Direct Calculation of the Transfer Map of Electrostatic Deflectors, and Comparison with the Codes *COSY INFINITY* and *GIOS*

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Abstract

COSY INFINITY uses a beamline coordinate system with a Frenet-Serret frame relative to the reference particle, and calculates differential algebra–valued transfer maps by integrating the ODEs of motion in the respective vector space over a differential algebra (DA).

We will describe and perform computation of the DA transfer map of an electrostatic spherical deflector in a laboratory coordinate system using two conventional methods: (1) by integrating the ODEs of motion using a 4th order Runge-Kutta integrator and (2) by computing analytically and in closed form the properties of the respective elliptical orbits from Kepler theory. We will compare the resulting transfer maps with (3) the DA transfer map of *COSY INFINITY*'s built-in electrostatic spherical deflector element `ESP` and (4) the transfer map of the electrostatic spherical deflector computed using the program *GIOS*.

In addition to the electrostatic spherical deflector, we study an electrostatic cylindrical deflector, where the Kepler theory is not applicable. We compute the DA transfer map by the ODE integration method (1), and compare it with the transfer maps by (3) *COSY INFINITY*'s built-in electrostatic cylindrical deflector element `ECL`, and (4) *GIOS*.

In addition to the code listings in the appendices, the codes to run the test cases are available at
`http://bt.pa.msu.edu/cgi-bin/display.pl?name=ELSPHTM17`

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1 Introduction

COSY INFINITY [4] uses a beamline coordinate system that comprises the phase space coordinates [3, p. 9]

$$
\begin{aligned}
&r_1 = x, &&r_2 = a = p_x/p_0, \\
&r_3 = y, &&r_4 = b = p_y/p_0, \\
&r_5 = l = -\left(t - t_0\right) v_0 \tfrac{\gamma}{1+\gamma}, &&r_6 = \delta_K = \tfrac{K - K_0}{K}.
\end{aligned}
$$

Coordinates x and y are the transversal Frenet-Serret position coordinates in meters, p is the momentum, K is the kinetic energy, v is the velocity, t is the time of flight, and γ is the Lorentz factor. The index 0 refers to the reference particle.

In *COSY INFINITY*, the equations of motion are integrated, once for each particle optical element comprising the lattice, in phase space represented as a vector space $\left({}_nD_v\right)^v$ over a differential algebra (DA) ${}_nD_v$ [2, pp. 86–100], where n is the computation order and v is the number of phase space coordinates. The result is a transfer map $\mathcal{M}$ that expresses the final coordinates z_{f} of any ray as $z_{\mathrm{f}} = \mathcal{M}\left(z_{\mathrm{i}}\right)$, where z_{i} are the initial coordinates [2, Chs. 4–5].

COSY INFINITY includes a built-in electrostatic spherical deflector particle optical element ESP. We will describe two conventional methods of computing the non-relativistic DA transfer map of an electrostatic spherical deflector: (1) by integrating the ODEs of motion using a 4th order Runge-Kutta integrator and (2) by computing and applying the transition matrix with elements as the Lagrange coefficients. We will calculate the transfer map of an electrostatic spherical deflector using these two methods and compare the results with (3) the DA transfer map of the non-relativistic version of *COSY INFINITY*'s built-in electrostatic spherical deflector element ESP and (4) the transfer map of the electrostatic spherical deflector computed using the program *GIOS* [5].

Additionally, we perform a similar study on an electrostatic cylindrical deflector, where the Kepler theory is not applicable, so we compute and compare the transfer maps by the methods (1), (3) using ECL, and (4).

2 Transfer Map Calculation by Integrating the ODEs in Laboratory Coordinates

Consider a bunch of non-relativistic charged particles launched with kinetic energy $K_0 = mv_0^2/2$ and zero potential energy, where m is the particle mass and v_0 is the reference velocity. Suppose that a circular reference orbit of radius r_0 is defined for the particle bunch in an electrostatic deflector. Now, for concreteness, consider one particle in this bunch. Following the convention of the potential energy as zero at the reference orbit, we calibrate the potential energy of the charged particle such that

$$
U(r_0) = 0.
$$

In the following section, we discuss the specific case of an electrostatic spherical deflector. The case of an electrostatic cylindrical deflector in Sub-Sec. 2.2.

2.1 The ODEs in an Electrostatic Spherical Deflector

The potential energy in an electrostatic spherical deflector is

$$U\left(r\right) = -\frac{\alpha}{r} + \frac{\alpha}{r_0}, \tag{1}$$

where the electrostatic potential V and the electric field are

$$U(r) = ZeV(r), \quad E_r(r) = -\frac{\partial V}{\partial r} = -\frac{\alpha}{Ze}\frac{1}{r^2}. \tag{2}$$

By energy conservation, an off-reference particle at initial radius r_i would upon entering the deflector have an initial velocity magnitude v_i such that

$$K_0 = \frac{mv_0^2}{2} = \frac{mv_i^2}{2} + U\left(r\right) = \frac{mv_i^2}{2} - \frac{\alpha}{r_i} + \frac{\alpha}{r_0}, \tag{3}$$

from where we obtain

$$v_i = \sqrt{v_0^2 - \frac{2\alpha}{m}\left(\frac{1}{r_0} - \frac{1}{r_i}\right)}, \tag{4}$$

or, expressing α in terms of the particle charge $q = Ze$, the reference orbit radius r_0, and the electric rigidity $\chi_e = \frac{pv}{Ze}$ as $\alpha = Ze\chi_e r_0$,

$$v_i = \sqrt{v_0^2 - \frac{2}{m}Ze\chi_e\left(1 - \frac{r_0}{r_i}\right)}. \tag{5}$$

COSY INFINITY's horizontal transversal coordinate x is defined relative to the circular reference orbit as $x = r - r_0$, where r is the length of the projection of a particle's radius vector on the plane of the reference particle's orbital motion. *COSY INFINITY*'s horizontal momentum component $a = p_x/p_0$ can be, in the non-relativistic case, expressed as $a = v_x/v_0$, where v_x is the x component of a particle's velocity and v_0 is the reference velocity.

