \\ > \;& \frac{\lambda(\eta)}{2} > 0. \end{aligned}$$

Hence

$$|h(y_1) - h(y_2)| \ll d^{\frac{\lambda(\eta)}{2}} |y_1 - y_2| \text{ for all } y_1, y_2 \in M,$$

where

$$d^{\frac{\lambda(\eta)}{2}} \ll 1.$$

So h is contracting on M, and hence h has a (unique) fixed point y_0 in M. Thus

$$P(\eta + y_0) = 0 \text{ with } \lambda(y_0) \geq 2\lambda(\eta) > \lambda(\eta).$$

Thus,

$$\eta + y_0 \approx \eta.$$

Since $P(\eta + y_0) = 0$, there exists $l \in \{1, \ldots, m_0\}$ such

$$\eta + y_0 = x_l.$$

Hence

$$\eta \approx x_l.$$

By uniqueness of η_l as a solution to

$$\begin{cases} g(x) = 0 \\ x \approx x_l \end{cases},$$

we obtain that

$$\eta = \eta_l,$$

which finishes the proof of the remark.

Corollary 6.5 *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$, let $\alpha < \beta$ be given in $[a, b]$ and let $f : [a, b] \to \mathcal{R}$ be expandable on $[a, b]$. Then f assumes on $[\alpha, \beta]$ every intermediate value between $f(\alpha)$ and $f(\beta)$.*

Proof. Let $t = \lambda(\beta - \alpha)$, and let $g : [d^{-t}\alpha, d^{-t}\beta] \to \mathcal{R}$ be given by $g(x) = f(d^t x)$. Then $t \geq 0$, and hence g is expandable on $[d^{-t}\alpha, d^{-t}\beta]$. Thus, by Theorem 6.1, g assumes on $[d^{-t}\alpha, d^{-t}\beta]$ every intermediate value between $g(d^{-t}\alpha)$ and $g(d^{-t}\beta)$. Now let S be an intermediate value between $f(\alpha)$ and $f(\beta)$; then S is an intermediate value between $g(d^{-t}\alpha)$ and $g(d^{-t}\beta)$. Hence there exists $\gamma \in [d^{-t}\alpha, d^{-t}\beta]$ such that $g(\gamma) = S$. Let $\eta = d^t\gamma$. Then $\eta \in [\alpha, \beta]$ and $f(\eta) = g(d^{-t}\eta) = g(\gamma) = S$.

Corollary 6.6 *Let $a < b$ in $\mathcal{R}$, let $\alpha < \beta$ in $[a, b]$ be given and let $f : [a, b] \to \mathcal{R}$ be expandable on $[a, b]$. Then f assumes on $[\alpha, \beta]$ every intermediate value between $f(\alpha)$ and $f(\beta)$.*

Proof. Let $t = \lambda(b-a)$ and let $g : [d^{-t}a, d^{-t}b] \to \mathcal{R}$ be given by $g(x) = f(d^t x)$. Then g is expandable on $[d^{-t}a, d^{-t}b]$ by definition, where $0 < d^{-t}b - d^{-t}a \sim 1$. By Corollary 6.5, g assumes on $[d^{-t}\alpha, d^{-t}\beta]$ every intermediate value between $g(d^{-t}\alpha)$ and $g(d^{-t}\beta)$. It follows that f assumes on $[\alpha, \beta]$ every intermediate value between $f(\alpha)$ and $f(\beta)$.

6.2.2 Maximum Theorem and Mimimum Theorem

We start this section by stating the following theorem whose proof follows directly from that of Theorem 5.20.

Theorem 6.2 *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$, and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$ with $i(f) = 0$. Then f is infinitely often differentiable on $[a,b]$, and for all $m \in Z^+$, we have that $f^{(m)}$ is expandable on $[a,b]$. Moreover, if f is given locally around $x_0 \in [a,b]$ by $f(x) = \sum_{n=0}^{\infty} a_n(x_0)(x-x_0)^n$, then $f^{(m)}$ is given by*

$$f^{(m)}(x) = g_m(x) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)\, a_n(x_0)(x-x_0)^{n-m}.$$

In particular, we have that $a_m(x_0) = f^{(m)}(x_0)/m!$ for all $m = 0, 1, 2, \ldots$.

Corollary 6.7 *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Then f is infinitely often differentiable on $[a,b]$, and for all $m \in Z^+$, we have that $f^{(m)}$ is expandable on $[a,b]$. Moreover, if f is given locally around $x_0 \in [a,b]$ by $f(x) = \sum_{n=0}^{\infty} a_n(x_0)(x-x_0)^n$, then $f^{(m)}$ is given by*

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)\, a_n(x_0)(x-x_0)^{n-m}.$$

In particular, we have that $a_m(x_0) = f^{(m)}(x_0)/m!$ for all $m = 0, 1, 2, \ldots$.

The following theorem is again a generalization of the maximum theorem for normal functions [5]; the key step in the proof is to apply the intermediate value theorem, Theorem 6.1, to the derivative function which is itself an expandable function by Theorem 6.2.

Theorem 6.3 *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$, and let $f : [a, b] \to \mathcal{R}$ be expandable on $[a, b]$. Then f assumes a maximum on $[a, b]$.*

Proof. Without loss of generality, we may assume that $i(f) = 0$ and that f_R is not constant on $[a_R, b_R] \cap R$, where $a_R = \Re(a)$ and $b_R = \Re(b)$. Since f_R is continuous on $[a_R, b_R] \cap R$, f_R assumes a maximum M_R on $[a_R, b_R] \cap R$. Since f_R is analytic on $[a_R, b_R] \cap R$, there are only finitely many points $X_1, \ldots, X_k$ in $[a_R, b_R] \cap R$ where f_R assumes its maximum M_R. We look for a maximum of f in the infinitely small neighborhoods of the X_j's. So let $j \in \{1, \ldots, k\}$ be given. Assume $X_j \in (a_R, b_R)$. Then $f_R'(X_j) = 0$ and there exists $\delta_1 > 0$ in R such that

$$\begin{aligned} f_R'(X) &> 0 \text{ for } X \in (X_j - \delta_1, X_j) \cap R \text{ and} \\ f_R'(X) &< 0 \text{ for } X \in (X_j, X_j + \delta_1) \cap R. \end{aligned}$$

Using Theorem 6.2 and the fact that f is expandable at X_j, there exists $\delta \le \delta_1$ in R such that $0 \le |h| < \delta \Rightarrow$

$$f(X_j + h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(X_j)}{n!} h^n \text{ and } f'(X_j + h) = \sum_{n=1}^{\infty} \frac{f^{(n)}(X_j)}{(n-1)!} h^{n-1}.$$

Let

$$m = \min\left\{ n \in Z^+ : \lambda\left(f^{(n+1)}(X_j)\right) = 0 \right\}.$$

Using the intermediate value and its proof, and using Remark 6.3 and its proof, all applied to f', we obtain at least one and at most $(m-1)$ roots of f' that are infinitely close to X_j, and f' changes sign from positive to negative in going from the left to the right of at least one of the roots. Thus we obtain at least one and at most $(m-1)$ local maxima of f in the infinitely small neighborhood of X_j. Let

$$M_j = \max\left\{ f(X_j + h) : |h| \ll 1 \right\}.$$

Similarly we show that f has a maximum in the infinitely small neighborhood of a if $a_R \in \{X_1, \ldots, X_k\}$ and that f has a maximum in the infinitely small neighborhood of b if $b_R \in \{X_1, \ldots, X_k\}$. Let

$$M = \max\{M_j : 1 \leq j \leq k\}.$$

We show that $M = \max\{f(x) : x \in [a,b]\}$. So let $x \in [a,b]$ be given. Suppose x is finitely away from X_j for all $j \in \{1, \ldots, k\}$. Then

$$f(x) - M = (f(x) - f_R(\Re(x))) + (f_R(\Re(x)) - \Re(M)) + (\Re(M) - M).$$

Since $\Re(x) \notin \{X_1, \ldots, X_k\}$, we have that $f_R(\Re(x)) - \Re(M)$ is negative and finite in absolute value. Since $|f(x) - f_R(\Re(x))| \ll 1$ and since $|\Re(M) - M| \ll 1$, we obtain that $f(x) - M < 0$; that is $f(x) < M$.

Now suppose x is infinitely close to one of the X_j's, say X_{j_0}. Then $f(x) \leq M_{j_0} \leq M$. Thus $f(x) \leq M$ for all $x \in [a,b]$. Moreover, M is assumed on $[a,b]$. Hence

$$M = \max\{f(x) : x \in [a,b]\}.$$

Corollary 6.8 *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$, let $\alpha < \beta$ be given in $[a,b]$ and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Then f assumes a maximum on $[\alpha, \beta]$.*

Corollary 6.9 *Let $a < b$ in $\mathcal{R}$ be given, let $\alpha, \beta \in [a,b]$ be such that $\alpha < \beta$ and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Then f assumes a maximum on $[\alpha, \beta]$.*

Proof. Let $t = \lambda(b-a)$ and let $g : [d^{-t}a, d^{-t}b] \to \mathcal{R}$ be given by $g(y) = f(d^t y)$. Then g is expandable on $[d^{-t}a, d^{-t}b]$. Thus there exists $y_1 \in [d^{-t}\alpha, d^{-t}\beta]$ such that $g(y) \leq g(y_1)$ for all $y \in [d^{-t}\alpha, d^{-t}\beta]$. Let $x_1 = d^t y_1$, and let $x \in [\alpha, \beta]$ be given. Then $d^{-t}x \in [d^{-t}\alpha, d^{-t}\beta]$. Thus

$$f(x) = g\left(d^{-t}x\right) \leq g(y_1) = g\left(d^{-t}x_1\right) = f(x_1).$$

Hence $f(x) \leq f(x_1)$ for all $x \in [\alpha, \beta]$.

Corollary 6.10 *Let $a < b$ in $\mathcal{R}$ be given, let $\alpha, \beta \in [a, b]$ be such that $\alpha < \beta$ and let $f : [a, b] \to \mathcal{R}$ be expandable on $[a, b]$. Then f assumes a minimum on $[\alpha, \beta]$.*

Corollary 6.11 *Let $a < b$ in $\mathcal{R}$ be given, let $\alpha, \beta \in [a, b]$ be such that $\alpha < \beta$ and let $f : [a, b] \to \mathcal{R}$ be expandable on $[a, b]$. Then there exist $m, M \in \mathcal{R}$ such that*

$$f([\alpha, \beta]) = [m, M].$$

Proof. By Corollary 6.9 and Corollary 6.10, there exist $x_1, x_2 \in [\alpha, \beta]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [\alpha, \beta]$. Let $m = f(x_1)$ and $M = f(x_2)$. By Corollary 6.6, for each $y \in [m, M]$, there exists $x \in [x_1, x_2] \subset [\alpha, \beta]$ such that $f(x) = y$. Thus, $f([\alpha, \beta]) = [m, M]$.

