Analytical and Computational Methods for the Levi-Civita Field

Martin BERZ Department of Physics and Astronomy and National Superconducting Cyclotron Laboratory, Michigan State University, East Lansing, MI 48824

Abstract

A field extension $\mathcal{R}$ of the real numbers is presented. It has similar algebraic properties as R; for example, all roots of positive numbers exist, and the structure $\mathcal{C}$ obtained by adjoining the imaginary unit is algebraically complete. The set can be totally ordered and contains infinitely small and infinitely large quantities. Under the topology induced by the ordering, the set is Cauchy complete, and it is shown that $\mathcal{R}$ is the smallest totally ordered algebraically and Cauchy complete extension of R. Furthermore, There is a natural way to extend any other real function under preservation of its smoothness properties, and as shown in an accompanying paper, power series have identical convergence properties as in R. In addition to these common functions, delta functions can be introduced directly. A calculus involving continuity, differentiability and integrability is developed. Central concepts like the intermediate value theorem and Rolle's theorem hold under slightly stronger conditions. It is shown that, up to infinitely small errors, derivatives are differential quotients, i.e. slopes of infinitely small secants. While justifying the intuitive concept of derivatives of the fathers of analysis, it also offers a practical way of calculating exact derivatives numerically.

1 Introduction

In this paper we develop elements of a calculus on the Levi-Civita field, which is seen to be the smallest non-Archimedean extension of the real numbers that is Cauchy- and algebraically complete. We begin the discussion with a review of some properties of totally ordered fields. Let K be a totally ordered non-Archimedean field extension of the real numbers R, and $<$ its order, which introduces the K-valued absolute value $|\ |$. We introduce the following terminology.

Definition 1 *($\sim$, $\approx, <<, [\ ], H, \lambda$) For $x, y \in K$, we define*

$x \sim y$ iff there are $n, m \in N$ such that $n \cdot |x| > |y|$ and $m \cdot |y| > |x|$

$x << y$ iff for all $n \in N$, $n \cdot |x| < |y|$

$x \approx y$ iff $x \sim y$ and $(x - y) << x$.

We also set $[x] = \{y \in K | y \sim x\}$ as well as $H = \{[x] | x \in K\}$ and $\lambda(x) = [x]$.

Apparently the relation "$\sim$" is an equivalence relation; the set of classes H of all nonzero elements of K is naturally endowed with an addition via $[x]+[y]=[x\cdot y]$ and an order via $[x]<[y]$ iff $x<<y$, both of which are readily checked to be well-defined. The class [1] is a neutral element, and for $x\neq 0$, $[1/x]$ is an additive inverse of $[x]$; thus H forms a totally ordered group, often referred to as the Hahn- or skeleton group. The projection λ from K to H satisfies $\lambda(x\cdot y)=\lambda(x)+\lambda(y)$ and is a valuation.

The remarkable fundamental theorem of Hahn [4] provides a complete classification of any non-Archimedean extensions K of R in terms of their skeleton group H. In fact, envoking the axiom of choice it is shown that the elements of K can be written as formal power series over the group K with real coefficients, and the set of appearing "exponents" forms a well-ordered subset of K.

In order to admit roots of positive elements, it is apparently necessary that K be divisible, and hence the smallest choice for K are the rationals Q. The Levi-Civita field is characterized by well-ordered exponent sets that are particularly "small", indeed as we shall see later, minimally small, yet at the same time large enough to allow for Cauchy-completeness and the existence of roots.

Definition 2 ***(The Family of Left-Finite Sets)*** *A subset M of the rational numbers Q will be called left-finite iff for every number $r\in Q$ there are only finitely many elements of M that are smaller than r. The set of all left-finite subsets of Q will be denoted by $\mathcal{F}$.*

The next lemma gives some insight into the structure of left-finite sets:

Lemma 1 *Let $M\in\mathcal{F}$. If $M\neq\emptyset$, the elements of M can be arranged in ascending order, and there exists a minimum of M. If M is infinite, the resulting strictly monotonic sequence is divergent. Furthermore, let $M,N\in\mathcal{F}$; we have $X\subset M\Rightarrow X\in\mathcal{F}$, $\quad M\cup N\in\mathcal{F}$, $\quad M\cap N\in\mathcal{F}$, $\quad M+N\in\mathcal{F}$. For $x\in M+N$, there are only finitely many $(a,b)\in M\times N$ with $x=a+b$.*

The proofs are straightforward. Having discussed the family of left-finite sets, we introduce two sets of functions from the rational numbers into R and C:

Definition 3 ***(The Sets $\mathcal{R}$ and $\mathcal{C}$)*** *We define*

$$\mathcal{R}=\{\quad f:Q\to R\mid\{x|f(x)\neq 0\}\in\mathcal{F}\}$$

$$\mathcal{C}=\{\quad f:Q\to C\mid\{x|f(x)\neq 0\}\in\mathcal{F}\}$$

So the elements of $\mathcal{R}$ and $\mathcal{C}$ are those real or complex valued functions on Q that are nonzero only on a left-finite set, i.e. they have left-finite support.