Now, let (x_i, a_i) be the initial beamline coordinates of the particle in the electrostatic spherical deflector. In reality, the deflector would have a fringe field; in this model, we approximate the fringe field by an instant jump in the electrostatic potential from zero to the electrostatic potential of the electrostatic spherical deflector at radius $r_i = r_0 + x_i$. Thus, all particles in a bunch experience a "step down" from kinetic energy K_0 to a kinetic energy $K = K_0 - U\left(r\right)$ at time $t = 0$.

In a similar way, at the end of the element, the particle will experience a "step up" due to the change of the potential energy back to zero. Note that these changes of energy are essential even in the absence of a true fringe field treatment to preserve the actual beam energy.

For calculations, we will use the polar laboratory coordinate system (r, θ), with particles launched at polar angle $\theta = 0$. Additionally, we will use the Cartesian laboratory coordinate system

$$(\tilde{x}, \tilde{y}) = r\left(\cos\theta, \sin\theta\right).$$

In this Cartesian laboratory coordinate system, the particle has the initial position

$$\vec{r}_{\mathrm{i}} = (r_0 + x_{\mathrm{i}}, 0) \tag{6}$$

and the initial velocity

$$\vec{v}_{\mathrm{i}} = \left(a_{\mathrm{i}} v_0, \sqrt{v_{\mathrm{i}}^2 - (a_{\mathrm{i}} v_0)^2} \right), \tag{7}$$

where v_{i} is the initial particle velocity magnitude at the initial radius r_{i} obtained using eq. 5.

From the second Newton law, the radial acceleration of the particle in the electrostatic spherical deflector is

$$\frac{d^2}{dt^2} r = -\frac{\mu}{r^2} + \omega^2 r, \tag{8}$$

where $\mu = \alpha/m$ and ω is the particle's angular velocity.

Considering eqns. 6 and 7, the conservation of the angular momentum can be expressed in terms of massless angular momentum

$$h = \omega r^2 = |\vec{r} \times \vec{v}|$$

as

$$
\begin{aligned}
h &= |\vec{r}_{\mathrm{i}} \times \vec{v}_{\mathrm{i}}| = |(\vec{r}_{\mathrm{i}})_1 (\vec{v}_{\mathrm{i}})_2 - (\vec{r}_{\mathrm{i}})_2 (\vec{v}_{\mathrm{i}})_1| = \\
&= \left| (r_0 + x_{\mathrm{i}}) \sqrt{v_{\mathrm{i}}^2 - (a_{\mathrm{i}} v_0)^2} - 0 \cdot a_{\mathrm{i}} v_0 \right| = \\
&= (r_0 + x_{\mathrm{i}}) \sqrt{v_{\mathrm{i}}^2 - (a_{\mathrm{i}} v_0)^2}.
\end{aligned}
\tag{9}
$$

Taking the time derivative of $\omega = h/r^2$, we obtain

$$\frac{d^2}{dt^2} \theta = -2 \frac{h}{r^3} v_r.$$

The final position $\vec{r}_{\mathrm{f}}$ and velocity $\vec{v}_{\mathrm{f}}$ can be obtained by integrating the ODEs of motion in the polar coordinates

$$
\frac{d}{dt}
\begin{pmatrix}
r \\
v_r \\
\theta \\
\omega
\end{pmatrix}
=
\begin{pmatrix}
v_r \\
-\frac{\mu}{r^2} + \omega^2 r \\
\omega \\
-2 \frac{h}{r^3} v_r
\end{pmatrix},
\tag{10}
$$

with the initial condition

$$
\begin{pmatrix}
r \\
v_r \\
\theta \\
\omega
\end{pmatrix}_0
=
\begin{pmatrix}
(\vec{r}_{\mathrm{i}})_1 \\
(\vec{v}_{\mathrm{i}})_1 \\
0 \\
(\vec{v}_{\mathrm{i}})_2 / (\vec{r}_{\mathrm{i}})_1
\end{pmatrix}.
$$

Considering that $d\theta/dt = \omega$, applying the chain rule, this system of ODEs can be expressed in terms of polar angle θ as the independent variable as

$$\frac{d}{d\theta}\begin{pmatrix} r \\ v_r \\ \theta \\ \omega \end{pmatrix} = \begin{pmatrix} \frac{v_r}{\omega} \\ -\frac{\mu}{r^2}\frac{1}{\omega} + \omega r \\ 1 \\ -2\frac{h}{r^3}\frac{v_r}{\omega} \end{pmatrix} \tag{11}$$

with the same initial condition.

Having obtained the final position

$$\vec{r}_{\mathrm{f}} = r_{\mathrm{f}}\left(\cos\theta_{\mathrm{f}}, \sin\theta_{\mathrm{f}}\right)$$

and velocity $\vec{v}_{\mathrm{f}}$, the final (x, a) coordinates are

$$(x_{\mathrm{f}}, a_{\mathrm{f}}) = \left(r_{\mathrm{f}} - r_0, \frac{\vec{v}_{\mathrm{f}} \cdot \vec{r}_{\mathrm{f}}}{v_0 r_{\mathrm{f}}}\right).$$

We calculated the DA transfer map of the electrostatic spherical deflector, in (x, a) beamline coordinates, with reference orbit radius $r_0 = 1\,\mathrm{m}$, by integration of the ODEs of motion in polar laboratory coordinates using a 4th order Runge-Kutta integrator. This transfer map is listed in Sec. 4.

To obtain a DA-valued transfer map, we set the initial phase space coordinates as

$$(x_{\mathrm{i}}, a_{\mathrm{i}}) = (d_1, d_2) = \left(\mathtt{DA}\,(1), \mathtt{DA}\,(2)\right),$$

the respective DA generators [2, pp. 86–96]. We performed the calculations with the computation order 3.

A version of the *COSY INFINITY* program that uses a 4th order Runge-Kutta integrator to obtain the transfer map by solving the ODEs of motion in polar laboratory coordinates is listed in App. A. Additionally, App. B lists a *Mathematica* program that integrates the ODEs in polar laboratory coordinates for an individual orbit, plots one turn of the orbit, and outputs the final (x, a) coordinates.