6.2.3 Rolle's Theorem and the Mean Value Theorem

In this section, we prove Rolle's theorem and the mean value theorem for expandable functions, which will lead the way to an integration theory in Section 6.3.

Theorem 6.4 (Rolle's Theorem) *Let $a < b$ in $\mathcal{R}$ be given, let $\alpha, \beta \in [a, b]$ be such that $\alpha < \beta$ and let $f : [a, b] \to \mathcal{R}$ be expandable. Suppose $f(\alpha) = f(\beta)$. Then there exists $c \in (\alpha, \beta)$ such that $f'(c) = 0$.*

Proof. If $f(x) = f(\alpha)$ for all $x \in [\alpha, \beta]$, then $f'(x) = 0$ for all $x \in (\alpha, \beta)$ and we are done. So we may assume that f is not constant on $[\alpha, \beta]$. Then f has either a maximum or a minimum at some $c \in (\alpha, \beta)$. Using Corollary 6.7 and Lemma 5.9, we obtain that f is topologically differentiable at c. Using Theorem 5.7 and Corollary 5.6, we finally obtain that $f'(c) = 0$.

Like the intermediate value theorem and the maximum theorem, the following mean value theorem is a generalization of the corresponding result for the normal functions [5].

Theorem 6.5 (Mean Value Theorem) *Let $a < b$ in $\mathcal{R}$ be given, let $\alpha, \beta \in [a,b]$ be such that $\alpha < \beta$ and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Then there exists $c \in (\alpha, \beta)$ such that*

$$f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}.$$

Proof. Let $F : [a,b] \to \mathcal{R}$ be given by

$$F(x) = f(x) - f(\alpha) - \frac{f(\beta) - f(\alpha)}{\beta - \alpha}(x - \alpha).$$

Then F is expandable on $[a,b]$. Moreover,

$$F(\alpha) = F(\beta) = 0.$$

Thus, by Theorem 6.4, there exists $c \in (\alpha, \beta)$ such that $F'(c) = 0$; that is

$$0 = F'(c) = f'(c) - \frac{f(\beta) - f(\alpha)}{\beta - \alpha},$$

which finishes the proof of the theorem.

As a direct consequence of the Mean Value Theorem, we obtain the following important result.

Corollary 6.12 *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Then the following are true.*

(i) If $f'(x) \neq 0$ for all $x \in (a,b)$ then either $f'(x) > 0$ for all $x \in (a,b)$ and f is strictly increasing on $[a,b]$, or $f'(x) < 0$ for all $x \in (a,b)$ and f is strictly decreasing on $[a,b]$.

(ii) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Proof.

(i) Suppose $f'(x) \neq 0$ for all $x \in (a,b)$. Then applying the intermediate value theorem to f', we infer that either $f'(x) > 0$ for all $x \in (a,b)$ or $f'(x) < 0$ for all $x \in (a,b)$. First assume that $f'(x) > 0$ for all $x \in (a,b)$ and let $x, y \in [a,b]$ be such that $y > x$. By Theorem 6.5, there exists $c \in (x,y) \subset (a,b)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since $c \in (a,b)$, we have that $f'(c) > 0$; and hence $f(y) > f(x)$. Thus, f is strictly increasing on $[a,b]$.

Now assume that $f'(x) < 0$ for all $x \in (a,b)$ and let $x, y \in [a,b]$ be such that $y > x$. Then there exists $c \in (x,y) \subset (a,b)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since $c \in (a,b)$, we have that $f'(c) < 0$; and hence $f(y) < f(x)$. Thus, f is strictly decreasing on $[a,b]$.

(ii) Suppose $f'(x) = 0$ for all $x \in (a,b)$, and let $y \in [a,b]$ be given. There exists $c \in (a,y) \subset (a,b)$ such that

$$f'(c) = \frac{f(y) - f(a)}{y - a}.$$

Since $c \in (a,b)$, we have that $f'(c) = 0$; and hence $f(y) = f(a)$. Thus $f(y) = f(a)$ for all $y \in [a,b]$.

6.3 Integration

In this section, we develop an integration theory on the class of expandable functions, which is a generalization of the integration theory developed in [5] for the normal functions.

Definition 6.5 *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Then a function $F : [a,b] \to \mathcal{R}$ is said to be an expandable primitive of f on $[a,b]$ if and only if F is expandable on $[a,b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$.*

Lemma 6.4 *Let $a, b \in \mathcal{R}$ be given such that $0 < b - a \sim 1$, and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Then f has an expandable primitive F on $[a,b]$.*

Proof. Using the proof of Lemma 6.3, there exists $m \in Z^+$ and there exist $x_1, \ldots, x_m \in [a,b]$ such that

$$[a,b] \subset \cup_{j=1}^{m} \left(x_j - \delta(x_j), x_j + \delta(x_j)\right),$$

where for all $j \in \{1, \ldots, m\}$, $\delta(x_j)$ is a real domain of expansion of f around x_j. For all $j \in \{1, \ldots, m\}$, there exists a regular sequence $(a_n(x_j))$ in $\mathcal{R}$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n(x_j)(x - x_j)^n \text{ for all } x \in \left(x_j - \delta(x_j), x_j + \delta(x_j)\right).$$

Define $F : [a,b] \to \mathcal{R}$ by

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n(x_j)}{(n+1)!}(x - x_j)^{n+1} \text{ for all } x \in \left(x_j - \delta(x_j), x_j + \delta(x_j)\right);$$

for all $j \in \{1, \ldots, m\}$. Using Theorem 5.21, we obtain that F is expandable on $[a,b]$. Using Theorem 6.2, we obtain that

$$F'(x) = f(x) \text{ for all } x \in [a,b].$$

Hence F is an expandable primitive of f on $[a,b]$.

Corollary 6.13 *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Then f has an expandable primitive F on $[a,b]$.*

Lemma 6.5 *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Let F_1 and F_2 be two expandable primitives of f on $[a,b]$. Then there exists a constant $c \in \mathcal{R}$ such that $F_2(x) = F_1(x) + c$ for all $x \in [a,b]$.*

Proof. Let $F : [a,b] \to \mathcal{R}$ be given by $F(x) = F_2(x) - F_1(x)$. Then F is expandable on $[a,b]$ and $F'(x) = F_2'(x) - F_1'(x) = f(x) - f(x) = 0$ for all $x \in [a,b]$. By Corollary 6.12, F is constant on $[a,b]$.

Corollary 6.14 *Let $a < b$ in $\mathcal{R}$ be given, let $\alpha, \beta \in [a,b]$ be given and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. Let F_1 and F_2 be two expandable primitives of f on $[a,b]$. Then $F_2(\beta) - F_2(\alpha) = F_1(\beta) - F_1(\alpha)$.*

Definition 6.6 *Let $a < b$ in $\mathcal{R}$ be given, let $\alpha, \beta \in [a,b]$ be given and let $f : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$. We define the integral of f from α to β, denoted by $\int_\alpha^\beta f$, as follows: Let $F : [a,b] \to \mathcal{R}$ be an expandable primitive of f on $[a,b]$, which exists by Corollary 6.13 and let*

$$\int_\alpha^\beta f = F(\beta) - F(\alpha).$$

Remark 6.4 *By Corollary 6.14, the integral in Definition 6.6 is independent of the choice of the expandable primitive function F; it depends only on f, α, and β, and hence it is well defined.*

Theorem 6.6 *Let $a < b$ in $\mathcal{R}$ be given, let $\alpha, \gamma, \beta \in [a,b]$ be given, let $f, g : [a,b] \to \mathcal{R}$ be expandable on $[a,b]$ and let $\kappa \in \mathcal{R}$ be given. Then*

$$\begin{aligned} \int_\alpha^\beta (f + \kappa g) &= \int_\alpha^\beta f + \kappa \int_\alpha^\beta g; \text{ and} \\ \int_\alpha^\beta f &= \int_\alpha^\gamma f + \int_\gamma^\beta f. \end{aligned}$$

Proof. Let $F, G : [a,b] \to \mathcal{R}$ be expandable primitives of f and g on $[a,b]$, respectively. Then, using Lemma 6.1, we obtain that $F + \kappa G$ is expandable on $[a,b]$. Moreover, using Theorem 5.13, we obtain that

$$(F + \kappa G)'(x) = F'(x) + \kappa G'(x) = f(x) + \kappa g(x) = (f + \kappa g)(x) \text{ for all } x \in [a,b].$$

Thus, $F+\kappa G$ is an expandable primitive of $f+\kappa g$ on $[a,b]$; and hence

$$\begin{aligned}
\int_{\alpha}^{\beta}(f+\kappa g) &= (F+\kappa G)(\beta)-(F+\kappa G)(\alpha) \\
&= F(\beta)-F(\alpha)+\kappa\left(G(\beta)-G(\alpha)\right) \\
&= \int_{\alpha}^{\beta} f+\kappa\int_{\alpha}^{\beta} g.
\end{aligned}$$

Since F is an expandable primitive of f on $[a,b]$, we also have that

$$\begin{aligned}
\int_{\alpha}^{\beta} f &= F(\beta)-F(\alpha) \\
&= (F(\gamma)-F(\alpha))+(F(\beta)-F(\gamma)) \\
&= \int_{\alpha}^{\gamma} f+\int_{\gamma}^{\beta} f.
\end{aligned}$$

Chapter 7

Computer Functions

In this final chapter, we present one of the applications of the non-Archimedean field $\mathcal{R}$, namely the computation of derivatives of real functions that can be represented on a computer; see also [38, 39, 40, 43].

