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# Differential algebraic determination of high-order off-energy closed orbits, chromaticities, and momentum compactions

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For some modern particle accelerators, including the planned muon collider, the accurate analysis of non-linear time-of-flight effects in the form of momentum compactions is critical for the preservation of bunch structure. A Differential Algebra-based (DA) method is presented that allows the determination of off-energy closed orbits and chromaticities to any order. By performing a coordinate transformation to the off-energy closed orbit, it is possible to compute momentum compactions analytically. This method has been implemented in the code COSY INFINITY and is tested for two cases where analytical solutions can be obtained by hand; agreement to machine precision is found. By contrast, comparisons are made with several codes that use conventional numerical methods for the determination of momentum compaction, and it is found that these approaches sometimes yield rather inaccurate results, especially for higher orders. *Keywords: maps, differential algebraic methods, momentum compaction, off-energy closed orbit, chromaticity. PACS: 29.27, 41.75, 41.75L.*

## 1 Introduction

For some quasi-isochronous rings including the recently proposed high luminosity 2 $TeV$ muon-muon collider [8], it is important to keep the bunch length at minimum (3 $mm$) in the presence of momentum spread of 0.15%. Under these conditions, the higher order momentum compaction factors can induce a large spread in the time structure and hence spoil isochronicity of the bunches [9]. For realistic lattices one must rely on beam dynamics codes for the computation of higher order momentum compactions. In this paper we derive a Differential Algebraic method that allows the computation of off-energy closed orbits, chromaticities, and momentum compactions to arbitrary order without the need of invoking any normal form methods. The method is implemented in the code COSY INFINITY. By contrast, tracking codes such as MAD or SYNCH fail to give precise answers at higher orders due to facts that will be summarized in the last section of the paper. A comparison of results with a simple model that can be calculated analytically up to order 3 is presented.

## 2 Differential algebraic theory

The momentum compaction is defined as the relative orbit length variation of an off-momentum closed trajectory relative to the trajectory of the reference particle. It is a function of the momentum offset $\delta = (p - p_0)/p_0$ , and in Taylor expansion can be written as

$$C = C_0 \left(1 + \alpha_0 \delta + \alpha_1 \delta^2 + \alpha_2 \delta^3 + \dots\right) \qquad (1)$$

Our goal is to calculate the coefficients $\alpha_i$ for $i = 0, 1, 2, \dots$ This will be accomplished to any order by DA methods, and to $i = 2$ analytically for a simplified FODO cell ring described in the next section. Furthermore, the DA method allows computation to arbitrary order of the quantities that describe how the tunes depend on energy, the so-called chromaticities. Both algorithms use the calculation of the off-energy closed orbit of a map.

The DA approach consists of the following steps. First one determines the parameter-dependent fixed point of the map, and then performs a linear decoupling of the planes in case this is necessary. After these steps one obtains a map which has a block-diagonal Jacobian, each matrix element being a DA

vector containing the value of the element and its derivatives with respect to parameters. This form allows the computation of the parameter-dependent tune shifts and the chromaticities within the DA framework using a simple formula involving the trace and determinant of the block matrices. Finally, to calculate the momentum compaction, a last coordinate change is necessary that transforms from the canonical COSY variables that measure time-of-flight to TRANSPORT-like variables that measure path length. In the following, the respective steps are addressed in detail.

The parameter-dependent fixed point calculation relies on the map inversion algorithm developed in [3][4], and so an outline of the algorithm is in order. All the maps of interest have no constant parts, that is they are origin preserving. In this case it is possible to compute the $n$-th order inverse $\mathcal{N}_n$ of a map $\mathcal{M}_n$ as long the linear part $\mathcal{M}_1$, is invertible, which is always the case for symplectic maps. To this end, one writes $\mathcal{M}_n = \mathcal{M}_1 + \mathcal{M}_n^*$. Letting $\circ$ denote the composition of maps, we have to $n$th order that

$$\begin{aligned} \mathcal{I}_n &= (\mathcal{M}_1 + \mathcal{M}_n^*) \circ \mathcal{N}_n = \mathcal{M}_1 \circ \mathcal{N}_n + \mathcal{M}_n^* \circ \mathcal{N}_n \\ \mathcal{N}_n &= \mathcal{M}_1^{-1} \circ (\mathcal{I} - \mathcal{M}_n^* \circ \mathcal{N}_n)\,. \end{aligned}$$

This is a fixed point problem for $\mathcal{N}_n$. Beginning iteration with $\mathcal{N}_n = \mathcal{I}_n$ yields convergence to the exact result in $n$ steps because $\mathcal{M}_n^*$ is purely nonlinear.