Obviously, we have $\mathcal{R}\subset\mathcal{C}$. In the following, we will denote elements of $\mathcal{R}$ and $\mathcal{C}$ by x, y, etc. and identify their values at $q\in Q$ with brackets like $x[q]$. We also introduce the notation $x=_r y$ iff $x[q]=y[q]$ for all $q\leq r$; apparently $=_r$ is an equivalence relation. We now define arithmetic on $\mathcal{R}$ and $\mathcal{C}$ following the prescription of the Hahn theory:

Definition 4 ***(Addition and Multiplication on $\mathcal{R}$ and $\mathcal{C}$)*** *We define addition on $\mathcal{R}$ and $\mathcal{C}$ componentwise:*

$$(x+y)[q]\;=\;x[q]+y[q]$$

Multiplication is defined as follows. For $q \in Q$ we set

$$(x \cdot y)[q] \;=\; \sum_{\substack{q_x, q_y \in Q, \\ q_x + q_y = q}} x[q_x] \cdot y[q_y]$$

We remark that $\mathcal{R}$ and $\mathcal{C}$ are closed under addition; because of the left-finiteness of the support, only finitely many contributions appear in the sum; and finally, $\mathrm{supp}(x \cdot y) \subseteq \mathrm{supp}(x) + \mathrm{supp}(y) \in \mathcal{F}$.

It turns out that the operations $+$ and $\cdot$ we just defined on $\mathcal{R}$ and $\mathcal{C}$ make ($\mathcal{R}$,+,$\cdot$) and ($\mathcal{C}$,+,$\cdot$) into fields. It is straightforward albeit slightly tedious to show the ring structure.

Lemma 2 *($\mathcal{R}$,+,$\cdot$) and ($\mathcal{C}$,+,$\cdot$) are commutative rings with units.*

And via the embedding

$$\Pi(x)[q] = \begin{cases} x & \text{if} \quad q = 0 \\ 0 & \text{else} \end{cases}$$

they trivially extend the real and complex numbers, respectively. We also define an element $d \in \mathcal{R}$ as

$$d[q] = \begin{cases} 1 & \text{if} \quad q = 1 \\ 0 & \text{else} \end{cases}$$

Theorem 1 *The sets $\mathcal{C}$ and R have the same cardinality.*

Proof:
Since we constructed an injective mapping $\Pi : R \to \mathcal{C}$, we have $\mathrm{card}(R) = c \le \mathrm{card}(\mathcal{C})$. On the other hand, every element of $\mathcal{C}$ is uniquely determined by a sequence of support points and two sequences of function values (for the real and imaginary parts respectively). So $\mathcal{C}$ can be mapped injectively to a subset of the set of functions $N \to R^3$ (where we agree to append triplets of zeroes if the set of support points is finite). Thus by the laws for cardinal number arithmetic, it follows that

$$\mathrm{card}(\mathcal{C}) \le (c^3)^{\aleph_0} = c^{\aleph_0} = c = \mathrm{card}(R),$$

and altogether we obtain $\mathrm{card}(R)$=$\mathrm{card}(\mathcal{C})$.

In the following sections, we develop the foundation of a workable calculus for the Levi-Civita field.

2 Structure of the Levi-Civita field

In his work about $\mathcal{R}$, Levi-Civita [7, 8] showed that $\mathcal{R}$ is a field, more or less by straighforward but laborious construction of inverses through solving linear system, and then realized that any power series with real coefficients converges for infinitely small elements. The field was then later also studied by Ostrowski [12], Neder [11], Laugwitz [6], and in his later years by Robinson [9]. From general properties of formal power series fields it follows that $\mathcal{C}$ is algebraically closed [10, 14], and as a consequence also $\mathcal{R}$ is real-closed. For a general overview of the algebraic properties of formal power series fields, we refer to the excellent comprehensive overview by Ribenboim [15], and for an overview of the related valuation theory the book by Krull [5]. A thorough and complete treatment of orderered structures can also be found in [13].

We begin our discussion with a useful tool.

Theorem 2 ***(Fixed Point Theorem)*** *Let $q_M \in Q$ be given. Define $M \subset \mathcal{R}$ ($M \subset \mathcal{C}$) to be the set of all elements x of $\mathcal{R}$ ($\mathcal{C}$) such that $\lambda(x) \geq q_M$. Let $f : M \to \mathcal{C}$ satisfy $f(M) \subset M$. Suppose there exists $k \in Q$, $k > 0$ such that for all x_1, $x_2 \in M$ and all $q \in Q$, we have*

$$x_1 =_q x_2 \;\Rightarrow\; f(x_1) =_{q+k} f(x_2). \tag{2.1}$$

Then there is a unique solution $x \in M$ of the fixed point equation

$$x = f(x).$$

Proof:
Choose an arbitrary $a_0 \in M$ and defines recursively $a_i = f(a_{i-1})$, $i = 1, 2, ...$ Since f maps M into itself, this gives a sequence of elements of M that satisfies $a_i[p] = a_{i-1}[p]$ for all $p < (i-1) \cdot k + q_M$. Next we define a function $x : Q \to C$ in the following way: for $q \in Q$ choose $i \in N$ such that $(i-1) \cdot k + q_M > q$. Set $x[q] := a_i[q]$, and note that by the previous argument, this is independent of the choice of i.We see that x is indeed a unique fixed point of f.

