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Computer Assisted Proof of the Existence of High Order Periodic Points

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Abstract. We describe an efficient method to rigorously prove the existence and attractiveness of high period periodic points through verified numerical computations. The proof consists of two parts: first the existence and then the attractiveness. To prove the existence, we only require a non-verified, numerical approximation of the derivative of the map at the fixed point. In the second part of the proof we require knowledge of the exact derivative. The use of Taylor Models in COSY INFINITY to carry out the calculations very successfully controls the dependency problem commonly encountered in verified numerics.

In comparison, we also implemented the same proof using traditional interval arithmetic. This approach is more complicated, as it requires calculations to be carried out in higher precision than the standard double precision. Also, interval methods suffer significantly from the dependency problem, which requires splitting of the domain into smaller, more tractable pieces.

We then apply both algorithms to prove the existence of a period 15 point in a Hénon map very close to the standard parameters. Using high precision intervals, we obtain a very tight enclosure of the periodic point with a precision of up to 70 decimal digits. By examining the Jacobian of the 15th iterate of the map, we lastly establish uniqueness and attractiveness of this periodic point.

1 Introduction

Given a continuously differentiable map

$$M : \mathbb{R}^n \mapsto \mathbb{R}^n,$$

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we want to prove the existence of a periodic point $\vec{x}$ of order m of that map, i.e. a point $\vec{x} \in \mathbb{R}^n$ such that

$$M^m(\vec{x}) = \vec{x}.$$

The direct proof, of course, would be to obtain the exact periodic point $\vec{x}$, and apply the map M m times. If the result is the same as the initial argument, $\vec{x}$ is a periodic point. Unfortunately, this direct method is only possible for simple maps where the periodic point can be calculated analytically. As soon as this becomes impossible, that way of proving the existence of a periodic point cannot be employed anymore.

Fortunately, there are many other methods available to prove the existence of a periodic point of a given map M. In particular, we are interested in methods that do not require exact knowledge of the periodic point, but rather allow us to prove that a certain subset $K \in \mathbb{R}^n$ contains a periodic point. That way, if K is small enough, one can obtain very good approximations to the position of a periodic point, without having to calculate it exactly, which in general is intractable on a computer.

The probably most common theorem of that type is the Banach fixed point theorem[2]. It can be applied to prove the existence and uniqueness of an attractive periodic point. We will use this method in section 4 to prove attractiveness and uniqueness of the periodic point.

In our proof of existence, however, we want to use a weaker theorem for several reasons. The Banach fixed point theorem requires knowledge of a Lipschitz constant for the map. Often this is obtained through the derivatives of the map. It is not always convenient to obtain an analytic expression of the derivative of a map. But even if the derivative is known, evaluating the derivative of a high period iteration of a map by the chain rule is computationally expensive.

Fortunately, to only prove the existence of one or more fixed points we can make use of the following version of the well known Brouwer fixed point theorem[5]:

Theorem 1.1 (Brouwer Fixed Point Theorem) *Let a continuous map M : $\mathbb{R}^n \mapsto \mathbb{R}^n$ and a convex compact set $K \subset \mathbb{R}^n$ be given. If $M(K) \subset K$ then K contains at least one fixed point of M.*

Note that this statement of the theorem uses the well known fact that any convex compact set in $\mathbb{R}^n$ is homeomorphic to a closed unit ball of some dimension $k \leqslant n$[7].

The conditions of Brouwer's theorem are easily verified by rigorous computations, which makes this theorem very useful for our purposes. Furthermore, the conditions for the Banach Fixed Point Theorem include the verification that a certain closed set is mapped into itself. Taking that set to be the interval box used in Brouwer's theorem, it is relatively simple to apply the Banach fixed point theorem to prove uniqueness and attractiveness.