The parameter-dependent fixed point $\vec{z}(\vec{\delta})$ is a periodic orbit of the map $\mathcal{M}_n$ satisfying the condition $(\vec{z}(\vec{\delta}), \vec{\delta}) = \mathcal{M}_n(\vec{z}(\vec{\delta}), \vec{\delta})$. To make the map origin preserving, which in turn implies that the partial derivatives of the transfer map with respect to parameters alone vanish[4], we perform a coordinate transformation, which in fact is a non-linear translation depending only on the parameters. To this end we introduce the map $\mathcal{I}_n^{\vec{z}}$, containing a unity map in the upper block describing the phase space variables and zeros elsewhere. Substracting $\mathcal{I}_n^{\vec{z}}$ on both parts we have

$$(\vec{0}, \vec{\delta}) = (\mathcal{M}_n - \mathcal{I}_n^{\vec{z}})(\vec{z}(\vec{\delta}), \vec{\delta})$$

and thus,

$$(\vec{z}(\vec{\delta}), \vec{\delta}) = (\mathcal{M}_n - \mathcal{I}_n^{\vec{z}})^{-1}(\vec{0}, \vec{\delta}),$$

from which we read off $\vec{z}(\vec{\delta})$ in the non-parameter lines. If energy is treated as a parameter, then $\vec{z}(\vec{\delta})$ is the off-energy closed orbit. A closer inspection reveals that the inverse of the map $(\mathcal{M}_n - \mathcal{I}_n^{\vec{z}})$ exists if and only if the phase space part of $\mathcal{M}_n$ does not have 1 as an eigenvalue. However, this corresponds to a fundamental resonance that is always avoided in the design of repetitive systems.

Next, the linear decoupling of the phase planes is performed by diagonalization of the linear part of the map already expanded around the parameter-dependent fixed point [3]. This is always possible if we stipulate pairwise distinct eigenvalues; in practice this restriction is not important since if it is not met the system is again on a linear resonance. Also, since we are interested in stable systems, the eigenvalues lie on the complex unit circle. So the linear part of the map in the eigenvector basis has the form

$$\hat{L}_C = \begin{pmatrix} e^{i\mu_1} & & & & & \\ & e^{-i\mu_1} & & 0 & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & 0 & & & e^{i\mu_v} & \\ & & & & & e^{-i\mu_v} \end{pmatrix}$$

We note that the eigenvectors associated with these eigenvalues form complex conjugate pairs are complex because the underlying matrix was real.

Next we perform another change of basis after which the matrix is real. For each conjugate pair of eigenvalues, we choose the real and imaginary parts of the corresponding eigenvectors as two basis vectors. The result is a matrix in block-diagonal form, the matrix elements of which depend on the parameters $\vec{\delta}$, and which thus has the form (with $m = n-1$)

$$\hat{L}_R = \begin{pmatrix} [a_1]_m & [b_1]_m & & & & \\ [c_1]_m & [d_1]_m & & 0 & & \\ & & \ddots & & & \\ & & & & [a_v]_m & [b_v]_m \\ & 0 & & & [c_v]_m & [d_v]_m \end{pmatrix}.$$

In the light of the preceding paragraphs, the matrix elements are now actually DA vectors containing the derivatives with respect to the parameters $\vec{\delta}$ up to order $m$. We note that all the basis changes are in fact similarity transformations, and combined with the fact that a sufficient condition for a 2x2 matrix to be symplectic is to merely have determinant 1, all the transformation matrices can be scaled such that the final map is still symplectic if the underlying one

was symplectic. Now we can apply the simple formula involving just the trace and the determinant of the matrix [4][3]

$$\mu_j = \mathrm{sign}\,(b_j) \cdot \arccos\left(\frac{T_j}{2\sqrt{D}_j}\right)$$

where $T_j$ and $D_j$ are the trace and determinant of the 2x2 matrices, to compute the DA vector $[\mu_j]$ describing the derivatives of the tune with respect to the system parameters. We obtain

$$\left[\mu_j\right]_m = \mathrm{sign}\,(b_j) \cdot \arccos\left(\frac{[a_j]_m + [b_j]_m}{2\left([a_j]_m [d_j]_m - [b_j]_m [c_j]_m\right)^{1/2}}\right),$$

where all elementary operations and the computation of arccos are now DA operations. Again, if energy is a parameter, then the coefficients of the relative energy deviation in the expansions of $\left[\mu_j\right]_m$ are the chromaticities.