While we will mainly apply the fixed point theorem in a general setting for the study of analysis, in passing we note that it entails a quick proof of the field property of $\mathcal{R}$ and $\mathcal{C}$ as well as their algebraic properties.

Corollary 1 *($\mathcal{R}$,+,·) and ($\mathcal{C}$,+,·) are fields. Moreover, let $z \in \mathcal{R}$ be nonzero and set $q = \lambda(z)$. If $n \in N$ is even and $z[q]$ is positive, z has two n-th roots in $\mathcal{R}$. If n is even and $z[q]$ is negative, z has no n-th roots in $\mathcal{R}$. If n is odd, z has a unique n-th root in $\mathcal{R}$. Let $z \in \mathcal{C}$ be nonzero. Then z has n distinct n-th roots in $\mathcal{C}$. Moreover, for any $r \in Q$, the values of the inverse $z^{-1}[q]$ and roots $\sqrt[n]{z}[q]$ can be calculated in finitely many steps.*

For the proof one merely factors out the complex "leading term" $z[\lambda(z)]$ and is thus left with studying the problem for $z \approx 1$. Writing $z = 1 + y$, the inversion problem and root finding problem are equivalent to the fixed point problems

$$x = -yx - y \text{ and}$$
$$x = \frac{y}{n} - x^2 \cdot \frac{P(x)}{n},$$

on the set $M = \{x | \; \lambda(x) \geq \lambda(y)\}$, respectively, where P is a polynomial with integer coefficients; for details see [1, 2]. Iteration gives the described values in finitely many steps.

In a conceptually similar way it is also possible to determine roots of polynomials with coefficients from $\mathcal{C}$. One advantage of this approach is that it not only assures the existence of the inverses, roots etc, but it also allows their explicit construction to any desired depth q through finitely many iterations of the right hand side of the fixed point equation by virtue of eq. 2.1.

Theorem 3 ***(Cauchy Completeness of*** $\mathcal{R}$ ***and*** $\mathcal{C}$***)*** *(a_n) is a Cauchy sequence in $\mathcal{R}$ or $\mathcal{C}$ if and only if (a_n) converges with respect to the order topoloty.*

Proof:
Let (a_n) be a Cauchy sequence in $\mathcal{R}$ or $\mathcal{C}$. For $q \in Q$, choose N_q such that $|a_n - a_m| < d^{q+1}$ for all $n, m > N_q$, and set $a[q] = a_{N_q+1}$. Then apparently a is well defined, and it is a limit of the sequence (a_n). The other direction is proved analogously as in R: Let (a_n) converge strongly to the limit a. Let $\epsilon > 0$ be given. Choose $n \in N$ such that $|a_\nu - a| < \frac{\epsilon}{2}\ \forall\ \nu > n$. Let now $n_1, n_2 > n$ be given. Then we have $|a_{n_1} - a_{n_2}| \leq |a_{n_1} - a| + |a_{n_2} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. The proof for $\mathcal{C}$ is analogous.

Using the idea of strong convergence allows a simple representation of the elements of $\mathcal{R}$ and $\mathcal{C}$:

Theorem 4 ***(Expansion in Powers of Differentials)*** *Let $((q_i), (x[q_i]))$ be the support points and function values of $x \in \mathcal{R}$ or $\mathcal{C}$. Then the sequence*

$$x_n = \sum_{i=1}^{n} x[q_i] \cdot d^{q_i}$$

converges to the limit x with respect to the order topology. Hence we can write

$$x = \sum_{i=1}^{\infty} x[q_i] \cdot d^{q_i}$$

Proof:
Without loss of generality, let the set of support points $\{q_i\}$ be infinite. Let $\epsilon > 0$ in $\mathcal{R}$ be given. Choose $n \in N$ such that $d^n < \epsilon$. Since q_i diverges strictly according to lemma (1), there is $m \in N$ such that $q_\nu > n\ \forall\ \nu > m$. Hence we have $(x_\nu - x)[i] = 0$ for all $i \leq n$ and for all $\nu > m$. Thus $|x_\nu - x| < \epsilon$ for all $\nu > m$. Therefore, (x_n) converges strongly to x.