2.2   The ODEs in an Electrostatic Cylindrical Deflector

The motion in an electrostatic cylindrical deflector is described in a similar way as in the case of a spherical deflector. Corresponding to eqns. 1 and 2, the potential energy in an electrostatic cylindrical deflector is

$$U\left(r\right) = ZeV\left(r\right) = \alpha \log \frac{r}{r_0}, \tag{12}$$

and the electric field is

$$E_r(r) = -\frac{\partial V}{\partial r} = -\frac{\alpha}{Ze}\frac{1}{r}. \tag{13}$$

The energy conservation equation 3 is replaced by

$$K_0 = \frac{mv_0^2}{2} = \frac{mv_i^2}{2} + U(r) = \frac{mv_i^2}{2} + \alpha \log \frac{r_i}{r_0}, \tag{14}$$

resulting in the following initial velocity v_i, instead of the corresponding equations in 4 and 5.

$$v_i = \sqrt{v_0^2 - \frac{2\alpha}{m} \log \frac{r_i}{r_0}} = \sqrt{v_0^2 - \frac{2}{m} Ze\chi_e r_0 \log \frac{r_i}{r_0}}. \tag{15}$$

Corresponding to the radial equation of motion in 8, we have in the electrostatic cylindrical deflector

$$\frac{d^2}{dt^2} r = -\frac{\mu}{r} + \omega^2 r, \tag{16}$$

resulting in, instead of eq. 10,

$$\frac{d}{dt} \begin{pmatrix} r \\ v_r \\ \theta \\ \omega \end{pmatrix} = \begin{pmatrix} v_r \\ -\frac{\mu}{r} + \omega^2 r \\ \omega \\ -2\frac{h}{r^3} v_r \end{pmatrix}, \tag{17}$$

and arriving at a set of ODEs in terms of θ, corresponding to eq. 11, as

$$\frac{d}{d\theta} \begin{pmatrix} r \\ v_r \\ \theta \\ \omega \end{pmatrix} = \begin{pmatrix} \frac{v_r}{\omega} \\ -\frac{\mu}{r}\frac{1}{\omega} + \omega r \\ 1 \\ -2\frac{h}{r^3}\frac{v_r}{\omega} \end{pmatrix} \tag{18}$$

with the initial condition

$$\begin{pmatrix} r \\ v_r \\ \theta \\ \omega \end{pmatrix}_0 = \begin{pmatrix} (\vec{r}_i)_1 \\ (\vec{v}_i)_1 \\ 0 \\ (\vec{v}_i)_2 / (\vec{r}_i)_1 \end{pmatrix},$$

where v_i expressed in eq. 15 is to be used. The other equations stay the same as in the electrostatic spherical deflector case.

In a similar way to the electrostatic spherical deflector case, a DA transfer map is calculated and is listed in Sec. 5. A *COSY INFINITY* program is listed in App. A.

3 Transfer Map Calculation Using Lagrange Coefficients in Laboratory Coordinates

Motion in a central field with a potential energy of the form $U(r) = -\alpha/r + \text{const}$ is described by conventional Kepler theory; in particular, by the equation of orbit

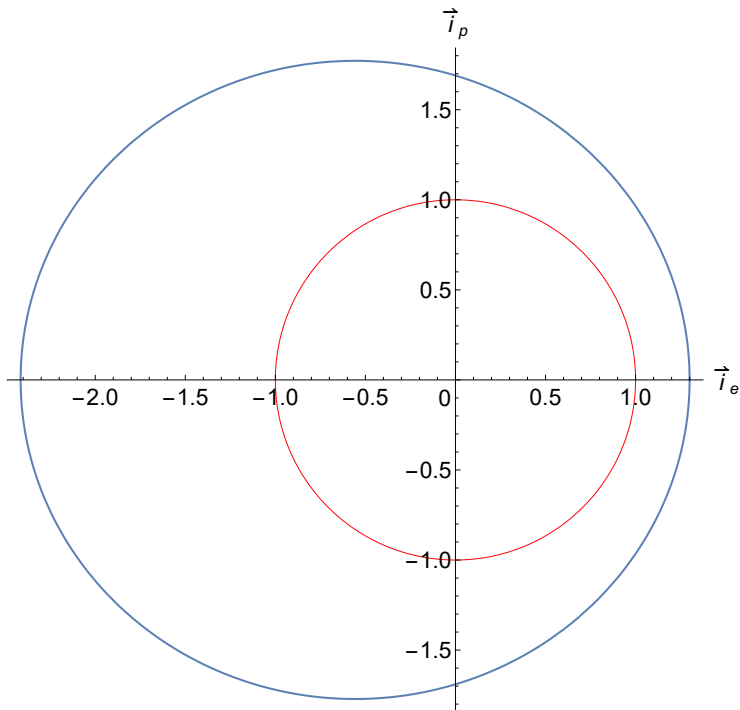$$r = \frac{p}{1 + e \cos f}, \tag{19}$$

Figure 1: Orbit of a particle launched counter-clockwise at polar angle $f_i = 0$ with initial beamline coordinates $(x_i, a_i) = (0.3, 0)$ through an electrostatic spherical deflector (blue). The reference orbit (red) has the radius $r_0 = 1\,\mathrm{m}$. The plot illustrates the basis vectors $\left(\vec{i}_e, \vec{i}_p\right)$ of the heliocentric coordinate system in relation to the orbit geometry. The plot was generated by the *Mathematica* notebook for integration of the ODEs for individual orbits in the polar laboratory coordinate system listed in App. B.

where p is the focal parameter, e is the orbit eccentricity, and polar angle f is called the true anomaly [1, p. 117]. True anomaly $f = 0$ corresponds to the direction of the perihelion, that is, the point of the orbit nearest to the orbital focus at the origin of the polar coordinate system (r, f).