7.1 Introduction

The general question of efficient differentiation is at the core of many parts of the work on perturbation and aberration theories relevant in Physics and Engineering; for an overview, see for example [11]. In this case, derivatives of highly complicated functions have to be computed to high orders. However, even when the derivative of the function is known to exist at the given point, numerical methods fail to give an accurate value of the derivative; the error increases with the order, and for orders greater than three, the errors often become too large for the results to be practically useful. On the other hand, while formula manipulators like Mathematica are successful in finding low-order derivatives of simple functions, they fail for high-order derivatives of very complicated functions. Consider, for example, the function

$$g(x) = \frac{\sin\left(x^3 + 2x + 1\right) + \frac{3+\cos(\sin(\ln|1+x|))}{\exp\left(\tanh\left(\sinh\left(\cosh\left(\frac{\sin(\cos(\tan(\exp(x))))}{\cos(\sin(\exp(\tan(x+2))))}\right)\right)\right)\right)}}{2 + \sin\left(\sinh\left(\cos\left(\tan^{-1}\left(\ln\left(\exp(x) + x^2 + 3\right)\right)\right)\right)\right)}. \tag{7.1}$$

Using the $\mathcal{R}$ numbers implemented in COSY INFINITY [8, 12], we find $g^{(n)}(0)$ for $0 \leq n \leq 19$. These numbers are listed in Table 7.1; we note that, for $0 \leq n \leq 19$,

Order n	$g^{(n)}(0)$	CPU Time
0	1.004845319007115	1.820 msec
1	0.4601438089634254	2.070 msec
2	-5.266097568233224	3.180 msec
3	-52.82163351991485	4.830 msec
4	-108.4682847837855	7.700 msec
5	16451.44286410806	11.640 msec
6	541334.9970224757	18.050 msec
7	7948641.189364974	26.590 msec
8	-144969388.2104904	37.860 msec
9	-15395959663.01733	52.470 msec
10	-618406836695.3634	72.330 msec
11	-11790314615610.74	97.610 msec
12	403355397865406.1	128.760 msec
13	$0.5510652659782951 \times 10^{17}$	168.140 msec
14	$0.3272787402678642 \times 10^{19}$	217.510 msec
15	$0.1142716430145745 \times 10^{21}$	273.930 msec
16	$-0.6443788542310285 \times 10^{21}$	344.880 msec
17	$-0.5044562355111304 \times 10^{24}$	423.400 msec
18	$-0.5025105824599693 \times 10^{26}$	520.390 msec
19	$-0.3158910204361999 \times 10^{28}$	621.160 msec

Table 7.1: $g^{(n)}(0), 0 \leq n \leq 19$, computed with $\mathcal{R}$ calculus

we list the CPU time needed to obtain all derivatives of g at 0 up to order n and not just $g^{(n)}(0)$. For comparison purposes, we give in Table 7.2 the function value and the first six derivatives computed with Mathematica. Note that the respective values listed in Table 7.1 and Table 7.2 agree. However, Mathematica used a much longer CPU time to compute the first six derivatives, and it failed to find the seventh derivative as it ran out of memory. We also list in Table 7.3 the first ten derivatives of g at 0 computed numerically using the numerical differentiation formulas

$$g^{(n)}(0) = (\Delta x)^{-n}\left(\sum_{j=0}^{n}(-1)^{n-j}\begin{pmatrix} n \\ j \end{pmatrix} g\left(j\Delta x\right)\right),\ \Delta x = 10^{-16/(n+1)},$$

for $1 \leq n \leq 10$, together with the corresponding relative errors obtained by comparing the numerical values with the respective exact values computed with $\mathcal{R}$ calculus.

Order n	$g^{(n)}(0)$	CPU Time
0	1.004845319007116	0.11 sec
1	0.4601438089634254	0.17 sec
2	-5.266097568233221	0.47 sec
3	-52.82163351991483	2.57 sec
4	-108.4682847837854	14.74 sec
5	16451.44286410805	77.50 sec
6	541334.9970224752	693.65 sec

Table 7.2: $g^{(n)}(0), 0 \leq n \leq 6$, computed with Mathematica

Order n	$g^{(n)}(0)$	Relative Error
1	0.4601437841866840	54×10^{-9}
2	-5.266346392944456	47×10^{-6}
3	-52.83767867680922	30×10^{-5}
4	-87.27214664649106	0.20
5	19478.29555909866	0.18
6	633008.9156614641	0.17
7	-12378052.73279768	2.6
8	-1282816703.632099	7.8
9	83617811421.48561	6.4
10	91619495958355.24	149

Table 7.3: $g^{(n)}(0), 1 \leq n \leq 10$, computed numerically

Furthermore, numerical methods and formula manipulators fail to find the derivatives of certain functions at given points even though the functions are differentiable at the respective points. For example, the functions

$$g_1(x) = |x|^{5/2} \cdot g(x) \text{ and } g_2(x) = \begin{cases} \frac{1-\exp(-x^2)}{x} \cdot g(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \tag{7.2}$$

where $g(x)$ is the function given in Equation (7.1) above, are both differentiable at 0; but the attempt to compute their derivatives using formula manipulators fails. This is not specific to g_1 and g_2, and is generally connected to the occurrence of nondifferentiable parts that do not affect the differentiability of the end result, of which case g_1 is an example, as well as the occurrence of branch points in coding as in IF-ELSE structures, of which case g_2 is an example.

One of the applications of the non-Archimedean field $\mathcal{R}$ deals with many of the general problems connected to computational differentiation [38, 39, 40]. Using the calculus on $\mathcal{R}$, we formulate a necessary and sufficient condition for the derivatives of functions from R into R representable on a computer to exist, and show how to find these derivatives whenever they exist.

7.2 Computer Functions of One Variable

At the machine level, a function $f : R \to R$ is characterized by what it does to the original set of memory locations. So f induces a function $\vec{F}(f) : R^m \to R^m$, where m is the number of memory locations affected in the process of computing f. We note here that, without compiler optimization, $\vec{F}(f)$ is unique up to flipping of the memory locations; on the other hand, with compiler optimization, $\vec{F}(f)$ is unique in the subspace describing the true variables. Moreover, at the machine level, any code constitutes solely of intrinsic functions, arithmetic operations and branches. In the following, we formally define the machine level representations of intrinsic functions, the Heaviside function, and the arithmetic operations.

Definition 7.1 *Let* $\mathcal{I}= \{H, \sin, \cos, \tan, \exp, \ldots\}$ *be the set consisting of the Heaviside function* H *and all the intrinsic functions on a computer, which for the sake of convenience are assumed to include the reciprocal function; and let* $\mathcal{O}= \{+, \cdot\}$.

Definition 7.2 *For* $f \in \mathcal{I}$*, define* $\vec{F}_{i,k,f} : R^m \to R^m$ *by*

$$\vec{F}_{i,k,f}(x_1, x_2, \ldots, x_m) = (x_1, \ldots, x_{k-1}, \underbrace{f(x_i)}_{k}, x_{k+1}, \ldots, x_m);$$

so the kth memory location is replaced by $f(x_i)$. Then $\vec{F}_{i,k,f}$ is the machine level representation of f. For $\otimes \in \mathcal{O}$, define $\vec{F}_{i,j,k,\otimes} : R^m \to R^m$ by

$$\vec{F}_{i,j,k,\otimes}(x_1, x_2, \ldots, x_m) = (x_1, \ldots, x_{k-1}, \underbrace{x_i \otimes x_j}_{k}, x_{k+1}, \ldots, x_m),$$

so the kth memory location is replaced by $x_i \otimes x_j$. Then $\vec{F}_{i,j,k,\otimes}$ is the machine level representation of $\otimes$. Finally, let

$$\mathcal{F} = \{\vec{F}_{i,k,f} : f \in \mathcal{I}\} \cup \{\vec{F}_{i,j,k,\otimes} : \otimes \in \mathcal{O}\}.$$

Definition 7.3 *A function $f : R \to R$ is called a computer function if and only if it can be obtained from intrinsic functions and the Heaviside function through a finite number of arithmetic operations and compositions. In this case, there are some $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_N \in \mathcal{F}$ such that $\vec{F}(f) = \vec{F}_N \circ \vec{F}_{N-1} \circ \cdots \circ \vec{F}_2 \circ \vec{F}_1$, and we call $\vec{F}(f) : R^m \to R^m$, already mentioned above, the machine level representation of f.*

Obviously, the so defined class of computer functions in a formal way describes all those functions that can be evaluated on a computer. Since we will be studying only computer functions, it will be useful to define the domain D_c of computer numbers as the subset of the real numbers representable on a computer.

We recall the following result, Corollary 4.10, which allows us to extend all intrinsic functions given by power series to $\mathcal{R}$.

Theorem 7.1 (Power Series with Purely Real Coefficients) *Let $\sum_{n=0}^{\infty} a_n X^n$, $a_n \in R$, be a power series with classical radius of convergence equal to η. Let $x \in \mathcal{R}$, and let $A_n(x) = \sum_{i=0}^{n} a_i x^i \in \mathcal{R}$. Then, for $|x| < \eta$ and $|x| \not\approx \eta$, the sequence $(A_n(x))$ converges absolutely weakly. We define the limit to be the continuation of the power series on $\mathcal{R}$.*

Remark 7.1 *The continuation $\bar{H}$ of the real Heaviside function H is defined for all $x \in \mathcal{R}$ by*

$$\bar{H}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} .$$

The functions $\sqrt[n]{x}$ and $1/x$ are continued to $\mathcal{R}$ via the existence of roots and multiplicative inverses on $\mathcal{R}$ (see Section 3.1).

Definition 7.4 *Let $f \in \mathcal{I}$, let D be the domain of definition of f in R, let $x_0 \in D$, and let $s \in \mathcal{R}$. Then we say that f is extendable to $x_0 + s$ if and only if $x_0 + s$ belongs to the domain of definition of $\bar{f}$, the continuation of f to $\mathcal{R}$, where $\bar{f}$ is given by Theorem 7.1 and Remark 7.1.*

Let $f_1, f_2 \in \mathcal{I}$ with domains of definition D_1 and D_2 in R respectively, let $x_0 \in D_1 \cap D_2$, let $s \in \mathcal{R}$, and let $\otimes \in \{+, \cdot\}$. Then we say that $f_2 \otimes f_1$ is extendable to $x_0 + s$ if and only if f_1 and f_2 are both extendable to $x_0 + s$.

Let $f_1, f_2 \in \mathcal{I}$ with domains of definition D_1 and D_2 in R respectively, let $x_0 \in D_1$ be such that $f_1(x_0) \in D_2$, and let $s \in \mathcal{R}$. Then we say that $f_2 \circ f_1$ is extendable to $x_0 + s$ if and only if f_1 is extendable to $x_0 + s$ and f_2 extendable to $f_1(x_0 + s)$.

Finally, let f be a real computer function, let D be the domain of definition of f in R, let $x_0 \in D$, and let $s \in \mathcal{R}$; then f is obtained in finitely many steps from functions in $\mathcal{I}$ via compositions and arithmetic operations. We define extendability of f to $x_0 + s$ inductively.