We will see that power series on $\mathcal{C}$ find a useful application in discussion of so called formal power series. As we show in the following theorem, any power series with purely complex coefficients converges for infinitely small arguments; furthermore, multiplication can be done term by term in the usual formal power series sense, and convergence is always assured. Therefore, formal power series with real or complex coefficients play a natural role as proper power series in the Levi-Civita fields. Power series with general coefficients, real or not, and over general regions, are studied in detail in an accompanying paper [18].

Theorem 5 ***(Formal Power Series)*** *Any Power series with purely complex coefficients converges strongly on any infinitely small ball, even if the classical radius of convergence is zero. Furthermore, on any infinitely small ball we have, again independently of the* *radius of convergence, that*

$$(\sum_{n=0}^{\infty} a_n x^n) \cdot (\sum_{n=0}^{\infty} b_n x^n) = (\sum_{n=0}^{\infty} c_n x^n),$$

where $c_n = \sum_{j=0}^{n} a_j \cdot b_{n-j}$

Proof:
Note that for infinitely small x and any $r \in Q$, we find an m with $x^i[r] = 0$ for any $i > m$. Hence for a fixed r, the above summation includes only finitely many terms, which may be resorted according to the distributive law.

To conclude the section, we have a theorem that establishes the uniqueness of $\mathcal{R}$ as the smallest useful field extension of the reals. We say the non-Archimedean field K_1 is smaller than K_2 if either the skeleton group H_1 of K_1 is contained in the skeleton group H_2 of K_2, or if $H_1 = H_2$ and $K_1 \subset K_2$. We have

Theorem 6 ***(Uniqueness of*** $\mathcal{R}$***)*** *The field* $\mathcal{R}$ *is the smallest totally ordered non archimedian field extension of* R *that is complete with respect to the order topology and that admits roots of positive elements.*

Proof:
Obviously, $\mathcal{R}$ satisfies the mentioned conditions. Also, the skeleton group of another such field must at least contain Q because if an nth root y of x exists, we must have $\lambda(y) = \lambda(x)/n$. It remains to show that $\mathcal{R}$ can be embedded in any other Cauchy complete field with skeleton group Q; let $\mathcal{S}$ be such a field.

Let $\delta \in \mathcal{S}$ be positive and infinitely small such that (δ^n) is a null sequence. Let $\delta^{1/n}$ be an n-th root of δ. Such a root exists according to the requirements. Now observe that $(\delta^{1/n})^m = (\delta^{1/n \cdot p})^{m \cdot p} \; \forall \; p \in N$. So let $q = \frac{m}{n} \in Q$, and let $\delta^q = (\delta^{1/n})^m$. This element is unique. Furthermore, δ^q is still infinitely small for $q > 0$. Let $q_1 < q_2$. Then we clearly have $\delta^{q_1} > \delta^{q_2}$. Now let $a \in R$. Since $\mathcal{S}$ is an extension of R, we also have $a \in \mathcal{S}$, and thus $a \cdot \delta^q \in \mathcal{S}$.

Now let $((q_i), (x[q_i]))$ be the table of an element x of $\mathcal{R}$. Consider the sequence

$$s_n = \sum_{i=1}^{n} x[q_i] \delta^{q_i}.$$

Then in fact this sequence converges in $\mathcal{S}$: Let $\epsilon > 0$ be given. Since, according to the requirements, (δ^n) converges to zero, there exists $n \in N$ such that $|\delta^\nu| < \epsilon \; \forall \; \nu \geq n$. Since the sequence (q_i) strictly diverges, there is $m \in N$ such that $q_\mu > n + 1 \; \forall \; \mu > m$. But then we have for arbitrary $\mu_1 > \mu_2 > m$:

$$\begin{aligned} |s_{\mu_1} - s_{\mu_2}| &= |\sum_{i=\mu_2+1}^{\mu_1} x[q_i] \delta^{q_i}| \leq \sum_{i=\mu_2+1}^{\mu_1} |x[q_i]| \delta^{q_i} \\ &\leq \left(\sum_{i=\mu_2+1}^{\mu_1} |x[q_i]| \right) \delta^{q_{\mu_2+1}} \leq \left(\sum_{i=\mu_2+1}^{\mu_1} |x[q_i]| \right) \delta^{n+1} \\ &< \delta^n < \epsilon, \end{aligned}$$

and thus the sequence converges because of the Cauchy completeness of $\mathcal{S}$. We now assign to every element $\sum_{i=1}^{\infty} x[q_i] \cdot d^{q_i}$ of $\mathcal{R}$ the element $\sum_{i=1}^{\infty} x[q_i] \cdot \delta^{q_i}$ of $\mathcal{S}$. This mapping is injective. Furthermore, we immediately verify that it is compatible with the algebraic operations and the order on $\mathcal{R}$.