1.1 Proof of Existence of Periodic Point. Consider now the application of Brouwer's theorem to the problem at hand. Let M be a continuously differentiable map. Without loss of generality, we assume the presumed periodic point of order m is near the origin. If that was not the case, we simply consider the new map

$$M_{new}(\vec{x}) = M(\vec{x} + \vec{z}) - \vec{z}, \tag{1.1}$$

where $\vec{z}$ is an approximation of the presumed periodic point. This new map is still continuously differentiable, and will have a presumed periodic point of the same order m near the origin.

Furthermore, if Q is a regular $n \times n$ matrix, then instead of M^m, we can consider the conjugated map

$$\widetilde{M}(\vec{x}) = Q^{-1}(M^m(Q(\vec{x}))). \tag{1.2}$$

By continuity, if $\widetilde{M}$ has a fixed point x_0 near the origin, then M has the periodic point $Q(x_0)$ of order m which is also near the origin.

Let K be a n-dimensional interval vector $K = ([-\epsilon, \epsilon], \ldots, [-\epsilon, \epsilon])$, where $\epsilon > 0$ is small. Being a Cartesian product of closed intervals, K is certainly compact and convex for any choice of ϵ.

We now want to show that $\widetilde{M}(K) \subset K$. Once this has been proven, then applying Brouwer's theorem leads to the conclusion that there is a fixed point of $\widetilde{M}$ in K. □

Note that, depending on the map M and the choice of ϵ, the proof may not yield a conclusive result. If $\widetilde{M}(K) \bigcap K \neq \emptyset$ but $\widetilde{M}(K) \not\subset K$, then no statement can be made about the existence or non-existence of a periodic point.

1.2 Preconditioning. Assume now the Jacobian J of M^m at the origin is known approximately through some non-verified, numerical computation. Assume, furthermore, all eigenvalues of J have norm less than 1. There then is a linear transformation Q, such that $Q^{-1}JQ$ contracts each face of the box K towards the origin.

This preconditioning by Q transforms, to first order, the attractive region around the origin to a rectangle, instead of a more general parallelogram. With this preconditioning, the proof will succeed in more cases that would otherwise not yield a result. Consider, for example, in dimension 2 the linear map Φ given by the matrix

$$\alpha \cdot \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

with $0 < \alpha < 1$. The eigenvalues are $\pm\alpha$, and thus less than one in absolute value, yet the point $(-1, 1)$ is mapped to $(3, 1)$. Thus no square centered at the origin with sides parallel to the coordinate axes will be mapped into itself. This only occurs after a change of coordinate such that the coordinate axes become eigenvectors.

2 Rigorous Computation

Standard floating point numbers available on computers only represent finite approximations to real numbers and are therefore not adequate for rigorous computations. For rigorous computations, typically interval arithmetic[12] is used. The problem with intervals is that they suffer from significant overestimation due to the dependency problem.

We will therefore use a more sophisticated method to perform rigorous numerical calculations. In this paper, remainder enhanced differential algebra expansions, also known as Taylor Models, are used to keep the dependency problem under control. It is well known that in dynamical systems those Taylor Models provide a very efficient way to control functional dependency[10].

Taylor Models are based on a differential algebra, which, combined with rigorous book keeping of all calculation errors and truncated orders, yields a Taylor expansion of a function up to arbitrary order valid on a given domain together with a rigorous remainder bound. Using Taylor Models, it is then possible to obtain rigorous enclosures of the range of a function over a given domain[3, 9].

2.1 Rigorous Coordinate Transformation. In the mathematical discussion above, we reduced the proof of the existence of any attractive periodic point to the proof for fixed points at the origin. In the proof, we used a preconditioning matrix Q such that the eigenvectors of the Jacobian of M^m at that point coincide with the coordinate axes. To reduce the map to that case, it generally is necessary to perform a translation and an affine transformation. While this is easy to do mathematically, some care has to be taken when this is done in rigorous computer arithmetic. The translation is simple, as the calculation of the inverse is just the negation of the periodic point coordinates, which is exact in floating point arithmetic.