The momentum compaction can now be readily calculated. Once the off-energy closed orbit $\vec{z}(\vec{\delta})$ is known, one last coordinate change is necessary. All DA map computation tools implemented in COSY INFINITY use symplectic coordinates, and in particular chromatic and longitudinal effects are described in terms of the time-of-flight and energy. Using the DA-based transformation to TRANSPORT-like coordinates [7] based on path length and momentum, we obtain the DA dependence of the closed orbit on momentum. The final step in the computation of the momentum compaction is to calculate the relative change of the length of the off-momentum closed orbit with respect to the on-momentum closed orbit.

## 3 Simplified FODO cell

As our model for the analytic calculation of the momentum compaction, we choose a simplified FODO cell for which the quantities of interest can be obtained analytically, and which is shown in Figure 1.

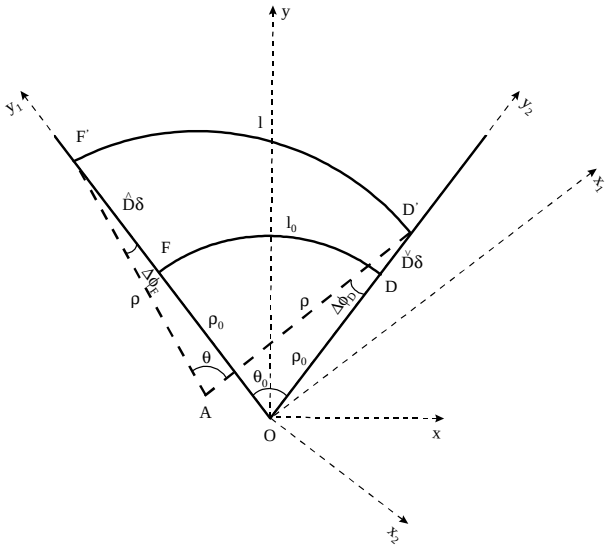


Fig 1. Outline of the half FODO cell. FF′- thin focusing quadrupole; DD′ - thin defocusing quadrupole; $\theta_0$- deflection angle for the homogeneous magnet between the quadrupoles. Shown are the reference orbit ($l_0$) and the off-momentum closed orbit ($l$).

We take the half cell as consisting of a half focusing thin quadrupole (located at $FF^{'}$) followed by a homogeneous magnetic dipole and a half defocusing thin quadrupole (located at $DD^{'}$). We repeat this cell until it forms a closed ring. Also, we neglect any fringe field effects. The two quadrupoles have the same integrated strength $S = KL$, where $L$ is the quadrupole length and $K$ is the normalized strength

$$K = \frac{B_t}{d \cdot \chi}$$

where $B_t$ denotes the flux at the pole tip, $d$ the aperture, and $\chi$ the rigidity of the reference particle. The thin lens approximation of the quadrupoles is characterized by simultaneously letting the length $L$ go to zero and the strength $K$ go to infinity in such a way that the product $K \cdot L$ stays constant.

The off-momentum closed orbit is characterized by $D(\delta)$ and $D^{'}(\delta)$, the dispersion function and its derivative with respect to the independent variable $s$, which satisfy

$$\begin{aligned} x &= D(\delta) \cdot \delta \text{ and} \\ x^{'} &= D^{'}(\delta) \cdot \delta. \end{aligned}$$

Expanding the dispersion function in terms of momentum, we have

$$\begin{aligned} D &= D_0 + D_1\delta + D_2\delta^2 + \dots \\ D' &= D_0' + D_1'\delta + D_2'\delta^2 + \dots \end{aligned}$$

Due to the symmetry of this cell, $D'(0) = D'(l_0) = 0$ at the entrance and exit of the half FODO cell. We denote by $\hat{D}$ and $\check{D}$ the value of the dispersion function at the position of the focusing and defocusing thin quadrupoles. Since the thin quadrupoles are zero length insertions, the dipoles fill all the space; correspondingly, the orbit length of the reference particle in the half FODO cell is given by $l_0 = \rho_0\theta_0$, and the orbit length of an off momentum particle is given by $l = \rho\theta$. In homogeneous magnetic fields, the radii of curvature $\rho$ and $\rho_0$ are related via $\rho = \rho_0(1+\delta)$.