The particle position $\vec{r}$ is expressed as

$$\vec{r} = \vec{i}_e r \cos f + \vec{i}_p r \sin f, \tag{20}$$

in terms of polar coordinates (r, f) and the basis vectors $\left(\vec{i}_e, \vec{i}_p\right)$ of the heliocentric coordinate system, where $\vec{i}_e$ is a unit vector in the direction of perihelion and unit vector $\vec{i}_p$ is chosen so that $\vec{i}_e \times \vec{i}_p$ is co-directional with the vector of the angular velocity of the particle, as Fig. 1 illustrates.

According to the second Newton law, zero radial acceleration at the reference

orbit requires

$$\frac{mv_0^2}{r_0} = \frac{\alpha}{r_0^2}.$$

Thus, the parameter $\mu = \alpha/m$ can be expressed as

$$\mu = \frac{\alpha}{m} = v_0^2 r_0 \tag{21}$$

in terms of the reference velocity v_0 and the reference orbit radius r_0.

The focal parameter p can be expressed in terms of μ and the massless angular momentum $h = |\vec{r} \times \vec{v}|$ as

$$p = \frac{h^2}{\mu}, \tag{22}$$

which is obtained in course of a standard derivation of the equation of orbit, eq. 19, as in [1, pp. 114–116].

We note that taking the time derivative of both sides of eq. 19 gives

$$\dot{r} = \dot{f} r \frac{e \sin f}{1 + e \cos f}. \tag{23}$$

Considering eqns. 23 and 22 and conservation of massless angular momentum $h = \dot{f} r^2$, taking the time derivative of the position vector $\vec{r}$ in eq. 20 yields for the particle velocity

$$\begin{aligned}
\vec{v} &= \vec{i}_e \left(\dot{r} \cos f - r\dot{f} \sin f \right) + \vec{i}_p \left(\dot{r} \sin f + r\dot{f} \cos f \right) = \\
&= \vec{i}_e \frac{h}{r} \left(\frac{e \sin f}{1 + e \cos f} \cos f - \sin f \right) + \vec{i}_p \frac{h}{r} \left(\frac{e \sin f}{1 + e \cos f} \sin f + \cos f \right) = \\
&= \vec{i}_e \frac{h}{r} \left(-\frac{\sin f}{1 + e \cos f} \right) + \vec{i}_p \frac{h}{r} \left(\frac{e + \cos f}{1 + e \cos f} \right) = \\
&= -\vec{i}_e \frac{\mu}{h} \sin f + \vec{i}_p \frac{\mu}{h} \left(e + \cos f \right).
\end{aligned} \tag{24}$$

Rewriting eqns. 20 and 24 in matrix form, we have

$$\begin{pmatrix} \vec{r} \\ \vec{v} \end{pmatrix} = A \begin{pmatrix} \vec{i}_e \\ \vec{i}_p \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} r \cos f & r \sin f \\ -\frac{\mu}{h} \sin f & \frac{\mu}{h} \left(e + \cos f \right) \end{pmatrix}. \tag{25}$$

We note that the determinant of the matrix A is, considering eq. 22,

$$\begin{aligned}
\det A &= r \frac{\mu}{h} \left(e + \cos f \right) \cos f + r \frac{\mu}{h} \sin^2 f = \\
&= r \frac{\mu}{h} \left(1 + e \cos f \right) = r \frac{\mu}{h} \frac{p}{r} = h.
\end{aligned} \tag{26}$$

Let f_i be the initial true anomaly of a particle in the spherical electrostatic deflector. Then, from the equation of orbit in eq. 19,

$$e \cos f_\mathrm{i} = \frac{p}{r_\mathrm{i}} - 1, \tag{27}$$

8

where r_i is the initial radius.

Taking the scalar product of initial position $\vec{r}_i$ and initial velocity $\vec{v}_i$, we obtain from eq. 25 that

$$\vec{r}_i \cdot \vec{v}_i = -r_i \frac{\mu}{h} \sin f_i \cos f_i + r_i \frac{\mu}{h} \sin f_i \left(e + \cos f_i\right) =$$

$$= r_i \frac{\mu}{h} e \sin f_i.$$

Hence,

$$e \sin f_i = \frac{\sqrt{p}\sigma_0}{r_i}, \tag{28}$$

where σ_0 is defined as [1, p. 130]

$$\sigma_0 = \frac{\vec{r}_i \cdot \vec{v}_i}{\sqrt{\mu}}.$$

Let $\theta = f_f - f_i$ be the true anomaly difference between the final and initial positions. Applying eqns. 27 and 28 to the equation of orbit in eq. 19 yields the final radius

$$
\begin{aligned}
r_f &= \frac{p}{1 + e \cos f_f} = \frac{p}{1 + e \cos \left(f_i + \theta\right)} = \\
&= \frac{p}{1 + \cos \theta \cos f_i - \sin \theta \sin f_i} = \\
&= \frac{p}{1 + \left(\frac{p}{r_i} - 1\right) \cos \theta - \frac{\sqrt{p}\sigma_0}{r_i} \sin \theta} = \\
&= r_i \frac{p}{r_i + (p - r_i) \cos \theta - \sqrt{p}\sigma_0 \sin \theta}.
\end{aligned}
\tag{29}
$$

Solving eq. 25 for the basis vectors $\left(\vec{i}_e, \vec{i}_p\right)$ of the heliocentric coordinate system in terms of the initial position $\vec{r}_i$, velocity $\vec{v}_i$, and true anomaly f_i, considering that $\det A = h$ according to eq. 26, we have

$$
\begin{aligned}
\begin{pmatrix} \vec{i}_e \\ \vec{i}_p \end{pmatrix} &= A_i^{-1} \begin{pmatrix} \vec{r}_i \\ \vec{v}_i \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} \frac{\mu}{h}\left(e + \cos f_i\right) & -r_i \sin f_i \\ \frac{\mu}{h} \sin f_i & r_i \cos f_i \end{pmatrix} \begin{pmatrix} \vec{r}_i \\ \vec{v}_i \end{pmatrix} = \\
&= \begin{pmatrix} \frac{\mu}{h^2}\left(e + \cos f_i\right) & -\frac{r_i}{h} \sin f_i \\ \frac{\mu}{h^2} \sin f_i & \frac{r_i}{h} \cos f_i \end{pmatrix} \begin{pmatrix} \vec{r}_i \\ \vec{v}_i \end{pmatrix}.
\end{aligned}
\tag{30}
$$