The more complicated part is the linear transformation into eigencoordinates, if applicable. To obtain this transformation, the approximate eigenvectors of the map at the periodic point have to be determined first. Here it is not necessary to be rigorous, as the proof above clearly is correct with any regular linear transformation. To obtain the expansion of the map to first order, we use non-verified, automatic differentiation through the DA data type in COSY INFINITY. Once we have obtained the matrix representation of the linear part of the map, any of the well known numerical methods to calculate eigenvalues and eigenvectors can be applied. This allows us to construct the matrix to transform eigencoordinates into Cartesian coordinates.

To complete the coordinate transformation, this matrix has to be inverted. This inversion has to be done rigorously to ensure that the resulting matrix really represents the inverse. In the example given below, the Hénon map is only two dimensional and thus the inversion of the matrix is easy. Instead of performing the calculations in floating point arithmetic, we use either interval or Taylor Model arithmetic. That way, we obtain a matrix of intervals or Taylor Models, which can be used to rigorously transform from Cartesian coordinates into eigencoordinates. For higher dimensions, more elaborate algorithms have to be used to obtain a rigorous inverse.

Once this is done, a rigorous representation of $\widetilde{M}$ can be constructed. In all further considerations, we will only use that map instead of the original map M.

2.2 Proof using Intervals. Following the discussion above, using intervals in principle is a fairly straightforward procedure. First one picks a small interval box K, as described above. Then the box is mapped by $\widetilde{M}$ and the result $\widetilde{M}(K)$ is verified to lie within K. With intervals, this test can easily be performed. All that has to be done is to compare the boundaries of the resulting intervals for each coordinate with the corresponding coordinates boundaries of the initial interval box K.

However, due to the overestimation in interval arithmetic, the initial box as a whole will most likely not be mapped into itself right away. In particular, if the map $\widetilde{M}$ is of high order, there will be significant dependency introduced in the interval arithmetic, and the result of $\widetilde{M}(K)$ will be several orders of magnitude larger than the smallest enclosure of the mathematically correct result. Nonetheless, as long as there is some overlap between $\widetilde{M}(K)$ and K, there is a chance that the boxes may actually map into each other.

To overcome this problem, it is necessary to split the starting box K into smaller boxes and map those. As the error in interval arithmetic decreases linearly with the box size, we can expect that, after a sufficient number of splits, the small boxes will eventually map into the original box K. Clearly, all of K is mapped into itself iff every small piece is mapped entirely into the original box.

For the actual implementation, there are a few considerations to be made. Certainly, one does not want to split the box into small pieces of a fixed, predefined size. We do not know the best box size, and, in fact, the optimal size of the boxes varies depending on the position within the starting box K. Pieces close to the center can be larger than pieces right at the boundary of K. Instead, an algorithm that automatically determines the optimal box size for the region under consideration is preferable. To implement this efficiently, it is best to use a stack based approach. Whenever a box fails to map fully into K, we split it and continue the process with the newly created boxes until all boxes were successfully mapped into K.

The complete algorithm for the verification procedure with automatic box size control is as follows:

1. Start with initial box K on the stack
2. Take the top box S off of the stack and evaluate $M(S)$
3. Compare $M(S)$ and K
    - If $M(S) \subset K$, discard S as it has been successfully mapped into K.
    - Otherwise, if $M(S) \bigcap K \neq \emptyset$, split S along each coordinate axis once, and push the resulting four pieces onto the stack.
    - If neither of the above is true, we have $M(S) \bigcap K = \emptyset$. In this case, declare failure and stop, as we just mapped a piece of K completely outside of K.
4. Continue with 2 until there are no more boxes on the stack.

While this looks trivial, there is a pitfall hidden in the implementation of this algorithm. For the interval algorithm to work, we require relatively small boxes. In all but the most simple cases, the split boxes have to be so small, that double precision intervals are not precise enough to carry out the operations anymore. In Equation 1.1 we need to calculate $\vec{x} + \vec{z}$, where $\vec{z}$ is the presumptive periodic point, and $\vec{x}$ is one of the small boxes.