We have the following result about the local form of computer functions, which will prove useful in studying the differentiability of computer functions.

Theorem 7.2 *Let f be a real computer function with domain of definition D, and let $x_0 \in D$ be such that f is extendable to $x_0 \pm d$. Then there exists $\sigma > 0$ in R such* *that, for* $0 < x < \sigma$,

$$f(x_0 \pm x) = A_0^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x), \tag{7.3}$$

where $A_i^{\pm}(x), 0 \leq i \leq i^{\pm}$, *is a power series in* x *with a radius of convergence no smaller than* σ, $A_i^{\pm}(0) \neq 0$ *for* $i = 1, \ldots, i^{\pm}$, *and the* $q_i^{\pm}$*'s are nonzero rational numbers that are not positive integers.*

Remark 7.2 *Noninteger rational powers may appear in Equation (7.3) as a result of the root function.*

Proof. The statement of the theorem can easily be verified for each $f \in \mathcal{I}$.

Let f_1 and f_2 be two computer functions with domains of definition D_1 and D_2 in R, respectively. Let $x_0 \in D_1 \cap D_2$, let f_1 and f_2 be both extendable to $x_0 \pm d$, and let f_1 and f_2 satisfy Equation (7.3) around x_0. For $\otimes \in \{+, \cdot\}$, let $F_{\otimes} = f_2 \otimes f_1$. Thus we have that

$$\begin{aligned} f_1(x_0 \pm x) &= A_0^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x) \text{ for } x \in (0, \sigma_1), \\ f_2(x_0 \pm x) &= B_0^{\pm}(x) + \sum_{j=1}^{j^{\pm}} x^{t_j^{\pm}} B_j^{\pm}(x) \text{ for } x \in (0, \sigma_2), \end{aligned}$$

where σ_1 and σ_2 are both positive real numbers; $A_i^{\pm}(x), 0 \leq i \leq i^{\pm}$, and $B_j^{\pm}(x), 0 \leq j \leq j^{\pm}$, are power series in x with radii of convergence no smaller than $\sigma = \min\{\sigma_1, \sigma_2\}$; $A_i^{\pm}(0) \neq 0$ for $i \in \{1, \ldots, i^{\pm}\}$ and $B_j^{\pm}(0) \neq 0$ for $j \in \{1, \ldots, j^{\pm}\}$; and the $q_i^{\pm}$'s and the $t_j^{\pm}$'s are nonzero rational numbers that are not positive integers. As a reminder, we note that σ_1, σ_2, the $A_i^{\pm}$'s, the $B_j^{\pm}$'s, the $q_i^{\pm}$'s, and the $t_j^{\pm}$'s depend on x_0.

For $0 < x < \sigma$, we have that

$$F_{\otimes}(x_0 \pm x) = f_2(x_0 \pm x) \otimes f_1(x_0 \pm x) = \left(\sum_{i=0}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x) \right) \otimes \left(\sum_{j=0}^{j^{\pm}} x^{t_j^{\pm}} B_j^{\pm}(x) \right), \tag{7.4}$$

where $q_0^{\pm} = t_0^{\pm} = 0$. It is easy to check that, for $\otimes = +$ or $\otimes = \cdot$, the result in Equation (7.4) is an expression of the form of Equation (7.3).

Now let f_1 and f_2 be two computer functions with domains of definition D_1 and D_2 in R, respectively. Let $x_0 \in D_1$, let f_1 be extendable to $x_0 \pm d$, let f_2 be extendable to $f_1(x_0 \pm d)$, and let f_1 and f_2 satisfy Equation (7.3) around x_0 and $f_1(x_0)$, respectively. Let $F_\circ = f_2 \circ f_1$. Thus we have that

$$\begin{aligned} f_1(x_0 \pm x) &= A_0^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x) \text{ for } x \in (0, \sigma_1), \\ f_2(f_1(x_0) \pm y) &= B_0^{\pm}(y) + \sum_{j=1}^{j^{\pm}} y^{t_j^{\pm}} B_j^{\pm}(y) \text{ for } y \in (0, \sigma_2), \end{aligned}$$

where σ_1 and σ_2 are positive real numbers; $A_i^{\pm}(x), 0 \leq i \leq i^{\pm}$ and $B_j^{\pm}(y), 0 \leq j \leq j^{\pm}$, are power series in x and y with radii of convergence no smaller than $\sigma = \min\{\sigma_1, \sigma_2\}$; $A_i^{\pm}(0) \neq 0$ for $i \in \{1, \ldots, i^{\pm}\}$ and $B_j^{\pm}(0) \neq 0$ for $j \in \{1, \ldots, j^{\pm}\}$; and the $q_i^{\pm}$'s and the $t_j^{\pm}$'s are nonzero rational numbers that are not positive integers. Without loss of generality, we may assume that at least one of the series $B_j^{\pm}(y)$ is infinite. It follows, since f_2 is extendable to $f_1(x_0 \pm d)$, that the $q_i^{\pm}$'s are all positive and that $A_0^{\pm}(0) = f_1(x_0)$. Let $A_{00}^{\pm}(x) = A_0^{\pm}(x) - A_0^{\pm}(0) = A_0^{\pm}(x) - f_1(x_0)$. Then $A_{00}^{\pm}(x)$ has no constant term, and we have, for $0 < x < \sigma_1$, that $f_1(x_0 \pm x) = f_1(x_0) + A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)$. Since $A_{00}^{\pm}(x)$ has no constant term and the $q_i^{\pm}$'s are all positive, there exists $\sigma \in R$, $0 < \sigma \leq \sigma_1$, such that $|A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)| < \sigma_2$ and $A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)$ has the same sign for all x satisfying $0 < x < \sigma$. To prove the last statement, note that since $g^{\pm}(x) = A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)$ is continuous at 0, there exists $\delta_1 \in R$, $0 < \delta_1 \leq \sigma_1$, such that $0 < x < \delta_1 \Rightarrow |g^{\pm}(x) - g^{\pm}(0)| = |A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)| < \sigma_2$. Now let $\alpha^{\pm} x^{q^{\pm}}$ be the leading term of $g^{\pm}(x)$. Write $g^{\pm}(x) = \alpha^{\pm} x^{q^{\pm}} \left(1 + g_1^{\pm}(x)\right)$, where $g_1^{\pm}(x)$ is continuous at 0 and $g_1^{\pm}(0) = 0$. Hence there exists $\delta_2 \in R$, $0 < \delta_2 \leq \sigma_1$, such that $0 < x < \delta_2 \Rightarrow |g_1^{\pm}(x)| < 1/2 \Rightarrow 1 + g_1^{\pm}(x) > 0 \Rightarrow g^{\pm}(x)$ has the same sign as $\alpha^{\pm}$. Let

$\sigma = \min\{\delta_1, \delta_2\}$. Then $0 < \sigma \leq \sigma_1$, and $0 < x < \sigma \Rightarrow |A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)| < \sigma_2$ and $A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)$ has the same sign as $\alpha^{\pm}$. Thus, for $0 < x < \sigma$, we have that

$$\begin{aligned}
F_{\circ}(x_0 \pm x) &= f_2\left(f_1(x_0 \pm x)\right) = f_2\left(f_1(x_0) + A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)\right) \\
&= E_0\left(A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)\right) \\
&\quad + \sum_{j=1}^{J}\left\{\left|A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)\right|^{s_j} E_j\left(A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)\right)\right\},
\end{aligned}$$

where $E_j, 0 \leq j \leq J$, are power series; $E_j(0) \neq 0$ for $1 \leq j \leq J$; and the s_j's are nonzero rational numbers that are not positive integers.

Note that for $1 \leq j \leq J$,

$$\begin{aligned}
\left|A_{00}^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)\right|^{s_j} &= \left|\alpha^{\pm}\right|^{s_j} x^{s_j q^{\pm}} \left(1 + g_1^{\pm}(x)\right)^{s_j} \\
&= \left|\alpha^{\pm}\right|^{s_j} x^{s_j q^{\pm}} S_j(g_1^{\pm}(x)),
\end{aligned}$$

where $g_1^{\pm}(x)$ is of the form of Equation (7.3), $g_1^{\pm}(0) = 0$, $|g_1^{\pm}(x)| < 1/2$, and $S_j(g_1^{\pm}(x)) = \left(1 + g_1^{\pm}(x)\right)^{s_j}$ is a power series in $g_1^{\pm}(x)$. Thus, it suffices to show that a power series of an expression of the form of Equation (7.3), in which the $q_i^{\pm}$'s are all positive and in which $A_0^{\pm}(0) = 0$, yields an expression of the same form.

So let $S(y) = \sum_{m=0}^{\infty} a_m y^m$ be a power series with positive radius of convergence η. Then, for x sufficiently small,

$$S\left(A_0^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)\right) = \sum_{m=0}^{\infty} a_m \left(A_0^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x)\right)^m. \tag{7.5}$$

For each $i \in \{1, \ldots, i^{\pm}\}$, write $q_i^{\pm} = m_i^{\pm}/n_i^{\pm}$, where $m_i^{\pm}$ and $n_i^{\pm}$ are positive and relatively prime. Expanding the powers in Equation (7.5), the only exponents of x

that may occur are of the form $k+s$, where k is a positive integer and

$$s \in T = \left\{ \frac{m_i^\pm}{n_i^\pm}, \ldots, (n_i^\pm - 1)\frac{m_i^\pm}{n_i^\pm} / i = 1, \ldots, i^\pm \right\},$$

a finite set. For each m let $S_m(x) = a_m \left(A_0^\pm(x) + \sum_{i=1}^{i^\pm} x^{q_i^\pm} A_i^\pm(x) \right)^m$. Then S_m is an infinite series

$$S_m(x) = \sum_{n=0}^{\infty} u_{mn}(x), \tag{7.6}$$

where $u_{mn}(x)$ is of the form $a_{mn}x^{k+s}$ with $a_{mn} \in R$, k a positive integer, and $s \in T$. Let η_1 be the radius of convergence of $A_0^\pm(x) + \sum_{i=1}^{i^\pm} x^{q_i^\pm} A_i^\pm(x)$, and let $0 < x < \eta_1/2$ be such that $\left| A_0^\pm(x) + \sum_{i=1}^{i^\pm} x^{q_i^\pm} A_i^\pm(x) \right| < \eta/2$. Then for each m, the sum in Equation (7.6) converges absolutely; so we can rearrange the terms in S_m. Moreover, the double sum $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{mn}(x)$ converges; so (see for example [31], pages 205-208) we obtain that

$$S\left(A_0^\pm(x) + \sum_{i=1}^{i^\pm} x^{q_i^\pm} A_i^\pm(x) \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{mn}(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{mn}(x).$$

Thus rearranging and regrouping the terms in Equation (7.5), we obtain an expression of the form $C_0^\pm(x) + \sum_{p=1}^{p^\pm} x^{r_p^\pm} C_p^\pm(x)$, where $C_p^\pm(x), 0 \le p \le p^\pm$, are power series, $C_p^\pm(0) \neq 0$ for $1 \le p \le p^\pm$, $p^\pm$ is finite, and the $r_p^\pm$'s are nonzero rational numbers which are not positive integers. Hence $S\left(A_0^\pm(x) + \sum_{i=1}^{i^\pm} x^{q_i^\pm} A_i^\pm(x) \right)$ is of the form of Equation (7.3). It follows that $F_\circ(x_0 \pm x)$ in Equation (7.5) is itself of the form of Equation (7.3).