An analytic expression of the final position $\vec{r}_f$ and velocity $\vec{v}_f$ in terms of the initial position $\vec{r}_i$ and velocity $\vec{v}_i$ is obtained by inserting eq. 30 in eq. 25, which gives

$$
\begin{aligned}
\begin{pmatrix} \vec{r}_f \\ \vec{v}_f \end{pmatrix} &= A_f A_i^{-1} \begin{pmatrix} \vec{r}_i \\ \vec{v}_i \end{pmatrix} = \begin{pmatrix} r_f \cos f_f & r_f \sin f_f \\ -\frac{\mu}{h} \sin f_f & \frac{\mu}{h}\left(e + \cos f_f\right) \end{pmatrix} \cdot \\
&\cdot \begin{pmatrix} \frac{\mu}{h^2}\left(e + \cos f_i\right) & -\frac{r_i}{h} \sin f_i \\ \frac{\mu}{h^2} \sin f_i & \frac{r_i}{h} \cos f_i \end{pmatrix} \begin{pmatrix} \vec{r}_i \\ \vec{v}_i \end{pmatrix} = \\
&= \begin{pmatrix} F & G \\ F_t & G_t \end{pmatrix} \begin{pmatrix} \vec{r}_i \\ \vec{v}_i \end{pmatrix}.
\end{aligned}
\tag{31}
$$

The matrix on the left is usually referred to as the transition matrix. Note that this is not the same as the so-called transfer matrix of beam physics, because the transition matrix contains all nonlinear effects by virtue of its elements on orbit parameters. Specifically, we obtain for the transition matrix elements: the transition matrix element F is

$$
\begin{aligned}
F &= \frac{\mu}{h^2} r_{\mathrm{f}} \left[\cos f_{\mathrm{f}} \left(e + \cos f_{\mathrm{i}} \right) + \sin f_{\mathrm{f}} \sin f_{\mathrm{i}} \right] = \\
&= \frac{\mu}{h^2} r_{\mathrm{f}} \left(e \cos f_{\mathrm{f}} + \cos \theta \right) = \\
&= \frac{\mu}{h^2} r_{\mathrm{f}} \left[\left(\frac{p}{r_{\mathrm{f}}} - 1 \right) + \cos \theta \right] = \\
&= 1 - \frac{r_{\mathrm{f}}}{p} \left(1 - \cos \theta \right),
\end{aligned}
$$

the transition matrix element F_t is

$$
\begin{aligned}
F_t &= \frac{\mu^2}{h^3} \left[- \sin f_{\mathrm{f}} \left(e + \cos f_{\mathrm{i}} \right) + \left(e + \cos f_{\mathrm{f}} \right) \sin f_{\mathrm{i}} \right] = \\
&= \frac{\mu^2}{h^3} \left[e \left(\sin f_{\mathrm{i}} - \sin f_{\mathrm{f}} \right) - \sin \theta \right] = \\
&= \frac{\mu^2}{h^3} \left[e \left(\sin f_{\mathrm{i}} - \sin f_{\mathrm{i}} \cos \theta - \sin \theta \cos f_{\mathrm{i}} \right) - \sin \theta \right] = \\
&= \frac{\mu^2}{h^3} \left[e \sin f_{\mathrm{i}} \left(1 - \cos \theta \right) - \sin \theta \left(1 + e \cos f_{\mathrm{i}} \right) \right] = \\
&= \frac{\mu^2}{h^3} \left[\frac{\sqrt{p}\sigma_0}{r_{\mathrm{i}}} \left(1 - \cos \theta \right) - \frac{p}{r_{\mathrm{i}}} \sin \theta \right] = \\
&= \frac{\sqrt{\mu}}{r_{\mathrm{i}} p} \left[\sigma_0 \left(1 - \cos \theta \right) - \sqrt{p} \sin \theta \right],
\end{aligned}
$$

the transition matrix element G is

$$
\begin{aligned}
G &= \frac{1}{h} r_{\mathrm{f}} r_{\mathrm{i}} \left(- \cos f_{\mathrm{f}} \sin f_{\mathrm{i}} + \sin f_{\mathrm{f}} \cos f_{\mathrm{i}} \right) = \\
&= \frac{r_{\mathrm{f}} r_{\mathrm{i}}}{h} \sin \theta = \frac{r_{\mathrm{f}} r_{\mathrm{i}}}{\sqrt{\mu p}} \sin \theta,
\end{aligned}
$$

and the transition matrix element G_t is

$$
\begin{aligned}
G_t &= \frac{\mu}{h^2} r_{\mathrm{i}} \left[\sin f_{\mathrm{f}} \sin f_{\mathrm{i}} + \left(e + \cos f_{\mathrm{f}} \right) \cos f_{\mathrm{i}} \right] = \\
&= \frac{\mu}{h^2} r_{\mathrm{i}} \left(e \cos f_{\mathrm{i}} + \cos \theta \right) = \\
&= \frac{\mu}{h^2} r_{\mathrm{i}} \left[\left(\frac{p}{r_{\mathrm{i}}} - 1 \right) + \cos \theta \right] = \\
&= 1 - \frac{r_{\mathrm{i}}}{p} \left(1 - \cos \theta \right).
\end{aligned}
$$

Thus, we obtained a transition matrix

$$\Phi = \begin{pmatrix} F & G \\ F_t & G_t \end{pmatrix}$$

that expresses the final coordinates $(\vec{r}_{\mathrm{f}}, \vec{v}_{\mathrm{f}})$ as a function of the initial coordinates $(\vec{r}_{\mathrm{i}}, \vec{v}_{\mathrm{i}})$ as

$$\begin{pmatrix} \vec{r}_{\mathrm{f}} \\ \vec{v}_{\mathrm{f}} \end{pmatrix} = \Phi \begin{pmatrix} \vec{r}_{\mathrm{i}} \\ \vec{v}_{\mathrm{i}} \end{pmatrix} \tag{32}$$

and comprises elements [1, pp. 128–131]

$$F(\theta) = 1 - \frac{r_{\mathrm{f}}}{p}(1 - \cos\theta),$$

$$F_t(\theta) = \frac{\sqrt{\mu}}{r_{\mathrm{i}}p}[\sigma_0(1 - \cos\theta) - \sqrt{p}\sin\theta],$$

$$G(\theta) = \frac{r_{\mathrm{f}}r_{\mathrm{i}}}{\sqrt{\mu p}}\sin\theta,$$

$$G_t(\theta) = 1 - \frac{r_{\mathrm{i}}}{p}(1 - \cos\theta).$$

The elements F, F_t, G, and G_t of the transition matrix Φ are called Lagrange coefficients. The Lagrange coefficients F_t and G_t are simply time derivatives of F and G, respectively.