In typical maps (see section 3), the coordinates of the periodic point $\vec{z}$ will be of order one, whereas the box size is of order 10^{-15}. It is evident, that this addition cannot be carried out precisely in double precision. As a result, as we keep splitting the search space into "smaller" pieces, we will quickly exhaust the available memory. The small pieces, however, are then effectively lost due to the limited precision in the coordinate transformation. That is the reason why an interval package with higher precision is required to carry out these calculations.

2.3 Proof using Taylor Models. The proof using Taylor models differs significantly from the interval version of the proof. Taylor Models significantly reduce the overestimation, that caused the intervals to grow rapidly. This is due to the drastic reduction of the dependency problem intrinsic to Taylor Model arithmetic. Thus, instead of mapping a very small box by successively splitting it, we will map just one single box of much larger size. With Taylor Models, this is already sufficient to map the box into itself and thus to conclude the proof.

To set up the calculation, we first generate the map $\widetilde{M}$ with an inverse Taylor Model matrix as described in subsection 2.1. Next, we create two Taylor Models x_i and y_i with only a linear coefficient of size 10^{-6} in the first and second independent variable, respectively. The image of the domain of these two Taylor Models then represents an initial box of width 10^{-6} centered around the origin. Last, we map these Taylor Models by $\widetilde{M}$, and verify that the image of the resulting Taylor Models x_f and y_f lies within that of x_i and y_i.

COSY INFINITY provides intrinsic functions to calculate an outer interval enclosure of the range of a Taylor Model, so we can easily obtain intervals containing the images of x_f and y_f. To verify that these two intervals actually lie within the range of the initial Taylor Models is a little more complicated. To test for enclosure, it first is necessary to obtain an inner interval enclosure of the image of the initial Taylor Models x_i and y_i. In this case, those enclosures can be obtained easily, since the initial Taylor Models are constructed to have a simple structure.

The initial Taylor Models only consist of one linear coefficient and a remainder bound. To obtain an inner enclosure one can simply take the remainder bound and add, in interval arithmetic, the linear coefficient extracted from the Taylor Model. Thus one obtains an enclosure of the upper bound of the image of the Taylor Model. Taking the lower bound of that interval yields an upper bound of the inner enclosure. Applying the same technique, but this time subtracting the linear coefficient from the remainder bound, and taking the upper bound of the resulting interval one obtains a lower bound for the inner enclosure (see Figure 1).

With the inner enclosure of the initial Taylor Models known, one compares those two inner interval bounds of the image of x_i and y_i to the outer interval enclosure of the image of x_f and y_f. If the outer range enclosures of the final coordinates are fully contained within the inner range enclosures of the initial coordinates, the initial box was mapped into itself.

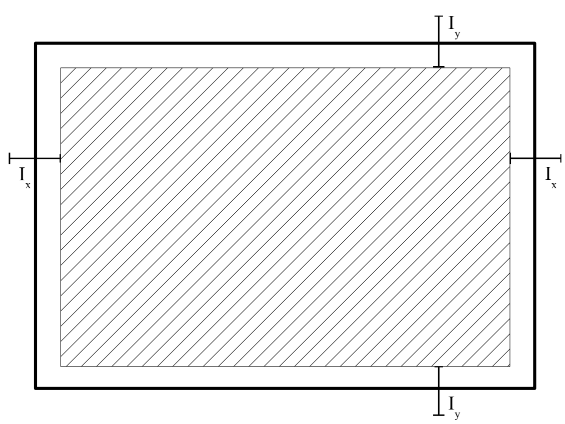

Figure 1 The image of a Taylor Model box (solid black), the interval bounds I_x and I_y, and the rigorous inner enclosure (hatched).