Now let f be a real computer function with domain of definition D, and let $x_0 \in D$ be such that f is extendable to $x_0 \pm d$. Then f is obtained in finitely many steps from functions in $\mathcal{I}$ via compositions and arithmetic operations. Using induction, we obtain the result immediately from the above.

Since the family of computer functions is closed under differentiation to any order n, Theorem 7.2 holds for derivatives of computer functions as well.

Definition 7.5 *(Continuation of Real Computer Functions) Let f be a real computer function with domain of definition D and let $x_0 \in D$ be such that f is extendable to $x_0 \pm d$. Then f is given around x_0 by a finite combination of roots and power series. Since roots and power series have already been extended to $\mathcal{R}$, f is extended to $\mathcal{R}$ around x_0 in a natural way similar to that of the extension of power series from R to C. That is, if $f(x_0 \pm x) = A_0^\pm(x) + \sum_{i=1}^{i_k^\pm} x^{q_i^\pm} A_i^\pm(x)$ for $0 < x < \sigma$, then we have for the continued function $\bar{f}$ that $\bar{f}(x_0 \pm x) = A_0^\pm(x) + \sum_{i=1}^{i_k^\pm} x^{q_i^\pm} A_i^\pm(x)$ for all $x \in \mathcal{R}$ satisfying $0 < x < \sigma$ and $x \not\approx \sigma$.*

Theorem 7.3 *Let f be a computer function that is differentiable at the point $x_0 \in R$ and extendable to $x_0 \pm d$. Then the continued function $\bar{f}$ is topologically differentiable at x_0, and the derivatives of f and $\bar{f}$ at x_0 agree.*

Proof. Since f is differentiable at x_0, there exists $\sigma > 0$ in R such that, for $x \in R$ and $0 < x < \sigma$, $f(x_0 \pm x) = f(x_0) \pm f'(x_0)x + \sum_{i=2}^{\infty} \alpha_i^\pm x^i + \sum_{j=1}^{J^\pm} x^{q_j^\pm} A_j^\pm(x)$; where $q_1^\pm, \ldots, q_{J^\pm}^\pm$ are noninteger rational numbers greater than 1, and $A_0^\pm, A_1^\pm, \ldots, A_{J^\pm}^\pm$ are power series in x. Thus we have for the continued function $\bar{f}$ that $\bar{f}(x_0 \pm x) = f(x_0) \pm f'(x_0)x + \sum_{i=2}^{\infty} \alpha_i^\pm x^i + \sum_{j=1}^{J^\pm} x^{q_j^\pm} A_j^\pm(x)$ for all $x \in \mathcal{R}$ satisfying $0 < x < \sigma$ and $x \not\approx \sigma$. Let

$$q^\pm = \begin{cases} \min\{q_j^\pm; 1 \le j \le J^\pm\} & \text{if } \{q_j^\pm; 1 \le j \le J^\pm\} \neq \emptyset \\ \infty & \text{if } \{q_j^\pm; 1 \le j \le J^\pm\} = \emptyset \end{cases},$$

let $q = \min(q^+, q^-)$, and let $k = \min\{1, q-1\}$. Then $0 < k \le 1$. We show that the continued function $\bar{f}$ is topologically differentiable at x_0, with derivative $\bar{f}'(x_0) = f'(x_0)$.

Let $x \in \mathcal{R}$ satisfy $0 < x < \sigma$ and $x \not\approx \sigma$. Then we have that

$$\left| \frac{\bar{f}(x_0 \pm x) - f(x_0)}{(\pm x)} - f'(x_0) \right| = \left| \pm \sum_{i=2}^{\infty} \alpha_i^\pm x^{i-1} \pm \sum_{j=1}^{J^\pm} x^{q_j^\pm - 1} A_j^\pm(x) \right|.$$

Let $\epsilon > 0$ be given in $\mathcal{R}$. As a first case, assume ϵ is not infinitely small, and let $\epsilon_r = \Re(\epsilon)$, the real part of ϵ. Since the real limit of $\left|\pm \sum_{i=2}^{\infty} \alpha_i^{\pm} y^{i-1} \pm \sum_{j=1}^{J^{\pm}} y^{q_j^{\pm}-1} A_j^{\pm}(y)\right|$, as $y \to 0^+, y \in R$, is equal to zero, there exists $\delta \in R$, $0 < \delta < \sigma/2$, such that

$$\left|\frac{f(x_0 \pm y) - f(x_0)}{(\pm y)} - f'(x_0)\right| < \frac{\epsilon_r}{2} \text{ whenever } y \in R \text{ and } 0 < y < 2\delta.$$

Now let $x \in \mathcal{R}$ be such that $0 < x < \delta$, and let $x_r = \Re(x)$. If $x_r = 0$, then x is infinitely small. Thus $\left|\{\bar{f}(x_0 \pm x) - f(x_0)\}/(\pm x) - f'(x_0)\right|$ is infinitely small, and hence smaller than ϵ. If $x_r \neq 0$, then $0 < x_r < 2\delta$. Therefore,

$$\left|\frac{\bar{f}(x_0 \pm x) - f(x_0)}{(\pm x)} - f'(x_0)\right| =_0 \left|\frac{f(x_0 \pm x_r) - f(x_0)}{(\pm x_r)} - f'(x_0)\right| < \frac{\epsilon_r}{2}.$$

Hence $\left|\{\bar{f}(x_0 + x) - f(x_0)\}/x - f'(x_0)\right| < \epsilon$ whenever $0 < |x| < \delta$.

As a second case, assume ϵ is infinitely small. Let

$$m^{\pm} = \begin{cases} \min\{i \geq 2 : \alpha_i^{\pm} \neq 0\} & \text{if } \{i \geq 2 : \alpha_i^{\pm} \neq 0\} \neq \emptyset \\ \infty & \text{if } \{i \geq 2 : \alpha_i^{\pm} \neq 0\} = \emptyset \end{cases} .$$

If $m^{\pm} = \infty$, let $\alpha_{m^{\pm}}^{\pm} = 0$. With the convention $1/0 = \infty$, let

$$\delta = \frac{1}{2} \min\left\{ \left(\epsilon/|A_1^+(0)|\right)^{1/k}, \left(\epsilon/|A_1^-(0)|\right)^{1/k}, \left(\epsilon/|\alpha_{m^+}^+|\right)^{1/k}, \left(\epsilon/|\alpha_{m^-}^-|\right)^{1/k} \right\}.$$

Then $\delta > 0$, and if $0 < |x| < \delta$ then $|\{\bar{f}(x_0 + x) - f(x_0)\}/x - f'(x_0)| < \epsilon$. Thus $\bar{f}$ is topologically differentiable at x_0, and $\bar{f}'(x_0) = f'(x_0)$.

In the rest of this chapter we will use f instead of $\bar{f}$ to represent the continuation of a real computer function f.

7.3 Computation of Derivatives

In this section, we develop a criterion that will allow us not only to check the continuity and the differentiability of a real computer function f at a point x_0, but also to obtain all existing derivatives of f at x_0.

Lemma 7.1 *Let f be a computer function. Then f is defined at $x_0 \in D_c$ if and only if $f(x_0)$ can be evaluated on a computer.*

This lemma of course hinges on a careful implementation of the intrinsic functions and operations, in particular in the sense that they should be executable for any floating point number in the domain of definition that produces a result within the range of allowed floating point numbers.

Lemma 7.2 *Let f be a computer function, let D be the domain of definition of f in R, let $x_0 \in D \cap D_c$, and let $s \in \mathcal{R}$. Then f is extendable to $x_0 + s$ if and only if $f(x_0 + s)$ can be evaluated on the computer.*

Lemma 7.3 *Let f be a computer function, and let x_0 be such that f is defined at x_0 and extendable to $x_0 \pm d$. Then f is continuous at x_0 if and only if*

$$f(x_0 - d) =_0 f(x_0) =_0 f(x_0 + d).$$

Proof. Since f is a computer function, defined at x_0 and extendable to $x_0 \pm d$, we have that

$$f(x_0 + x) = A_0(x) + \sum_{j=1}^{J_r} x^{q_j} A_j(x) \text{ and } f(x_0 - x) = B_0(x) + \sum_{j=1}^{J_l} x^{t_j} B_j(x)$$

for $0 < x < \sigma$, where σ is a positive real number; where the A_j's and the B_j's are power series in x, where $A_j(0) \neq 0$ for $1 \leq j \leq J_r$ and $B_j(0) \neq 0$ for $1 \leq j \leq J_l$; and where the q_j's and the t_j's are nonzero rational numbers that are not positive integers. Let $A_0(x) = \sum_{i=0}^{\infty} \alpha_i x^i$ and $B_0(x) = \sum_{i=0}^{\infty} \beta_i x^i$. Then f is continuous at x_0 if and only if $q_j > 0$ for all $j \in \{1, \ldots, J_r\}$, $t_j > 0$ for all $j \in \{1, \ldots, J_l\}$, and $\alpha_0 = \beta_0 = f(x_0)$; that is, if and only if $f(x_0 + d) =_0 f(x_0) =_0 f(x_0 - d)$.