Applying the transfer matrix from eq. 32 to the initial position $\vec{v}_{\mathrm{i}}$ and velocity $\vec{p}_{\mathrm{i}}$ from eqns. 6 and 7, we obtain the final position $\vec{r}_{\mathrm{f}}$ and velocity $\vec{v}_{\mathrm{f}}$.

The final (x, a) coordinates are obtained from the final position

$$\vec{r}_{\mathrm{f}} = r_{\mathrm{f}}(\cos\theta_{\mathrm{f}}, \sin\theta_{\mathrm{f}})$$

and velocity $\vec{v}_{\mathrm{f}}$ as

$$(x_{\mathrm{f}}, a_{\mathrm{f}}) = \left(r_{\mathrm{f}} - r_0, \frac{\vec{v}_{\mathrm{f}} \cdot \vec{r}_{\mathrm{f}}}{v_0 r_{\mathrm{f}}}\right).$$

We calculated the DA transfer map of the electrostatic spherical deflector, in (x, a) beamline coordinates, with reference orbit radius $r_0 = 1$ m, using the transition matrix with elements as the Lagrange coefficients in terms of the true anomaly difference. This transfer map is listed in Sec. 4.

The transfer map only depends on reference orbit radius r_0 and the central angle of the tracked sector of the electrostatic spherical deflector. As long as the reference orbit radius r_0 is kept the same by adjusting the charge of the inner sphere of the deflector, the transfer map does not depend on the charged particle's kinetic energy, mass, or charge. Indeed, considering eqns. 22, 9, 6, and

21, the focal parameter p is

$$p = \frac{h^2}{\mu} = \frac{|\vec{r}_i \times \vec{v}_i|^2}{v_0^2 r_0} =$$

$$= \frac{(r_0 + x_i)^2 \left[v_i^2 - (a_i v_0)^2 \right]}{v_0^2 r_0} =$$

$$= \frac{(r_0 + x_i)^2 \left[v_0^2 - \frac{2\alpha}{m} \left(\frac{1}{r_0} - \frac{1}{r_0 + x_i} \right) - (a_i v_0)^2 \right]}{v_0^2 r_0} =$$

$$= \frac{(r_0 + x_i)^2 \left[v_0^2 - 2v_0^2 r_0 \left(\frac{1}{r_0} - \frac{1}{r_0 + x_i} \right) - (a_i v_0)^2 \right]}{v_0^2 r_0} =$$

$$= \frac{(r_0 + x_i) \left[2r_0 - \left(1 + a_i^2 \right) (r_0 + x_i) \right]}{r_0},$$

and the eccentricity e is [1, p. 116] such that, considering eqns. 6, 4, and 21,

$$1 - e^2 = p \left(\frac{2}{r} - \frac{v^2}{\mu} \right) = p \left(\frac{2}{r_0 + x_i} - \frac{v_i^2}{\mu} \right) =$$

$$= p \left[\frac{2}{r_0 + x_i} - \frac{v_0^2 - 2v_0^2 r_0 \left(\frac{1}{r_0} - \frac{1}{r_0 + x_i} \right)}{v_0^2 r_0} \right] =$$

$$= p \left[\frac{2}{r_0 + x_i} - \frac{1 - 2r_0 \left(\frac{1}{r_0} - \frac{1}{r_0 + x_i} \right)}{r_0} \right] =$$

$$= p \left[\frac{2}{r_0 + x_i} - \frac{2r_0 - (r_0 + x_i)}{r_0 (r_0 + x_i)} \right] = \frac{p}{r_0}.$$

Thus, the focal parameter p and the eccentricity e depend only on reference orbit radius r_0 and initial beamline coordinates (x_i, a_i).

To obtain a DA-valued transfer map, we set the initial phase space coordinates as

$$(x_i, a_i) = (d_1, d_2) = (\texttt{DA}\,(1), \texttt{DA}\,(2)),$$

the respective DA generators [2, pp. 86–96]. We performed the calculations with the computation order 3. A *COSY INFINITY* program to obtain the transfer map is listed in App. C.

4 Transfer Maps of an Electrostatic Spherical Deflector and Comparison

Here, we list and compare the DA transfer maps for particles passing through a $45°$ sector of the electrostatic spherical deflector, calculated

1. by integration of the ODEs of motion in polar laboratory coordinates (eq. 11) using a 4th order Runge-Kutta integrator;

2. using the transition matrix with elements as the Lagrange coefficients in terms of the true anomaly difference;

3. for *COSY INFINITY*'s built-in electrostatic spherical deflector element `ESP`; and

4. using the code sequence `E S` in the program *GIOS*.

In all cases, the reference orbit radius is $r_0 = 1\,\text{m}$, and non-relativistic equations of motion were used. For definiteness, the particles in the bunch were set to kinetic energy 1 MeV, mass 1 amu, and charge 1 e; however, as noted above, this setting has no impact on the orbit geometry as long as the reference orbit radius is kept the same by adjusting the voltages of the inner and outer spherical shells of the electrostatic spherical deflector. For visual transfer map comparison purposes, the computation order 3 was used in all cases, except for *GIOS*, where the computation order 2 was used.