3 Application to Near Standard Hénon Map

We now apply the methods developed above to an example. In the following, we consider one of the simplest maps exposing chaotic behaviour, the Hénon map[8]

$$H : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ (x, y) \mapsto (1 + y - Ax^2, Bx). \end{cases}$$

In the original map studied by Hénon himself, and many others after him, the parameters were chosen to be $A = 1.4$ and $B = 0.3$. We consider a slightly different value $A = 1.422$ while keeping $B = 0.3$.

In this map, the existence of an attracting period 15 point was suggested by non-verified numerical experiments. The coordinates of this suggested periodic point are approximately

$$\begin{aligned} x &\approx -0.0869282203452939, \\ y &\approx 0.2391536750716747. \end{aligned}$$

As this point seems to be attractive, simple iteration of the Hénon map, starting with those coordinates, improves the approximate coordinates until the change between iterations is of the size of floating point errors. In standard double precision calculations this will yield about 15 valid decimal digits.

Using non-verified floating point calculations, we want to gain some insight into the general behavior of this periodic point. Simply iterating the coordinates given above by the map H^{15} a few times in double precision yields the following noteworthy results.

$$\begin{array}{lll} H^{15}: & x = -0.0869282203454\underline{4442} & y = 0.2391536750716\underline{964} \\ H^{30}: & x = -0.0869282203452\underline{2939} & y = 0.2391536750716\underline{747} \\ H^{45}: & x = -0.0869282203454\underline{4442} & y = 0.2391536750716\underline{964} \\ H^{60}: & x = -0.0869282203452\underline{2939} & y = 0.2391536750716\underline{747} \end{array}$$

Note how the last few underlined digits alternate between two different points. Looking at this result naively, one may suspect that the periodic point is actually of period 30, instead of period 15. It will turn out, however, that these oscillations are caused by floating point roundoff errors, and have no mathematical significance. We just happened to find an orbit for which this effect appears.

3.1 Proof of Existence. Using the environment for rigorous numerical calculations provided by COSY INFINITY[11], we implemented the algorithm for an automated proof using Taylor Models as described in subsection 2.3. The implementation is straightforward, since our example only has two dimensions we can perform the calculation of eigenvectors and eigenvalues, as well as the matrix inversion, analytically. To make sure the inversion is rigorous, we carry out those calculations using Taylor Models with only a constant part instead of floating point numbers. The result is a matrix of Taylor Models entries.

In the following, we will denote an interval of the form $[1.234\underline{5678}, 1.234\underline{6789}]$ by 1.234^{6789}_{5678} to conserve space and make the width of the interval more clearly visible.

Theorem 3.1 (Existence of Period 15 Point in Hénon Map) *Given the Hénon map with parameters $A = 1.422$ and $B = 0.3$, there exists at least one periodic point of period 15 within the interval*

$$\begin{aligned} X &= 1.1957^{80721557596}_{58008577504}, \\ Y &= 0.050^{52194963414509}_{49328335698421}, \end{aligned}$$

Proof By repeated iteration of the origin by the map H in double precision arithmetic, until the change between to iterates is less than 10^{-12}, we obtain a candidate for the presumptive periodic point. Mathematically it is clear, that there is no preference for any particular point in the orbit of the periodic point candidate. Numerically, however, it is favorable to choose a point that has a well conditioned eigenvector matrix. If that is the case, the inversion can be carried out nicely, and will not produce much overestimation. We heuristically determined the eigenvectors at each point in the orbit and based on that information choose the following periodic point candidate

$$\begin{aligned} x &\approx 1.195769365067588, \\ y &\approx 0.05050761649554453. \end{aligned}$$

For the following calculation, we initialized the DA engine of COSY INFINITY to carry out operations up to order ten in two independent variables. Using automatic differentiation, we calculated the linearization of H^{15} and derived the following approximation of the eigenvector matrix

$$\begin{pmatrix} 0.8852161763463854 & 0.2504328280425341 \\ -0.4651798804061557 & 0.9681339776284161 \end{pmatrix},$$

which is then rigorously inverted, using Cramer's rule on Taylor Models with only constant coefficients.