Theorem 7.4 *Let f be a computer function that is continuous at x_0 and extendable to $x_0 \pm d$. Then f is differentiable at x_0 if and only if $(f(x_0+d)-f(x_0))/d$ and $(f(x_0)-f(x_0-d))/d$ are both at most finite in absolute value, and their real parts agree. In this case,*

$$\frac{f(x_0+d)-f(x_0)}{d} =_0 f'(x_0) =_0 \frac{f(x_0)-f(x_0-d)}{d}.$$

If f is differentiable at x_0 and extendable to $x_0 \pm d$, then f is twice differentiable at x_0 if and only if $(f(x_0+2d)-2f(x_0+d)+f(x_0))/d^2$ and $(f(x_0)-2f(x_0-d)+f(x_0-2d))/d^2$ are both at most finite in absolute value, and their real parts agree. In this case

$$\frac{f(x_0+2d)-2f(x_0+d)+f(x_0)}{d^2} =_0 f^{(2)}(x_0) =_0 \frac{f(x_0)-2f(x_0-d)+f(x_0-2d)}{d^2}.$$

In general, if f is $(n-1)$ times differentiable at x_0 and extendable to $x_0 \pm d$, then f is n times differentiable at x_0 if and only if $d^{-n}\left(\sum_{j=0}^{n}(-1)^{n-j}\begin{pmatrix} n \\ j \end{pmatrix} f\left(x_0+jd\right)\right)$ and $d^{-n}\left(\sum_{j=0}^{n}(-1)^{j}\begin{pmatrix} n \\ j \end{pmatrix} f\left(x_0-jd\right)\right)$ are both at most finite in absolute value, and their real parts agree. In this case

$$\begin{aligned} d^{-n}\left(\sum_{j=0}^{n}(-1)^{n-j}\begin{pmatrix} n \\ j \end{pmatrix} f\left(x_0+jd\right)\right) \quad &=_0 \quad f^{(n)}(x_0) \\ &=_0 \quad d^{-n}\left(\sum_{j=0}^{n}(-1)^{j}\begin{pmatrix} n \\ j \end{pmatrix} f\left(x_0-jd\right)\right). \end{aligned}$$

Proof. Since f is continuous at x_0, we have that

$$\begin{aligned} f(x_0+x) &= f(x_0)+\sum_{i=1}^{\infty}\alpha_i x^i+\sum_{j=1}^{J_r}x^{q_j}A_j(x) \\ f(x_0-x) &= f(x_0)+\sum_{i=1}^{\infty}\beta_i x^i+\sum_{j=1}^{J_l}x^{t_j}B_j(x) \end{aligned} \tag{7.7}$$

for $0<x<\sigma$, where σ is a positive real number, where the A_j's and the B_j's are power series in x that do not vanish at $x=0$, and where the q_j's and the t_j's are noninteger positive rational numbers. Observe that f is n times differentiable at x_0 if and only if

$$q_j > n \text{ for } 1 \leq j \leq J_r,\ t_j > n \text{ for } 1 \leq j \leq J_l, \text{ and } \alpha_j = (-1)^j \beta_j \text{ for } 1 \leq j \leq n. \tag{7.8}$$

Assume f is differentiable at x_0. Then, using Equation (7.8), we have that

$$q_j > 1 \text{ for all } j \in \{1, \ldots, J_r\},\ t_j > 1 \text{ for all } j \in \{1, \ldots, J_l\}, \text{ and } \alpha_1 = -\beta_1 = f'(x_0).$$

Hence,

$$\frac{f(x_0+d) - f(x_0)}{d} = \sum_{i=1}^{\infty} \alpha_i d^{i-1} + \sum_{j=1}^{J_r} d^{q_j - 1} A_j(d) =_0 \alpha_1 = f'(x_0).$$

Similarly,

$$\frac{f(x_0) - f(x_0-d)}{d} = -\sum_{i=1}^{\infty} \beta_i d^{i-1} - \sum_{j=1}^{J_l} d^{t_j - 1} B_j(d) =_0 -\beta_1 = f'(x_0).$$

Combining the above two equations, we obtain that

$$\frac{f(x_0+d) - f(x_0)}{d} =_0 f'(x_0) =_0 \frac{f(x_0) - f(x_0-d)}{d}.$$

Now assume that $(f(x_0+d) - f(x_0))/d$ and $(f(x_0) - f(x_0-d))/d$ are both at most finite in absolute value, and their real parts agree. Then, using Equation (7.7), $|\sum_{i=1}^{\infty} \alpha_i d^{i-1} + \sum_{j=1}^{J_r} d^{q_j-1} A_j(d)|$ and $|-\sum_{i=1}^{\infty} \beta_i d^{i-1} - \sum_{j=1}^{J_l} d^{t_j-1} B_j(d)|$ are both at most finite, and

$$\sum_{i=1}^{\infty} \alpha_i d^{i-1} + \sum_{j=1}^{J_r} d^{q_j - 1} A_j(d) =_0 -\sum_{i=1}^{\infty} \beta_i d^{i-1} - \sum_{j=1}^{J_l} d^{t_j - 1} B_j(d).$$

Hence,

$$q_j > 1 \text{ for all } j \in \{1, \ldots, J_r\},\ t_j > 1 \text{ for all } j \in \{1, \ldots, J_l\}, \text{ and } \alpha_1 = -\beta_1,$$

from which we infer, using Equation (7.8), that f is differentiable at x_0 with

$$f'(x_0) = \alpha_1 = -\beta_1 =_0 \frac{f(x_0+d) - f(x_0)}{d} =_0 \frac{f(x_0) - f(x_0-d)}{d}.$$

This finishes the proof of the first part of the theorem.

Since the second part of the theorem is only a special case of the last one, with $n = 2$, we will go directly to proving the last part of the theorem. Note that since f is $(n-1)$ times differentiable at x_0,

$$\begin{aligned} f(x_0 + x) &= \sum_{i=0}^{n-1} \frac{f^{(i)}(x_0)}{i!} x^i + \sum_{i=n}^{\infty} \alpha_i x^i + \sum_{j=1}^{J_r} x^{q_j} A_j(x) \\ f(x_0 - x) &= \sum_{i=0}^{n-1} (-1)^i \frac{f^{(i)}(x_0)}{i!} x^i + \sum_{i=n}^{\infty} \beta_i x^i + \sum_{j=1}^{J_l} x^{t_j} B_j(x) \end{aligned}$$

for $0 < x < \sigma$, where σ is a positive real number, where the A_j's and the B_j's are as before, and where the q_j's and the t_j's are noninteger rational numbers greater than $n-1$.

Assume f is n times differentiable at x_0. Then

$$q_j > n \text{ for all } j \in \{1, \ldots, J_r\}, \ t_j > n \text{ for all } j \in \{1, \ldots, J_l\},$$

and

$$n!\ \alpha_n = (-1)^n n!\ \beta_n = f^{(n)}(x_0).$$

It can be shown by induction on n that

$$\begin{aligned} d^{-n} \left(\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x_0 + jd) \right) &=_0 \quad n!\ \alpha_n \text{ and} \\ d^{-n} \left(\sum_{j=0}^{n} (-1)^{j} \binom{n}{j} f(x_0 - jd) \right) &=_0 \quad (-1)^n n!\ \beta_n. \end{aligned}$$

Therefore,

$$\begin{aligned} d^{-n} \left(\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x_0 + jd) \right) &=_0 \quad f^{(n)}(x_0) \\ &=_0 \quad d^{-n} \left(\sum_{j=0}^{n} (-1)^{j} \binom{n}{j} f(x_0 - jd) \right). \end{aligned}$$

Now assume that

$$d^{-n}\left(\sum_{j=0}^{n}(-1)^{n-j}\left(\begin{array}{c}n\\ j\end{array}\right)f\left(x_0+jd\right)\right) \text{ and } d^{-n}\left(\sum_{j=0}^{n}(-1)^{j}\left(\begin{array}{c}n\\ j\end{array}\right)f\left(x_0-jd\right)\right)$$

are both at most finite in absolute value, and their real parts agree. Then

$$q_j > n \text{ for all } j\in\{1,\ldots,J_r\},\ t_j>n \text{ for all } j\in\{1,\ldots,J_l\}, \text{ and } n!\ \alpha_n=(-1)^n n!\ \beta_n,$$

from which we infer, again using Equation (7.8), that f is n times differentiable at x_0 with

$$\begin{aligned} f^{(n)}(x_0)=n!\ \alpha_n=(-1)^n n!\ \beta_n \quad &=_0\quad d^{-n}\left(\sum_{j=0}^{n}(-1)^{n-j}\left(\begin{array}{c}n\\ j\end{array}\right)f\left(x_0+jd\right)\right)\\ &=_0\quad d^{-n}\left(\sum_{j=0}^{n}(-1)^{j}\left(\begin{array}{c}n\\ j\end{array}\right)f\left(x_0-jd\right)\right).\end{aligned}$$

This finishes the proof of the theorem.

Since knowledge of $f(x_0-d)$ and $f(x_0+d)$ gives us all the information about a computer function f extendable to $x_0\pm d$, in a real positive radius σ around x_0, we have the following result which states that, from the mere knowledge of $f(x_0-d)$ and $f(x_0+d)$, we can find at once the order of differentiability of f at x_0 and the accurate values of all existing derivatives.

Theorem 7.5 *Let f be a computer function that is continuous at x_0 and extendable to $x_0\pm d$. Then f is n times differentiable at x_0 if and only if there exist real numbers $\alpha_1,\ldots,\alpha_n$ such that*

$$f(x_0-d)=_n f(x_0)+\sum_{j=1}^{n}(-1)^j\alpha_j d^j \text{ and } f(x_0+d)=_n f(x_0)+\sum_{j=1}^{n}\alpha_j d^j.$$

Moreover, in this case $f^{(j)}(x_0)=j!\ \alpha_j$ for $1\le j\le n$.

7.4 Examples

As a first example, we consider a simple function and study its differentiability at 0. Let $f(x) = x\sqrt{|x|} + \exp(x)$. It is easy to see that f is differentiable at 0 with $f(0) = f'(0) = 1$ and that f is not twice differentiable at 0. We will show now how using the result of Theorem 7.4 will lead us to the same conclusion. First we note that f is defined at 0 and extendable to $\pm d$.

It is useful to look at what goes on inside the computer for this simple example. Altogether, we need six memory locations to store the variable, the intermediate values, and the function value. These six memory locations are

$$\begin{array}{lll} x, & S_1 = \text{abs}(x), & S_2 = \text{sqrt}(S_1), \\ S_3 = x * S_2, & S_4 = \exp(x), & a = S_3 + S_4. \end{array}$$

So we can look at $\vec{F}(f)$ as a function from R^6 into R^6. Let

$$\left\{ \begin{array}{ll} \vec{E}: R \to R^6; & \vec{E}(x) = (x, 0, 0, 0, 0, 0) \\ \vec{F}: R^6 \to R^6; & \vec{F}(x, p_2, p_3, p_4, p_5, p_6) = (x, S_1, S_2, S_3, S_4, a) \\ P: R^6 \to R; & P(x, S_1, S_2, S_3, S_4, a) = a \\ G: R \to R; & G(x) = P \circ \vec{F} \circ \vec{E}(x) \end{array} \right. .$$

Then $G(x) = a =_M f(x)$, where M is an upper bound of the support points that can be obtained on the computer.