The Jacobian $M = \text{Jac}(\mathcal{M})$ of the transfer map $\mathcal{M}$ of any Hamiltonian system satisfies the symplecticity condition $M \cdot \hat{J} \cdot M^{\mathrm{T}} = \hat{J}$ [2, pp. 155–159], where the phase space coordinates are ordered as $z = (q_1, \ldots, q_m, p_1, \ldots, p_m)$, $(q_1, \ldots, q_m)$ are the canonical position coordinates and $(p_1, \ldots, p_m)$ are the conjugate momentum coordinates,

$$\hat{J} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

I_m is an $m \times m$ identity matrix, and m is the phase space dimension.

For each transfer map calculation case, we computed deviations from the conditions of symplecticity [2, pp. 155–159][6] for the first and second order aberration coefficients and motion in the x-a plane. These conditions of symplecticity are as follows:

$$
\begin{aligned}
g_1 &= (x|x)(a|a) - (a|x)(x|a) - 1 = 0, \\
g_2 &= (x|x)(a|xa) - (a|x)(x|xa) + (x|xx)(a|a) - (a|xx)(x|a) = 0, \\
g_3 &= (x|x)(a|aa) - (a|x)(x|aa) + (x|xa)(a|a) - (a|xa)(x|a) = 0,
\end{aligned}
\tag{33}
$$

where the aberration coefficient $(z_i | z_{j_1} \cdots z_{j_n})$ is the partial derivative

$$(z_i | z_{j_1} \cdots z_{j_n}) = \left(\frac{\partial^n (\mathcal{M}(z))_i}{\partial z_{j_1} \cdots \partial z_{j_n}} \right)_{z=0}$$

of the i-th component of the respective transfer map $\mathcal{M}$ applied to a coordinate vector $z = (z_1 \cdots z_{2m})$, and $2m$ is the number of phase space coordinates.

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4.1 Transfer Map Obtained by Integrating the ODEs in Laboratory Coordinates

The transfer map from integration of the ODEs of motion in polar laboratory coordinates using a 4th order Runge-Kutta integrator is as follows.

```
TRANSFER MAP OBTAINED IN LAB COORDINATES
BY INTEGRATION OF THE ODEs OF MOTION
X_f
     I    COEFFICIENT              ORDER EXPONENTS
     1    0.7071067811865617          1    1 0  0
     2    0.7071067811865485          1    0 1  0
     3    −.4999999999998140          2    2 0  0
     4    0.9999999999999932          2    1 1  0
     5    0.2071067811865112          2    0 2  0
     6    −.3535533905929528          3    3 0  0
     7    0.6066017177973747E−01      3    1 2  0
     8    0.2928932188133920          3    0 3  0
     ─────────────────────────────────────────

A_f
     I    COEFFICIENT              ORDER EXPONENTS
     1    −.7071067811864190          1    1 0  0
     2    0.7071067811865486          1    0 1  0
     3    −.7071067811865216          2    0 2  0
     4    −.3535533905931392          3    3 0  0
     5    −1.060660171779751          3    1 2  0
     ─────────────────────────────────────────
```

DA coefficients with absolute values less than 10^{-11} are omitted.

The deviations g_1, g_2, and g_3 from the conditions of symplecticity listed in eq. 33 in this case were as follows:

$$g_1 = -0.7949196856316121 \times 10^{-13},$$
$$g_2 = -0.2514743315828557 \times 10^{-12},$$
$$g_3 = -0.1188442801551391 \times 10^{-12}.$$

The 4th order Runge-Kutta integrator code for calculation of this transfer map is listed in App. A.

4.2 Transfer Map Obtained Using Lagrange Coefficients in Laboratory Coordinates

The transfer map obtained in laboratory coordinates using the Lagrange coefficients transition matrix is as follows.

```
TRANSFER MAP OBTAINED IN LAB COORDINATES
USING LAGRANGE COEFFICIENTS
X_f
     I    COEFFICIENT              ORDER EXPONENTS
     1   0.7071067811865475          1    1 0   0
     2   0.7071067811865475          1    0 1   0
     3   −.5000000000000000          2    2 0   0
     4    1.000000000000000          2    1 1   0
     5   0.2071067811865475          2    0 2   0
     6   −.3535533905932737          3    3 0   0
     7   0.6066017177982122E−01      3    1 2   0
     8   0.2928932188134523          3    0 3   0
     ─────────────────────────────────────────────

A_f
     I    COEFFICIENT              ORDER EXPONENTS
     1   −.7071067811865475          1    1 0   0
     2   0.7071067811865476          1    0 1   0
     3   −.7071067811865475          2    0 2   0
     4   −.3535533905932737          3    3 0   0
     5   −1.060660171779821          3    1 2   0
     ─────────────────────────────────────────────
```

DA coefficients with absolute values less than 10^{-11} are omitted.

The deviations g_1, g_2, and g_3 from the conditions of symplecticity listed in eq. 33 in this case were as follows:

$$g_1 = -0.1110223024625157 \times 10^{-15},$$
$$g_2 = 0.3119771259853192 \times 10^{-16},$$
$$g_3 = 0.1110223024625157 \times 10^{-15}.$$

A *COSY INFINITY* code for calculation of this transfer map is listed in App. C.

4.3 Transfer Map of *COSY INFINITY*'s Built-In Electrostatic Spherical Deflector Element `ESP`

The transfer map of *COSY INFINITY*'s built-in electrostatic spherical deflector element `ESP`, obtained using non-relativistic equations of motion, is as follows.

```
TRANSFER MAP OF COSY INFINITY'S ESP ELEMENT

X_f
      I    COEFFICIENT              ORDER EXPONENTS
      1    0.7071067811865475         1    1 0  0
      2    0.7071067811865475         1    0 1  0
      3   −.4999999999999999          2    2 0  0
      4    1.000000000000000          2    1 1  0
      5    0.2071067811865475         2    0 2  0
      6   −.3535533905932738          3    3 0  0
      7    0.6066017177982123E−01     3    1 2  0
      8    0.2928932188134525         3    0 3  0
      ─────────────────────────────────────────────

A_f
      I    COEFFICIENT              ORDER EXPONENTS
      1   −.7071067811865475          1    1 0  0
      2    0.7071067811865475         1    0 1  0
      3   −.7071067811865475          2    0 2  0
      4   −.3535533905932737          3    3 0  0
      5   −1.060660171779821          3    1 2  0
      ─────────────────────────────────────────────
```