Remark 1 In the proof of the uniqueness, we noted that δ was only required to be positive and infinitely small and such that (δ^n) is a null sequence. But besides that, its actual magnitude was irrelevant. Thus, none of the infinitely small quantities is significantly different from the others. In particular, there is a natural automorphism of $\mathcal{R}$ given by the mapping $x \mapsto x'$, where $x'[q] = b^q x[a \cdot q]$; $a \in Q$, $b \in R$, $a, b > 0$ fixed. This property has no analogy in R.

3 Continuity and Differentiability

We will introduce the concepts of continuity and differentiability on $\mathcal{R}$ and $\mathcal{C}$ in this section. This is done as in R via the $\epsilon - \delta$- method. Because without further restrictions, ϵ and δ may be of a completely magnitude resulting in rather weak requirements, a stronger condition is also introduced.

Definition 5 ***(Continuity and Equicontinuity)*** *The Function $f : D \subset \mathcal{R} \to \mathcal{R}$ is called continuous at the point $x_0 \in D$, if for any positive $\epsilon \in \mathcal{R}$ there is a positive $\delta \in \mathcal{R}$ such that*

$$|f(x) - f(x_0)| < \epsilon \text{ for any } x \in D \text{ with } |x - x_0| < \delta.$$

The function is called equicontinuous at the point x_0, if for any ϵ it is possible to choose the δ in such a way that $\delta \sim \epsilon$.

We note that the stronger condition of equicontinuity is automatically satisfied in R, since there we always have $\epsilon \sim \delta$.

Theorem 7 ***(Rules about Continuity)*** *Let $f, g : D \subset \mathcal{R} \to \mathcal{R}$ be (equi)continuous at the point $x \in D$ (and there ~ 1). Then $f + g$ and $f \cdot g$ are (equi)continuous at the point x. Let h be (equi)continuous at the point $f(x)$, then $h \circ f$ is (equi)continuous at the point x.*

Proof:
The proof is analogous to the case of R.

Definition 6 ***(Differentiability, Equidifferentiability)*** *The function $f : D \subset \mathcal{R} \to \mathcal{R}$ is called differentiable with derivative g at the point $x_0 \in D$, if for any positive $\epsilon \in \mathcal{R}$, we can find a positive $\delta \in \mathcal{R}$ such that*

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g \right| < \epsilon \text{ for any } x \in D \backslash \{x_0\} \text{ with } |x - x_0| < \delta.$$

If this is the case, we write $g = f'(x_0)$. The function is called equidifferentiable at the point x_0, if for any at most finite ϵ it is possible to choose δ such that $\delta \sim \epsilon$.

Analogously, we define differentiability on $\mathcal{C}$ using absolute values.

Theorem 8 ***(Rules about Differentiability)*** *Let $f, g : D \to \mathcal{R}$ be (equi)differentiable at the point $x \in D$ (and not infinitely large there). Then $f+g$ and $f \cdot g$ are (equi)differentiable at the point x, and the derivatives are given by $(f+g)'(x) = f'(x) + g'(x)$ and $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$. If $f(x) \neq 0$ ($f(x) \sim 1$), the function $1/f$ is (equi)differentiable at the point x with derivative $(1/f)'(x) = -f'(x)/f^2(x)$. Let h be differentiable at the point $f(x)$, then $h \circ f$ is differentiable at the point x, and the derivative is given by $(h \circ f)'(x) = h'(f(x)) \cdot f'(x)$.*

Proof:
The proofs are done as in the case of R. For equidifferentiability we also get $\epsilon \sim \delta$.

Functions that are produced by a finite number of arithmetic operations from constants and the identity have therefore the same properties of smoothness as in R and C. In particular, we obtain

Corollary 2 ***(Differentiability of Rational Functions)*** *A rational function (with purely complex coefficients) is (equi)differentiable at any (finite) point where the denominator does not vanish (is ~ 1) .*

However, for functions that cannot be expressed only in terms of algebraic operations and limits, this method is not applicable, and other methods to define continuations are needed. In particular, we are interested in preserving as many of the original smoothnesss properties as possible.

Definition 7 ***(Analytic Continuation on $\mathcal{R}$ and $\mathcal{C}$)*** *Let f be an analytic function on the region $D \subset R$ or C. To the function f, we construct an analytic continuation $\bar{f}$ on all points infinitely close to D as follows: Write $\bar{x} = X + x$, with $X \in D$, $|x|$ at most infinitely small, and define $\bar{f}(\bar{x})$ as:*

$$\bar{f}(\bar{x}) = \sum_{i=0}^{\infty} f^{(i)}(X) \cdot \frac{x^i}{i!}$$

Theorem 9 ***(Continuation of Differentiable Functions)*** *Let f be an analytic function on $D \subset R$ or C. Then the continued function $\bar{f}$ is infinitely many times equidifferentiable, and for real or complex points in D, the derivatives of f and $\bar{f}$ agree.*

Proof:
Let $x \in [a, b]$. We will first consider the case of finite ϵ. We choose a δ such that for all real h with $|h| < 2\delta$, the difference quotient $(f(Re(x) + h) - f(Re(x)))/h$ does not differ from the derivative by more than $\epsilon/2$. Let now $h \in \mathcal{R}$ be positive with $|h| < \delta$, and let h_c be its real part. For $h_c = 0$, the difference between the derivative and the difference quotient is infinitely small, and therefore certainly smaller than the finite ϵ. Otherwise, since $|h_c| < 2\delta$, we infer that the difference quotient does not disagree with the derivative by more than ϵ.