If we input the value $x = -d$, then the six memory locations will be filled as follows:

$$\begin{array}{lll} x = -d, & S_1 = d, & S_2 = d^{1/2}, \\ S_3 = -d^{3/2}, & S_4 = \sum_{j=0}^{M} (-1)^j d^j / j!, & a = -d^{3/2} + \sum_{j=0}^{M} (-1)^j d^j / j!. \end{array}$$

So the output will be $G(-d) = 1 - d - d^{3/2} + d^2/2! + \sum_{j=3}^{M} (-1)^j d^j / j! =_M f(-d)$. If we input the value $x = 0$, the output is $G(0) = 1$. Since $f(0)$ is real and $f(0) =_M G(0)$, we infer that $f(0) = 1$. Similarly, we find that

$$\begin{aligned} G(d) &= 1 + d + d^{3/2} + d^2/2! + \sum_{j=3}^{M} d^j/j! =_M f(d) \\ G(-2d) &= 1 - 2d - 2^{3/2}d^{3/2} + 2d^2 + \sum_{j=3}^{M}(-2)^j d^j/j! =_M f(-2d) \\ G(2d) &= 1 + 2d + 2^{3/2}d^{3/2} + 2d^2 + \sum_{j=3}^{M} 2^j d^j/j! =_M f(2d). \end{aligned}$$

Note that $f(-d) =_0 1 = f(0) =_0 f(d)$; hence f is continuous at 0. Since

$$\frac{f(d) - f(0)}{d} =_0 1 =_0 \frac{f(0) - f(-d)}{d},$$

we infer that f is differentiable at 0, with $f'(0) = 1$. However,

$$\frac{f(2d) - 2f(d) + f(0)}{d^2} =_0 \left(2^{3/2} - 2\right) d^{-1/2} + 1,$$

which implies that $|(f(2d) - 2f(d) + f(0))/d^2|$ is infinitely large. Hence f is not twice differentiable at 0.

Next, we consider the two functions already mentioned in the introduction, Equation (7.2), which are clearly computer functions. Consider first the function $g_1(x)$. If we input the values $x = -3d, -2d, -d, 0, d, 2d, 3d$, we obtain the following output up to depth 3

$$\begin{aligned} g_1(\pm 3d) &=_3 15.66398831641272 d^{5/2} \\ g_1(\pm 2d) &=_3 5.684263512907927 d^{5/2} \\ g_1(\pm d) &=_3 1.004845319007115 d^{5/2} \\ g_1(0) &= 0. \end{aligned}$$

Since $g_1(-d) =_0 g_1(0) =_0 g_1(d)$, g_1 is continuous at 0. A simple computation shows that $\{g_1(d) - g_1(0)\}/d =_0 0 =_0 \{g_1(0) - g_1(-d)\}/d$, from which we infer that g_1 is differentiable at 0, with $g_1'(0) = 0$. Also $\{g_1(2d) - 2g_1(d) + g_1(0)\}/d^2 =_0 0 =_0 \{g_1(0) - 2g_1(-d) + g_1(-2d)\}/d^2$, from which we conclude that g_1 is twice differentiable at 0, with $g_1^{(2)}(0) = 0$. On the other hand, $\{g_1(3d) - 3g_1(2d) + 3g_1(d) - g_1(0)\}/d^3 \sim d^{-1/2}$, which entails that $|(g_1(3d) - 3g_1(2d) + 3g_1(d) - g_1(0))/d^3|$ is infinitely large. Hence g_1 is not three times differentiable at 0.

By evaluating $g_2(-d)$ and $g_2(d)$ up to any fixed depth and applying Theorem 7.5,

Order n	$g_2^{(n)}(0)$	CPU Time
0	0.	3.400 msec
1	1.004845319007115	4.030 msec
2	0.9202876179268508	5.710 msec
3	-18.81282866172102	8.240 msec
4	-216.8082597872205	12.010 msec
5	-364.2615904917884	17.570 msec
6	101933.1724529188	25.150 msec
7	3798311.370563978	35.700 msec
8	60765353.84260825	49.790 msec
9	-1441371402.871872	67.210 msec
10	-156736847166.3961	89.840 msec
11	-6725706835826.155	118.950 msec
12	-131199307184575.8	154.530 msec
13	5770286440090848.	200.660 msec
14	$0.7837443136320079 \times 10^{18}$	256.460 msec
15	$0.4850429351252696 \times 10^{20}$	321.630 msec
16	$0.1734774579876559 \times 10^{22}$	400.140 msec
17	$-0.1757849296527536 \times 10^{23}$	478.940 msec
18	$-0.9350429649226352 \times 10^{25}$	582.150 msec
19	$-0.9521402181303937 \times 10^{27}$	702.390 msec

Table 7.4: $g_2^{(n)}(0), 0 \leq n \leq 19$, computed with $\mathcal{R}$ calculus

we obtain that g_2 is differentiable at 0 up to arbitrarily high orders. In Table 7.4, we list only the function value and the first nineteen derivatives of g_2 at 0, together with the CPU time needed to compute all derivatives up to the respective order. The numbers in Table 7.4 were obtained using the implementation of $\mathcal{R}$ in COSY INFINITY [8, 12].

7.5 Computer Functions of Many Variables

Since we know now how to compute the nth order derivative of a real computer function of one variable at a given real point x_0 whenever the nth order derivative exists and the function is extendable to $x_0 \pm d$, the following lemma shows how to find all nth order partial derivatives at a given real point $\vec{p}_0$ of a function $f : R^m \to R$ which can be represented on a computer whenever all the nth order partial derivatives exist and are continuous in a neighborhood of $\vec{p}_0$ and extendable to $\vec{p}_0 + (\pm d, \pm d, \ldots, \pm d)$.

Lemma 7.4 *Let $f : R^m \to R$ be a function representable on a computer whose nth order partial derivatives exist and are continuous in the neighborhood of the point $\vec{p}_0 = (x_{01}, x_{02}, \ldots, x_{0m})$. Then the nth order partial derivatives of f at $\vec{p}_0$ can always be computed in terms of nth order derivatives of real computer functions of one variable.*

Proof. Let l be the number of nth order partial derivatives of f; we note in passing that it can be shown [2] by induction on n and m that $l = (n+m-1)!/(n!\,(m-1)!)$. Let $k = l \cdot m$ and let $p_1, p_2, \ldots, p_k$ denote the first k prime numbers. For $j = 1, \ldots, k$, let $\alpha_j = \sqrt[n+1]{p_j}$. For $i = 1, \ldots, l$, let

$$f_i(x) = f(x_{01} + \alpha_{(i-1)m+1}x, x_{02} + \alpha_{(i-1)m+2}x, \ldots, x_{0m} + \alpha_{im}x).$$

Then $f_i, i = 1, \ldots, l$, are l real computer functions of x, n times differentiable at 0. Evaluating $(d^n f_i/dx^n)|_{x=0}$ for $i = 1, \ldots, l$ yields l equations in the l unkowns

$$\left.\frac{\partial^n f}{\partial x_1^{n_1}, \partial x_2^{n_2}, \ldots, \partial x_m^{n_m}}\right|_{\vec{p}=\vec{p}_0}, \text{ with } \left\{ \begin{array}{l} n_1, n_2, \ldots, n_m \in \{0, 1, \ldots, n\} \\ \text{and} \\ n_1 + n_2 + \cdots + n_m = n \end{array} \right. .$$

The matrix $\hat{M}$ of the coefficients has as entries products of the different α's raised to exponents between 0 and n. In the ith row, we have only products of the form

$c_{n;n_1,n_2,\ldots,n_m}\alpha_{(i-1)m+1}^{n_1}\alpha_{(i-1)m+2}^{n_2}\ldots\alpha_{im}^{n_m}$, where $c_{n;n_1,n_2,\ldots,n_m}$ is a positive integer. The determinant of $\hat{M}$ is the sum of $l!$ terms, each of which is the product of a positive integer and the α's raised to exponents less than or equal to n, and such that not all the exponents in any one term agree with those in any of the remaining $(l!-1)$ terms. By our choice of the α's, no cancellation in the evaluation of the determinant could occur; hence $\det \hat{M} \neq 0$.

It is worth noting that the choice of the α's above is far from being the only one possible. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be any set of k real numbers. We look at $\det \hat{M}$ as a function from R^k into R. A purely statistical argument shows that it is very unlikely that $\det \hat{M}$ be zero for a given choice of numbers. We are led to believe that there exist even uncountably many choices of $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in R^k$ that give a nonvanishing determinant. Here we provide simple choices of the α's only in the case $m = 2$: For $m = 2$, we have that $l = n+1$ and $k = 2(n+1)$. For $i = 1, 2, \ldots, n+1$, let $\alpha_{2i-1} = 1$ and $\alpha_{2i} = \beta_{i-1}$, where $\beta_0 = 0$ and $\beta_{j_1} \neq \beta_{j_2}$ if $j_1 \neq j_2$ in $\{0, 1, \ldots, n\}$. Then

$$\hat{M} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & n\beta_1 & \frac{n(n-1)}{2}\beta_1^2 & \ldots & n\beta_1^{n-1} & \beta_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & n\beta_n & \frac{n(n-1)}{2}\beta_n^2 & \ldots & n\beta_n^{n-1} & \beta_n^n \end{pmatrix}.$$

Therefore,

$$\det \hat{M} = C_n \begin{vmatrix} \beta_1 & \beta_1^2 & \ldots & \beta_1^n \\ \beta_2 & \beta_2^2 & \ldots & \beta_2^n \\ \vdots & \vdots & \vdots & \vdots \\ \beta_n & \beta_n^2 & \ldots & \beta_n^n \end{vmatrix} = C_n V(\beta_1, \beta_2, \ldots, \beta_n), \text{ where}$$

$$C_n = \prod_{j=1}^{n} \binom{n}{j}, \text{ and } V(\beta_1, \beta_2, \ldots, \beta_n) = \left(\prod_{k_1=1}^{n} \beta_{k_1}\right) \prod_{k_2=1}^{n-1} \prod_{k_3=k_2+1}^{n} (\beta_{k_3} - \beta_{k_2})$$

is the well-known Vandermonde determinant. Hence

$$\det \hat{M} = \left(\prod_{j_1=1}^{n} \binom{n}{j_1} \right) \left(\prod_{j_2=1}^{n} \beta_{j_2} \right) \prod_{j_3=1}^{n-1} \prod_{j_4=j_3+1}^{n} (\beta_{j_4} - \beta_{j_3}) \neq 0.$$