The x and y coordinates, in eigencoordinates, of the initial box are chosen to be Taylor Models with only a linear coefficient of 10^{-5} in the first and second independent variable respectively. This initial box is then transformed from eigencoordinates into Cartesian coordinates to yield the periodic point enclosure in Cartesian coordinates. This Taylor Model enclosure is then bounded, yielding the interval representation given in the statement of Theorem 3.1.

To conclude the proof, the initial box in Cartesian coordinates is mapped 15 times by H, converted back to eigencoordinates, and then tested for inclusion in the initial box according to the method presented in subsection 2.3. □

Note that, while strange at first glance, using an initial box as big as 10^{-5} currently is necessary to successfully carry out the proof. In fact, due to the specific structure of Taylor Models, the verification process fails if the initial box is chosen significantly smaller. At the moment, Taylor Models in COSY INFINITY are implemented based on double precision floating point numbers. One of the great features of Taylor Models is, that the coefficients scale with width of the initial box raised to the order of the coefficient. That is, if the initial Taylor Model has linear coefficients of order 10^{-10} the resulting Taylor Model will have second order coefficients of order 10^{-20}. But since the constant part of the resulting Taylor Model will be of order 1, any contribution by the second order coefficients is lost, as the constant part only carries about 15 significant digits.

Thus, if the initial box is chosen too small, this effectively reduces the calculation order to just the linear case. While normally in verified numerics it is desirable to operate on large boxes without much overestimation, in this case the strength of Taylor Models in that field limits our ability to precisely pinpoint the periodic point. A solution to this is to store the coefficients of the lower orders of the Taylor Model with a higher precision. Then also higher order coefficients will contribute to the Taylor Model. We are currently implementing such high precision Taylor Models, which will allow us to overcome the constraints posed by double precision floating point numbers. Once this effort is completed, it will be possible to use much smaller initial Taylor Models in this process, and thus obtain much smaller enclosures of the periodic point.

It is also worth noting that the computational time required for this proof was negligible. The whole initial box is mapped at once, requiring only 15 consecutive evaluations of the map H and a few simple operations afterwards to verify the enclosure.

3.2 High Precision Enclosure. To obtain a more precise rigorous enclosure of the periodic point, we also implemented an interval version of the proof. As described in subsection 2.2, it is necessary to use a high precision interval package to successfully carry out this proof using intervals. While there are many different implementations of rigorous interval packages available, most of those rely on the underlying floating point numbers provided by the processor. On modern computers, this usually means floating point numbers according to the IEEE 754[1] standard, so that the precision of those packages is limited to about 15 decimal digits.

As stated before, this is insufficient for this proof. To mitigate the situation, we implemented our own rigorous high precision intervals in COSY INFINITY[14]. For outputting these high precision numbers as decimal fractions, we currently rely on the high precision interval package MPFI[13]. With these intervals it is possible to implement, and carry out, the proof of the following theorem.

Theorem 3.2 (High Precision Enclosure of Period 15 Point in Hénon Map) *Given the Hénon map with parameters $A = 1.422$ and $B = 0.3$, there exists exactly one periodic point of period 15 within the interval*

$$\begin{aligned} x &= 1.19576936506755033604110098396554 89 \\ &\quad 352337235594806801053003707350839 68^{32853}_{10139}, \end{aligned}$$

$$\begin{aligned} y &= 0.05050761649556464888828848017561 61 \\ &\quad 01684142680828370628141055516578229^{4397960}_{1531331}. \end{aligned}$$

Before proving this theorem, we will make an estimate of the computational complexity of the proof. Mapping a single interval box around the presumptive periodic point from eigencoordinates into Cartesian coordinates, then through H^{15}, and back to eigencoordinates, results a blowup in the size of the box by a factor of about 1300 in x, and 1100 in y. In interval arithmetic this is not uncommon, and is to be expected with this type of map. The reason is the high period of the periodic point.