On the other hand, for $\epsilon \ll 1$, observe that since δ has to be chosen with $\delta \sim \epsilon$, it is sufficient to study only the points that are infinitely near to x; but for those points, the function $\bar{f}$ is given by a power series, which is differentiable to the advertised values.

As mentioned before, functions defined by algebraic operations and limits, especially rational functions and power series, can also be continued directly by virtue of their algebraic and convergence properties. However, in this case the same result is obtained. In an accompanying paper [18], we show that also power series are differentiable.

For all equidifferentiable functions, we obtain a fundamental

Theorem 10 ***(Derivatives are Differential Quotients)*** *Let $f : D \to \mathcal{R}$ be a function that is equidifferentiable at the point $x \in D$. Let $|h| \ll d^r$, and $x + h \in D$. Then the derivative of f satisfies*

$$f'(x) =_r \frac{f(x+h) - f(x)}{h}$$

In particular, the real part of the derivative can be calculated exactly from the differential quotient for any infinitely small h.

Proof:
Let h be as in the requirement, $h = h_0 \cdot d^{r_h}(1 + h_1)$, with $h_0 \in R$, h_1 as before, and therefore $r_h > r$. Choose now $\epsilon = d^{(r+r_h)/2}$; since f is equidifferentiable, we can find a positive $\delta \sim \epsilon$ such that for any Δx with $|\Delta x| < \delta$, the differential quotient differs less than ϵ from the derivative and hence $\left| \frac{f(x+\Delta x)-f(x)}{\Delta x} - f'(x) \right|$ is infinitely smaller than d^r. But apparently, the above h satisfies $|h| < \delta$.

This is a central theorem, because it allows the calculation of derivatives of functions on R by simple arithmetic on $\mathcal{R}$. It forms the basis of much of the work on the computational differentiation of computer functions [3, 17].

The following consequence is often important for practical purposes.

Corollary 3 ***(Remainder Formula)*** *Let f be a function equidifferentiable at x, let $|h| \ll 1$. Then we obtain:*

$$f(x+h) = f(x) + h \cdot f'(x) + r(x,h) \cdot h^2,$$

with an at most finite remainder $r(x,h)$.

Proof:
Let $q = \lambda(h)$. Then we have by the above theorem

$$f'(x) =_r \frac{f(x+h) - f(x)}{h} \;\forall\; r < q,$$

from which we get by multiplication with h and rearrangement of terms

$$f(x+h) =_{r+q} f(x) + f'(x) \cdot h \;\forall\; r < q.$$

Let D be the difference between the left and the right hand side. Clearly $D[r] = 0 \,\forall\, r < 2q$. Let $r(x,h) = D/h^2$. Then we have $r(x,h)[r] = 0 \;\; \forall\; r < 0$, and therefore the expected result

$$f(x+h) = f(x) + f'(x) \cdot h + r(x,h) \cdot h^2,$$

as claimed.

Example 1 ***(Calculation of Derivatives with Differentials)*** *Let us consider the function $f(x) = x^2 - 2x$. f is differentiable on R, and we have $f'(x) = 2x - 2$. As we know, we can get certain approximations to the derivative at the position x by calculating the difference quotient*

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

at the position x. Roughly speaking, the accuracy increases if Δx gets smaller. In the enlarged field $\mathcal{R}$, infinitely small quantities are available, and thus it is natural to calculate the difference quotient for such infinitely small numbers. For example let $\Delta x = d$; we obtain

$$\frac{f(x+d) - f(x)}{d} = \frac{(x^2 + 2xd + d^2 - 2x - 2d) - (x^2 - 2x)}{d} = 2x - 2 + d$$

We realize that, in agreement with the last theorem, the difference quotient differs from the exact value of the derivative by only an infinitely small error. If all we are interested in is the usual real derivative of the real function $f : R \to R$, then this is given exactly by the 'real part' of the difference quotient.

This observation is of great fundamental and practical importance, since the operations on $\mathcal{R}$ can be implemented directly on a computer. Thus we are now able to determine exact derivatives numerically. This is a drastic improvement compared to all numerical methods operating with differences.

There is also a generalization of the concept of equidifferentiability based on the method of derivates. This method has the consequence that any C^∞ functions can locally be expanded in power series and thus assume a particularly clear structure.

