Taylor-Model-based Enclosure of Invariant Manifolds for Planar Diffeomorphisms and Applications

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Fundamental questions in the study of interesting dynamics of planar diffeomorphisms like the Hénon map involve homoclinic phenomena, topological entropy and strange attractors. Inherently, answering these questions requires knowledge about the stable and unstable manifolds, which in the typical case in the plane are smooth curves. We present a method to find highly accurate Taylor Model enclosures of the invariant curves near hyperbolic fixed points. Successive iteration of these local enclosures yields similarly accurate enclosures of pieces of the global manifold tangle. Applications presented include the automatic computation of rigorous enclosures of all homoclinic points up to finite iterates. This allows to find symbolic dynamics in the original system and consequently compute rigorous lower bounds for its topological entropy.

1 Taylor Model Enclosure of Invariant Manifolds

Taylor Models [1] combine high-order polynomial approximations [2] of functions with interval arithmetic [3] to rigorously bound truncation errors to achieve fully verified and sharp functional inclusions. This naturally leads to the application of Taylor Models in the computation of invariant manifolds for planar diffeomorphism, since techniques to compute truncated power series expansions of the manifolds locally near fixed points have long been known [4–6]. We present an extension of these techniques to outfit the polynomials with intervals that can be shown to rigorously bound the truncation errors.
In the following, let $f : R^2 \longrightarrow R^2$ be a C^r-diffeomorphism, $r \geq 1$, and assume that the origin is a hyperbolic fixed point. Let $\gamma : [-1,1] \longrightarrow R^2$ be a polynomial curve approximating the local unstable manifold, obtained via any of the standard methods, and let $P : [-1,1]^2 \longrightarrow R^2$ be a narrow 'thickening' of γ via the introduction of a small transverse parameter.

Theorem 1.1 *Let* $P = (P_1, P_2)$ *be a two-dimensional bijective polynomial on* U *which satisfies* $P(0,0) = (0,0)$. *Let* $(\tilde{P}, \tilde{I}) := f(P, I)$ *evaluated in Taylor model arithmetic, where* I *is the trivial interval* $[0,0]^2$. *Let*

$$R = P(U),\ \tilde{R} = \tilde{P}(U) + \tilde{I}, B_u = P([-1,1] \times \{1\}),\ \textit{and}\ B_l = P([-1,1] \times \{-1\})$$

denote the ranges of P *and* $\tilde{P} + \tilde{I}$ *and the 'upper' and 'lower' boundaries of the range* R*, respectively. Assume now that*

$$(B_u \cup B_l) \cap \tilde{R} = \emptyset. \tag{1}$$

Then the unstable manifold does not leave R *through* B_u *or* B_l.

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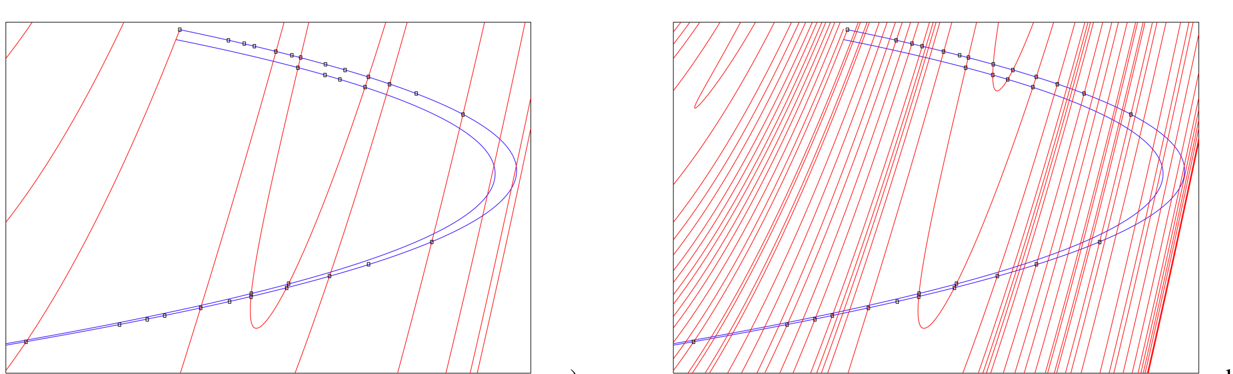

Fig. 1 5th (**a**) and 9th (**b**) preimage of a local stable manifold piece (red) of the Hénon map $\mathcal{H}_{a,b}$ for $a = 1.4$, $b = 0.3$, together with the unstable manifold (blue). The actual Taylor Model enclosures are several orders of magnitude below printer resolution in size. Also shown are magnified interval enclosures of selected homoclinic points (*see Sec.2*).