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Index

$=_r$, 35

Q, 1, 18

Q^+, 18

Q^-, 18

R, 1, 18

R^+, 18

R^-, 18

Z, 18

Z^+, 18

Z^-, 18

$\approx$, 35

cos, 89

cosh, 90

cot, 95

exp, 89

$\gg$, 47

λ, 35

$\ll$, 47

∂, 43

$\sim$, 35

sin, 89

sinh, 90

tan, 94

d, 36

n-times differentiability, 128

  rules about, 131

nth derivate function, 128

nth derivative, 128

  of computer functions, 195

$\mathcal{R}$, 32

  a field, 38

  a non-Archimedean field extension of R, 32

  addition and multiplication on, 37

  algebraic structure of, 32

  calculus on; see Calculus on $\mathcal{R}$, 6

  Cauchy complete in order topology, 60

  definition of, 34

  different from R, 63, 65, 99, 112–114, 128

  differential algebraic structure of, 43

  embedding of R in, 37

  existence of roots in, 41

expandable functions on, 167
is the Levi-Civita field, 4
not Cauchy complete in weak topology, 74
order structure of, 45
power series on, 82, 167
strong convergence in, 58
topological structure of, 49
uniqueness of, 55

Absolute value, 49
Algebraic numbers, 1
Alling, N. L., 4
Analysis, 1
Non-Archimedean; see Non-Archimedean Analysis, 4
Nonstandard; see Nonstandard Analysis, 3
Automorphism
field; see Field automorphism, 23
group; see Group automorphism, 27

Berz, M., 4

Calculus
non-Archimedean; see Non-Archimedean calculus, 98
Calculus on $\mathcal{R}$, 6, 98, 196
Calculus on the expandable functions, 174
Cauchy, 40
strongly; see Strongly Cauchy, 60
weakly; see Weakly Cauchy, 74
Cauchy complete, 1, 40, 60, 74
Cauchy sequence, 40
Chain rule, 110, 123
Composition
of continuous functions, 118
of differentiable functions, 123
of expandable functions, 170
of topologically continuous functions, 102
of topologically differentiable functions, 110
Computational differentiation, 2, 6, 197
Computer functions, 6, 194
n-times differentiable, 207, 210
computation of derivatives of, 7, 195, 205
example, 211, 213
continuation from R to $\mathcal{R}$, 204
continuous, 206
differentiable, 207
extendable, 199, 206

local form of, 200
machine level representation of, 198
of many variables, 214
of one variable, 197
Continuation
of computer functions, 204
of real functions, 4
of real power series, 89, 199
Continuity, 115
⇒boundedness, 115
⇒topological uniform continuity, 115
rules about, 117
topological; see Topological continuity, 99
Continuum, 2
Contracting function, 40–42, 162, 180, 183
Convergence
strong; see Strong convergence, 57
weak; see Weak convergence, 53
Conway, J., 4, 8
COSY INFINITY, 195, 213
Criterion for strong convergence
of infinite series, 64
of power series, 83
of sequences, 61
Criterion for weak convergence
of power series, 86
of sequences, 72
Davis, M., 3
Delta functions, 3
Derivate function, 107, 120
nth, 128
Derivation, 43
Derivative, 107, 120
nth, 128
computation of, 205
of a composition, 110, 123
of a product, 109, 122
of a sum, 109, 122
of computer functions, 195
Derivatives are differential quotients, 2, 6, 121
Differentiability, 120
⇒ topological differentiability, 120
n-times, 128
infinitely often, 132
rules about, 122
test for, 125
topological; see Topological differentiability, 107
Differential, 47

Differential Algebras, 13

Differential quotients, 2, 48, 121

Distributions, 2

Examples

- $\mathcal{R}$ not Cauchy complete in weak topology, 74
- computation of derivatives of computer functions, 211–213
- function on a closed interval
  - topologically continuous; multiple primitives, 114
  - topologically continuous; no maximum, 114
  - topologically continuous; not all intermediate values assumed, 113
  - topologically continuous; not bounded, 99
  - topologically differentiable; multiple primitives, 114
  - topologically differentiable; no maximum, 114
  - infinitely often differentiable, 134
  - infinitely often differentiable; Taylor series converges nowhere to the function, 134
  - topologically differentiable with zero derivative; not constant, 113
  - topologically differentiable; not all intermediate values assumed, 113
- nontrivial field automorphism on $\mathcal{R}$, 47
- nontrivial order preserving field automorphism on L, 31
- strong convergence of power series undecided, 85

Expandable functions, 167

- algebraic properties of, 168
- calculus on, 174
- definition of, 167
- infinitely often differentiability of, 185
- integration of, 190
- intermediate value theorem for, 175, 184, 185
- maximum theorem for, 186, 187
- mean value theorem for, 189
- Rolle's theorem for, 188

Field automorphism, 23, 24, 56

- only one on Q, 25
- only one on R, 26, 47, 56
- order preserving, 25

on L, 28

on $\mathcal{R}$, 28

Finite, 47

Fixed point theorem, 38, 41, 42, 160, 183

Formula manipulators, 194, 196

Frobenius, 46

Function

nth derivate, 128

computer; see Computer functions, 194

contracting; see Contracting function, 40

delta; see Delta functions, 3

derivate; see Derivate function, 107

expandable; see Expandable functions, 167

Heaviside, 197

intrinsic, 197

quasi-linear; see Quasi-linear function, 147

reciprocal, 199

root, 199

transcendental; see Transcendental functions, 1

Game theory, 4

Geometry, 1

Gonshor, H., 4

Group automorphism, 27

Hahn theorem, 8

Heaviside function, 197

Implementation, 13, 35, 53, 206

Index of a function, 173

Infinitely large, 2, 47

Infinitely often differentiable, 133, 185

Infinitely small, 2, 47

Infinitesimal, 36, 47

Integration, 190

Intermediate value theorem, 6, 7, 113, 147, 157, 175, 184–186

Intrinsic functions, 197

Inverse function, 23, 164

Inverse function theorem, 147, 164

Isomorphism, 22–24

order preserving, 22, 24

Iteration, 40, 157

Knuth, D., 4

Laugwitz, D., 3

Laws of nature, 1

Left-finite sets, 33

properties of, 33

Leibniz, 48

Levi-Civita, 4, 32

Lightstone, A. H., 4

Lipschitz constant, 115

Luxemburg, W. A. J., 3

Mathematica, 195, 196

Maximum theorem, 6, 7, 186, 187

Mean value theorem, 7, 189

Measure topology, 54

Measurement, 1

Non-Archimedean Analysis, 4, 5

Non-Archimedean calculus, 7, 98, 196

Non-Archimedean field, 2, 7, 21, 22, 25, 27, 32, 34, 47, 54, 56, 128, 194, 197

Nonstandard Analysis, 3, 13

Numbers

- algebraic, 1
- rational, 1
- real, 1
- surreal, 4, 8

Numerical differentiation, 195

Order topology, 5, 50

Ostrowski, A., 4

Power series, 1, 5, 82, 140, 143, 167, 198

- Continuation from R to $\mathcal{R}$, 198
- radius of convergence of, 86, 144
- reexpansion of, 144
- strong convergence of, 83
- weak convergence of, 86, 167
- with purely real coefficients, 89, 198
  - radius of convergence of, 89, 198

Quasi-linear function, 147

- intermediate value theorem for, 157
- inverse function theorem for, 164
- properties of, 151, 153, 154
- scaled function of, 151
  - properties, 151, 153, 154
  - strictly increasing, 155
- strictly monotone, 155

Radius of convergence of power series, 86, 144, 167

- with purely real coefficients, 89, 198

Rational numbers, 1

Real numbers, 1

Reexpansion of power series, 144

Regular sequence, 57, 62, 83, 167

Remainder formula

of order 0, 116

of order 1, 121

of order n, 130

Robinson, A., 3

Rolle's theorem, 7, 188

Roots, 1, 3, 5, 22, 34, 36, 41, 199

Schmieden, C., 3

Secants, 2, 128, 129, 147

Semi-norm, 51, 70

Sequence

Cauchy; see Cauchy sequence, 40

regular; see Regular sequence, 57

strongly Cauchy, 60

strongly convergent, 58

weakly Cauchy, 74

weakly convergent, 70

with purely real members, 69, 77

Skeleton group, 19

$S_L = Z$, 20

$S_R = \{0\}$, 20

$S_{\mathcal{R}} = Q$, 21

group automorphism on, 27

Strong convergence, 57

$\Rightarrow$ weak convergence, 81

of infinite series, 64, 65

criterion, 64

of power series, 83

criterion, 83

of sequences, 58

criterion, 61

Strongly Cauchy, 60

Strongly convergent, 58, 62

$\Rightarrow$ regular, 62

Stroyan, K. D., 3

supp, 35

Support, 35

Surreal numbers, 4, 8

The number d, 36

Theorem

fixed point, 38, 41, 42, 160, 183

Frobenius, 46

Hahn, 8

intermediate value, 6, 7, 113, 147, 157, 175, 184–186

inverse function, 147, 164

maximum, 6, 7, 186, 187

mean value, 7, 189

Rolle's, 7, 188

Topological continuity, 6, 99

at a point, 99

on a set, 99

rules about, 101

Topological differentiability, 6, 107, 204
 ⇒topological continuity, 108
 at a point, 107, 204
 on a set, 107
 rules about, 109
Topological uniform continuity, 103
Topology
 induced from $\mathcal{R}$ into R, 50, 53, 54, 68, 77
 measure, 54
 order, 50
 weak, 53
Transcendental functions, 89

Uniqueness of R, 1
Uniqueness of $\mathcal{R}$, 55

Weak convergence, 53, 69, 72, 167
 of infinite series, 78
 of power series, 86, 167
  criterion, 86
  with purely real coefficients, 89
 of sequences, 70, 72
  criterion, 72
Weak topology, 5, 53
Weakly bounded, 75
Weakly Cauchy, 74
Weakly convergent, 70