4 Improper Functions

Clearly the class of rational or continued functions is rather small compared to the class of all possible smooth functions on $\mathcal{R}$ or $\mathcal{C}$. In particular, we are interested in certain functions that cannot be obtained by continuation from R or C, like Delta Functions.

Definition 8 ***(Scaled Continued Functions)*** *Let f be a function on D in $\mathcal{R}$ or $\mathcal{C}$. Then we will call f a scaled continued function if f can be written as*

$$f = l_1 \circ f_c \circ l_2,$$

where $l_1(x) = a_1 + b_1 \cdot x$ and $l_2(x) = a_2 + b_2 \cdot x$ are linear functions with coefficients from $\mathcal{R}$ or $\mathcal{C}$ and where f_c is a continued function.

We will see that while enhancing our pool of interesting functions substantially, the above introduced scaled normal functions behave very similarly to the normal functions.

Another interesting class of improper functions are the delta functions:

Definition 9 ***(Delta Functions)*** *Let $\bar{f}_d$:$R \to R$ be continuous, n times differentiable with $\int_{-\infty}^{\infty} \bar{f}_d\,(x)\,dx = 1$. Let f_d be the order n continuation of $\bar{f}_d$, and let $c \gg 1$. Then the function f with*

$$f\,(x) = \begin{cases} 0 & \text{for } |x| \gg 1/c \\ c \cdot f_d(c \cdot x) & \text{else} \end{cases}$$

is called a delta function.

Lemma 3 *Delta Functions vanish for all arguments with finite or infinitely large absolute value, and there are points infinitely close to the origin where they assume infinitely large values.*

So apparently the definition of delta functions just follows the intuitive concept. Within theories of integration on the Levi-Civita fields developed elsewhere one sees that they can be integrated, and satisfy the famous integral projection property.

Example 2 ***(Some Delta Functions)*** *The following functions are delta functions:*

$$\begin{aligned}
\delta_1(x) &= \begin{cases} 1/d & \textit{for } x \in [-d/2, d/2] \\ 0 & \textit{else} \end{cases} \\
\delta_2(x) &= \begin{cases} (1-|x|/d)/d & \textit{for } x \in [-d, d] \\ 0 & \textit{else} \end{cases} \\
\delta_3(x) &= \begin{cases} (1-x^2/2d^2)/2d & \textit{for } x \in [-d, d] \\ (|x|-2d)^2/4d^3 & \textit{for } d < |x| \leq 2d \\ 0 & \textit{else} \end{cases} \\
\delta_4(x) &= \begin{cases} \exp[-x^2/d^2]/\sqrt{2\pi}d & \textit{for } |x|/d \textit{ not infinite} \\ 0 & \textit{else} \end{cases}
\end{aligned}$$

The second example is continuous on $\mathcal{R}$, the third and fourth even differentiable on $\mathcal{R}$.

5 Intermediate Values and Rolle's Theorem

In this section we will discuss certain fundamental and important concepts of analysis, namely those of intermediate values and of extrema of functions. In the case of real functions, continuity is sufficient for the function to assume intermediate values and extrema. In an accompanying paper [18] and in [16], similar results are obtained for a more special class of functions, the power series, but without restrictions on magnitudes on derivatives. However, in $\mathcal{R}$, somewhat stronger conditions are required. We begin by demonstrating that in $\mathcal{R}$, continuity is not enough to guarantee that intermediate values be assumed.

Example 3 ***(Continuous Functions and Intermediate Values)*** *Let us consider two functions, defined on the interval* $[-1, +1]$:

$$f_1(x) = \begin{cases} -1 & \textit{if } x \leq 0 \textit{ or } (x > 0 \textit{ and } x \ll 1) \\ 1 & \textit{else} \end{cases}$$
$$f_2(x) = Re(x).$$

We refer to f_2 as the Micro Gauss bracket, as it determines the (unique) real part of x.

Both f_1 and f_2 are continuous; for any ϵ just choose $\delta = d$ and utilize that both functions are constant on the d neighborhood around x for any $x \in [-1, +1] \subset \mathcal{R}$. The function f_2 is even equicontinuous: for any $\epsilon > 0$ in $\mathcal{R}$, choose $\delta = \epsilon/2$.

But the function f_1 does not assume the value 0 which certainly lies between $f_1(-1)$ and $f_1(+1)$.The values of the function f_2 are purely real, which implies that d will not be assumed, while it is obviously an intermediate value. On the other hand, f_2 at least comes infinitely close to any intermediate value.

The next theorem will show that intermediate values are assumed under a condition on the derivative.

Theorem 11

(Intermediate Value Theorem) *Let f be a function defined on the finite interval $[a, b]$, and let f be equidifferentiable there. Furthermore, assume $f(x)$ is finite, $f'(x) \sim 1$ in $[a, b]$. Then f assumes every intermediate value between $f(a)$ and $f(b)$.*

Proof:
Let S be an intermediate value between $f(a)$ and $f(b)$. We begin by determining an $X \in [a, b]$ such that $|S - f(X)|$ is infinitely small.