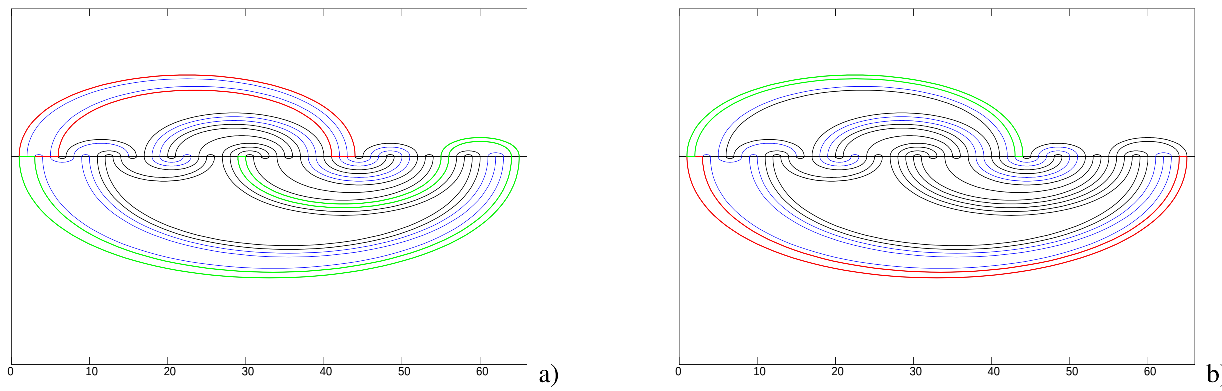

Fig. 2 A piece of the homoclinic tangle of the standard Hénon map with selected curvilinear rectangles (red) and their images (green).

The assumption (1) can easily be checked using Taylor Model bounds on the derivatives of the exactly known curve P. In practice, a thickness of the Taylor Model enclosure of 10^{-12}-10^{-15} can be achieved. Naturally, the same result holds for the stable manifold with the inverse map.
After the local enclosures of the invariant manifolds have been established, subsequent iteration through f or its inverse in Taylor Model arithmetic generate enclosures of significant pieces of the global manifold tangle. Via bisection and reexpansion the Taylor Model enclosures can maintain their sharpness through the iteration procedure.

As an example we consider the Hénon map $\mathcal{H}_{a,b}(x,y) := (1 + y - a\,x^2, b\,x)$ with the standard parameters $a = 1.4$ and $b = 0.3$. Figure (1) depicts different preimages of the stable manifold in the vicinity of the fixed point near $(0.63, 0.18)$.

2 Automatic construction of symbolic dynamics

Once a Taylor Model enclosure of the global manifold tangle has been constructed, a verified global optimization routine allows to locate and sharply bound all its homoclinic intersections (cf. Fig.(1)). Additionally, one is able to order the enclosed homoclinic points along the manifolds, compute their handedness (i.e. the orientation of the manifolds at the intersection in the sense of their parametrizations) and determine image-preimage-pairs within the set of homoclinic points.

Previous works [7,8] show that lower bounds for the topological entropy of a system can be obtained from this information, which is sufficient to automatically construct curvilinear 'rectangles' with homoclinic points at their vertices and boundaries that are alternatingly pieces of the un-/stable manifolds. Once these rectangles are constructed, the image-preimage-pairs within their corner points allow to determine Markov crossings of the rectangles under iteration. The rectangle crossings generate an incidence matrix which induces a subshift of finite type on the symbol space enumerating the rectangles. Entropy estimates can be inferred form the spectral radius of the incidence matrix.

In figure (2) examples are shown for a homoclinic tangle of the Hénon map with 65 homoclinic points ordered along the stable (horizontal) and unstable (curved) manifolds. A larger computation with 707 homoclinic points and 329 rectangles yields the result in the following theorem. We note for comparison that [9] proved $h_{top}(\mathcal{H}_{a,b}) \geq 0.430$ and suggested numerically $h_{top}(\mathcal{H}_{a,b}) \approx 0.464$.

Theorem 2.1 *For the standard parameter values $a = 1.4$ and $b = 0.3$, the topological entropy of the Hénon map $\mathcal{H}_{a,b}$ satisfies $h_{top}(\mathcal{H}_{a,b}) \geq 0.4536$.*

References

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