If the ring consist of $2N$ half cells, the total orbit lengths are

$$\begin{aligned} C_0 &= 2N \cdot l_0 \\ C &= 2N \cdot l \end{aligned}$$

which implies the relation

$$\frac{C}{C_0} = (1+\delta)\frac{\theta}{\theta_0}.$$

From this relation it is possible to determine the first few momentum compaction factors, and we obtain

$$\alpha_0 = 1 - \frac{S\left(\hat{D}_0 - \check{D}_0\right)}{\theta_0}, \quad \alpha_1 = -\frac{S\left(\hat{D}_1 - \check{D}_1\right)}{\theta_0},$$

$$\alpha_2 = -\frac{S\left(\hat{D}_2 - \check{D}_2\right)}{\theta_0} - \frac{S^3\left(\hat{D}_0^3 - \check{D}_0^3\right)}{6\theta_0}.$$

We refer to [6] for details. The dispersion functions are calculated by a geometric approach, which yields for the momentum compaction coefficients

$$\alpha_0 = 1 - \frac{2S^2 t}{\theta_0\left(\theta_0^2 + S^2\right)}, \quad \alpha_1 = \frac{S^4 t\left(3\theta_0^2 + S^2 t^2\right)}{\theta_0\left(\theta_0^2 + S^2\right)^3},$$

$$\alpha_2 = -\frac{S^4 t}{3\theta_0\left(\theta_0^2 + S^2\right)^5} \cdot \left[12\theta_0^6 + S^2\left(39 + 4t^2\right)\theta_0^4 + 26S^4t^2\theta_0^2 + S^6t^2\left(4 + 3t^2\right)\right]. \quad (2)$$

Here $S$ was replaced with $S/(2\pi\text{meter})$ in order to make the parameter dimensionless, and $t = \tan(\theta_0/2)$. Again, for details we refer to [6].

# 4 Comparison between various tracking codes and COSY

As pointed out in [9],[10] comparison of $\alpha_i$ with tracking codes such as MAD and SYNCH gives agreement for $i = 0, 1$ but not $i = 2$. The reasons of disagreement are attributed to various factors related to inaccurate tracking of off-momentum particles (in the kick approximation) and numerical errors due to numerical differentiations. Table 1 shows an example for $S = 0.5$ and $N = 150$.

| $\alpha$ | SYNCH | MAD | Theory |
|---|---|---|---|
| $\alpha_0$ | 0.00171503 | 0.00171503 | 0.0017150314 |
| $\alpha_1$ | 0.00267272 | 0.00267421 | 0.0026727345 |
| $\alpha_2$ | 0.00105371 | 0.00064879 | -0.0000931587 |

Table 1. Comparison of momentum compactions up to order 3 for $N = 150$ and $S = 0.5$ among SYNCH, MAD and Theory [9]

In contrast, the comparison of $\alpha_2$ calculated in this report and given in Table 2, and the results from the code COSY ([2],[5],[1]) show excellent agreement up to at least 10 digits. Moreover, the agreement is over the full range of the involved variables, namely the integrated quadrupole strengths $S$ and the number of the cells $N$. Table 2 shows the results for a small ring with $N = 15$ and a large ring with $N = 150$, and values of $S$ in the range $[0.0, 0.9]$

| N | S | COSY | Theory | Difference |
|---|---|---|---|---|
| 15 | 0.01 | -.103908718E-04 | -.103908711E-04 | .722374551E-12 |
| 15 | 0.1 | -.651254349E-01 | -.651254349E-01 | .152496071E-11 |
| 15 | 0.3 | -.249285370734 | -.249285370735 | .793004550E-12 |
| 15 | 0.5 | -.101510514395 | -.101510514393 | .151861856E-11 |
| 15 | 0.9 | -.226833835E-01 | -.226833835E-01 | .581569514E-12 |
| 150 | 0.01 | -.648111606E-01 | -.648111606E-01 | .232450170E-11 |
| 150 | 0.1 | -.103033154E-01 | -.103033154E-01 | .286317844E-12 |
| 150 | 0.3 | -.224557080E-03 | -.224557079E-03 | .408824721E-13 |
| 150 | 0.5 | -.931586556E-04 | -.931586560E-04 | .337685893E-12 |
| 150 | 0.9 | -.750827110E-04 | -.750827104E-04 | .661708426E-12 |

Table 2. Comparison of $\alpha_2$ between COSY and Theory as a function of number of cells, $N$, and integrated and normalized quadrupole strengths, $S$

We conclude that the method outlined in this paper and implemented in COSY INFINITY allows the computation of high order momentum compactions to essentially machine precision, without limitations due to inaccuracies of tracking, numerical schemes to find the fixed point, and numerical differentiation to determine derivatives of the fixed point with respect to momentum.

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