In case S lies infinitely near $f(a)$, choose $X = a$; otherwise, if S lies infinitely near $f(b)$, choose $X = b$. Otherwise, let S_R, a_R, b_R be the real parts of S, a, b, respectively. Define a real function $f_R : [a_R, b_R] \to R$ as follows:

$$f_R(r) = \begin{cases} \Re(f(r)) & \text{if } r \in (a_R, b_R) \\ \Re(f(a)) & \text{if } r = a_R \\ \Re(f(b)) & \text{if } r = b_R \end{cases}$$

where "$\Re$" denotes the real part. Then as a real function, f_R is continuous on $[a_R, b_R]$. Since S is not infinitely near $f(a)$ or $f(b)$, we infer that S_R lies between $f_R(a_R)$ and $f_R(b_R)$, and hence there is a real $X \in (a_R, b_R)$ such that $f_R(X) = S_R$. Because $a_R < X < b_R$ and all three numbers are real, we have $X \in [a, b]$. Furthermore, $|S - f(X)| \leq |S - S_R| + |f_R(X) - f(X)|$ is infinitely small as desired.

Now let $s = S - f(X)$. We try to find an infinitely small x such that $X + x \in [a, b]$ and $S = f(X + x)$. Because of equidifferentiability of f, we get according to the remainder formula:

$$S = f(X + x) = f(X) + f'(X) \cdot x + r(X, x) \cdot x^2,$$

where $r(X, x)$ is at most finite, and by assumption $f'(X)$ is finite as well. Transforming the condition on x to a fixed point problem, we obtain

$$x = \frac{s}{f'(X)} - \frac{r(X, x)}{f'(X)} \cdot x^2 = F(x).$$

Choose now $M = \{x | \lambda(x) \geq \lambda(s), X + x \in [a, b]\}$. Then $r(X, x)$ and hence F are defined on M. And we have $F(M) \subset M$: Clearly on M, $\lambda(F(x)) = \lambda(s)$. Furthermore, if $X = a$, s has the same sign as the derivative in $[a, b]$ and hence as $f'(X)$; thus x is positive, entailing $X + x \in [a, b]$; if $X = b$, s and $f'(X)$ have opposite signs, and hence x is negative, entailing $X + x \in [a, b]$; and otherwise, X is finitely far away from both a and b, entailing $X + x \in [a, b]$. We now show that F is contracting on M for any infinitely small q that satisfies $q >> d^{\lambda(s)}$. Let such a q be given; we first observe that because of differentiability and the finiteness of $f'(X)$, we have for all $x \in M$ that $|(f\,(X + x) - f(X))/x - f'(X)| < q \cdot |f'(X)|/4$, but also $|(f\,(X + x) - f(X))/x - f'(X + x)| < q \cdot |f'(X)|/4$, and thus by the triangle inequality $|f'(X) - f'(X+x)| < q \cdot |f'(X)|/2$.Using the remainder formula and again differentiability, we have that

$$|F(x_1) - F(x_2)| = \left| \frac{r(X,x_1)x_1^2 - r(X,x_2)x_2^2}{f'(X) \cdot (x_1 - x_2)} \right| \cdot |x_1 - x_2|$$
$$= \left| \frac{f(X+x_1) - f(X+x_2) - f'(X)(x_1 - x_2)}{f'(X)(x_1 - x_2)} \right| \cdot |x_1 - x_2|$$
$$\leq \left(\left| \frac{f(X+x_1) - f(X+x_2)}{(x_1 - x_2)} - f'(X+x_1) \right| + |f'(X+x_1) - f'(X)| \right) \cdot$$
$$\frac{|x_1 - x_2|}{|f'(X)|}$$
$$< \left(\frac{q}{2} + \frac{q}{2} \right) \cdot |x_1 - x_2| = q \cdot |x_1 - x_2|.$$

Thus F is contracting and hence has a fixed point, assuring that the intermediate value is assumed.

Two remarks are in order. First, while the theorem is stated here for finite functions with finite derivative in finite domains, a simple re-scaling of domains or function values and observation that the underlying linear transformations do not change the existence of intermediate values allow its applicability for a wider variety of situations. Furthermore, the proof shows that, again at the expense of clarity, the requirements of the theorem can be reduced to asking that the derivative not vanish at the real intermediate value. For the most important application of the intermediate value theorem in practice, namely the construction of inverse functions, this however does not represent a major restriction, since inverses are usually needed over extended ranges.

As a corollary, by applying the intermediate value theorem to f',one also obtains a special version of an analogue to Rolle's Theorem:

Corollary 4 ***(Rolle's Theorem)*** *Let f be a function on the finite interval $[a,b]$. Let f be equidifferentiable twice, and let $f'' \sim 1$ on $[a,b]$. Then there exists $\xi \in [a,b]$ with $f'(\xi) = 0$.*

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