Mathematically H^{15} is a polynomial of order $2^{15} = 32768$. We, however, do not operate with the full expansion of H^{15}, but instead let H act on the argument 15 times. While this reduces the total number of operations carried out significantly, in interval arithmetic this still results in a significant amount of overestimation because of dependency.

With these heuristics it is possible to estimate the minimum number of boxes necessary to map the complete initial box into itself. Since the splitting always happens in both x and y, the maximal length of the split boxes is about 1/1300 of the initial size. Thus 1300^2, or about 1.7 million, is a lower bound for the number of boxes required.

Proof The calculations for this proof were carried out with intervals set to a precision of about 75 significant decimal digits. In the first step, we again obtained a high precision approximation of the presumptive periodic point by repeatedly iterating the origin by the map H, until the change between two iterates became less than 10^{-74}. Then, by manually iterating H until the same point as in Theorem 3.1 was reached, we obtained the following periodic point candidate

$$
\begin{aligned}
x &\approx 1.195769365067550336041100983965548935\\
&\quad 233723559480680105300370735083968214 95,\\
y &\approx 0.050507616495564648888288480175616101 6\\
&\quad 841426808283706281410555165782292964645.
\end{aligned}
$$

Using a double precision approximation of this point, we again employed automatic differentiation to calculate the linearization of H^{15} around this point. We obtained the following approximate eigenvector matrix

$$
\begin{pmatrix} 0.8852161763463209 & 0.2504328280405625 \\ -0.4651798804062782 & 0.9681339776289262 \end{pmatrix}.
$$

This step is still done in double precision, because the underlying data type for DA vectors in COSY INFINITY is currently only available in double precision. This restriction, however, does not pose any problem, as the transformation only needs to approximate the eigenvectors. The eigenvector matrix is then rigorously inverted using Cramer's rule on high precision intervals. The resulting inverse matrix has very tight high precision intervals as coefficients, which is essential for the proof to succeed.

The initial box in eigencoordinates is chosen to be the interval $[-10^{-70}, 10^{-70}]$, both in x and y. This initial box is then transformed from eigencoordinates to Cartesian coordinates to yield the actual periodic point enclosure given in the statement of Theorem 3.2.

To conclude the proof, the splitting-mapping-cycle described in subsection 2.2 is initiated with the initial box and iterated until all boxes on the stack have been successfully mapped into the initial box. □

Note that, unlike the Taylor Model version of the proof, this process requires about 12 to 13 splits of the initial box in each direction in order to compensate the effects of overestimation in the interval arithmetic. This is consistent with the estimate done earlier, as it results in a box size of about $1/2^{12} \approx 1/4000$. The proof takes about 130 minutes on a 2 GHz MacBook with 2 GB of RAM to complete. In that time, about 70 million boxes are successfully mapped into the initial box, while about 24 million boxes failed to map correctly and had to be split.

4 Uniqueness and Attractiveness

As stated before, the Brouwer fixed point theorem only establishes the existence of at least one periodic point within the intervals given in the above theorems. In order to prove the uniqueness and attractiveness of the point, some more work is necessary. We will use the Banach fixed point theorem[7] to prove uniqueness and attractiveness.

Theorem 4.1 (Banach Fixed Point Theorem) *Let a continuous map $M : \mathbb{R}^n \mapsto \mathbb{R}^n$ and a compact set $K \subset \mathbb{R}^n$ be given. If $M(K) \subset K$ and M is a contraction on K, then K contains exactly one unique, attractive fixed point of M.*

Note that the Banach fixed point theorem can be viewed as a natural extension of the Brouwer fixed point theorem in this context. If the set K in the Banach theorem is chosen to be the convex compact set used in the above theorems, the first condition of the Banach fixed point theorem, $M(K) \subset K$, is already established. The only additional condition that has to be shown is the contraction map property.

In order to show that M is indeed a contraction map on this convex compact set K, it is sufficient to bound the operator norm of the Jacobian of M on K. If it is less than one, then M is a contraction on K[6, p. 38].

To calculate a bound of the operator norm of the Jacobian matrix $J(M)$ of the map M on a box K, we make use of the following identity for real valued square matrices $A = J(M)$:

$$||A|| = \sqrt{\lambda_{\max}\left(A^T A\right)},$$

where λ_{max} indicates the largest eigenvalue[4, p. 269]. The eigenvalues of the 2×2 matrix $A^T A$ can be computed using the analytic expression for the roots of the characteristic polynomial

$$\lambda^2 - tr(A^T A)\lambda + \det(A^T A) = 0$$

Theorem 4.2 (Uniqueness and Attractiveness of Periodic Point) *The enclosure of the period 15 point given in Theorem 3.2 contains exactly one periodic point of order 15. Furthermore, this periodic point is unique and attractive in the enclosure given in Theorem 3.1.*

Proof Using the method described above, the Jacobian matrix of H^{15} on the same box K used in the Taylor Model proof of the existence of a fixed point, given in Theorem 3.1, is computed. For simplicity, the calculation is preformed in Cartesian coordinates.

The rigorous calculation of the Jacobian is performed using a Taylor Model matrix. Taylor Models are the natural choice for this problem, since they allow for calculations to be performed over large volumes without significant overestimation.

The Jacobian of the Hénon map H is given explicitly by

$$JH(x,y) = \begin{pmatrix} -2Ax & 1 \\ B & 0 \end{pmatrix}$$

To evaluate the Jacobian of the fifteenth iterate, $J(H^{15})(x,y)$, the chain rule is applied repeatedly, yielding:

$$\begin{aligned} J(H^{15})(x,y) &= J(H^{14})(H(x,y)) \cdot JH(x,y) \\ &= JH(H^{14}(x,y)) \cdot JH(H^{13}(x,y)) \cdots JH(H(x,y)) \cdot JH(x,y) \end{aligned}$$

By performing all operations in this calculation in Taylor Model arithmetic, the resulting Taylor Model matrix represents a rigorous enclosure of the correct Jacobian of H^{15} over the entire box K. From there it is trivial to compute the matrix product

$$A = J\left(H^{15}\right)^T J\left(H^{15}\right).$$

The eigenvalues of the resulting two dimensional matrix A are then calculated directly as Taylor Models and bounded by intervals. The resulting enclosures of the two eigenvalues e_1 and e_2 over the box given in Theorem 3.1 are:

$$\begin{aligned} e_1 &= [-0.8210105861696513 \cdot 10^{-9}, 0.8210017711776557 \cdot 10^{-9}] \\ e_2 &= [0.9002450764758135, 0.9809161063129307] \end{aligned}$$

Since both are bounded from above by a value less than 1, so is $||J(H^{15})||$, and the map indeed represents a contraction in the box K. Together with the previously proven self mapping property from Theorem 3.1, this establishes all requirements for the Banach fixed point theorem, thus concluding the proof. □

Conclusion

From the computation time for the proof of existence of the periodic point it is clear that Taylor Models perform significantly better than intervals. The dependency problem is virtually eliminated compared to traditional interval methods. The only advantage of intervals at this point is the availability of high precision interval implementations. This allows for a much more precise enclosure of the periodic point than the Taylor Models. However, this advantage will become obsolete with the advent of the high precision Taylor Models we are developing right now.

In the proof of uniqueness and attractiveness of the fixed point, the ability of Taylor Models to function properly over large volumes turns into a desirable feature. Taylor Models allow for fast bounding of the eigenvalues of the Jacobian over a relatively large area. Thus it is possible to establish an inner bound of the attractive basin of the fixed point.

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