The deviations g_1, g_2, and g_3 from the conditions of symplecticity listed in eq. 33 were as follows:

$$g_1 = -0.2220446049250313 \times 10^{-15},$$
$$g_2 = 0.2220446049250313 \times 10^{-15},$$
$$g_3 = 0.3330669073875470 \times 10^{-15}.$$

The *COSY INFINITY* code for calculation of this transfer map of the `ESP` element is listed in App. D. We note that for simplicity, essentially the same result can be obtained using the relativistic equations of motion that are by default used in *COSY INFINITY* by simply using a very low kinetic energy for the calculation. For example, the transfer map computed using the relativistic equations with the kinetic energy 10^{-7} agrees with the above listed transfer map to about 10^{-10}. In this case, the values of the deviations g_1, g_2, and g_3 are $\sim 10^{-15}$.

4.4 Transfer Map Computed Using the Electrostatic Spherical Deflector Element in the Program *GIOS*

The 2nd order transfer map of the electrostatic spherical deflector computed using the code sequence E S in the program *GIOS* is as follows.

```
TRANSFER MAP COMPUTED USING THE PROGRAM GIOS


X_f
      I    COEFFICIENT              ORDER EXPONENTS
      1    0.7071067812               1    1 0   0
      2    0.7071067812               1    0 1   0
      3    −.5000000000               2    2 0   0
      4    0.2928932188               2    1 1   0
      5    0.2071067812               2    0 2   0




      ─────────────────────────────────────────────

A_f
      I    COEFFICIENT              ORDER EXPONENTS
      1    −.7071067812               1    1 0   0
      2    0.7071067812               1    0 1   0
      3    −.5000000000               2    2 0   0
      4    −.7071067812               2    1 1   0
      5    −.2071067812               2    0 2   0
      ─────────────────────────────────────────────
```

The deviations g_1, g_2, and g_3 from the conditions of symplecticity listed in eq. 33 were as follows:

$$g_1 = 0.3804934145534844 \times 10^{-10},$$
$$g_2 = -0.2928932188380493,$$
$$g_3 = 0.7071067812000000.$$

We note that the deviations g_2 and g_3 are significant in magnitude and indicate error(s) in the program *GIOS*.

The *GIOS* input for calculation of this transfer map is listed in App. E. We also note that these differences are not due to the fact that *GIOS* uses momentum-like coordinates that differ from those of *COSY INFINITY*. The respective effects manifest themselves only in order three in x and a terms, which we are not comparing here.

5 Transfer Maps of an Electrostatic Cylindrical Deflector and Comparison

In this section, we list and compare the DA transfer maps for particles passing through a 45° sector of the electrostatic cylindrical deflector, calculated

1. by integration of the ODEs of motion in polar laboratory coordinates (eq. 18) using a 4th order Runge-Kutta integrator;

2. for *COSY INFINITY*'s built-in electrostatic spherical deflector element `ECL`; and

3. using the code sequence `E S` in the program *GIOS*.

In all cases, the reference orbit radius is $r_0 = 1\,\mathrm{m}$, and non-relativistic equations of motion were used. For definiteness, the particles in the bunch were set to kinetic energy $1\,\mathrm{MeV}$, mass $1\,\mathrm{amu}$, and charge $1\,e$; however, as noted above, this setting has no impact on the orbit geometry as long as the reference orbit radius is kept the same by adjusting the voltages of the inner and outer cylindrical shells of the electrostatic cylindrical deflector. For visual transfer map comparison purposes, the computation order 3 was used in all cases, except for *GIOS*, where the computation order 2 was used.

5.1 Transfer Map Obtained by Integrating the ODEs in Laboratory Coordinates

The transfer map from integration of the ODEs of motion in polar laboratory coordinates using a 4th order Runge-Kutta integrator is as follows.

```
TRANSFER MAP OBTAINED IN LAB COORDINATES
BY INTEGRATION OF THE ODEs OF MOTION
X_f
     I   COEFFICIENT              ORDER EXPONENTS
     1   0.4440158403262955         1     1  0   0
     2   0.6335810656653742         1     0  1   0
     3   −1.029322282408186         2     2  0   0
     4   0.4452197131126942         2     1  1   0
     5   0.9767302144878563E−01     2     0  2   0
     6   −.9310536195453081         3     3  0   0
     7   −.7814348139392595         3     2  1   0
     8   −.7214969045085385         3     1  2   0
     9   0.1172683765076191         3     0  3   0
     _____

A_f
     I   COEFFICIENT              ORDER EXPONENTS
     1   −1.267162131330700         1     1  0   0
     2   0.4440158403261798         1     0  1   0
     3   −.3987403747459493         2     2  0   0
     4   −.3499052358015541         2     1  1   0
     5   −.7510014111250763         2     0  2   0
     6   −.6758776475462285         3     3  0   0
     7   −.2919765941780568         3     2  1   0
     8   −1.233526213798069         3     1  2   0
     9   −.2301781799921400         3     0  3   0
     _____
```

The deviations g_1, g_2, and g_3 from the conditions of symplecticity listed in eq. 33 in this case were as follows:

$$g_1 = -0.7316369732279782 \times 10^{-13},$$
$$g_2 = 0.1615374500829603 \times 10^{-12},$$
$$g_3 = -0.2078615057854449 \times 10^{-12}.$$

The 4th order Runge-Kutta integrator code for calculation of this transfer map is listed in App. A.

5.2 Transfer Map of *COSY INFINITY*'s Built-In Electrostatic Cylindrical Deflector Element `ECL`

The transfer map of *COSY INFINITY*'s built-in electrostatic spherical deflector element `ECL`, obtained using non-relativistic equations of motion, is as follows.

```
TRANSFER MAP OF COSY INFINITY'S ESP ELEMENT

X_f
```

I	COEFFICIENT	ORDER	EXPONENTS		
1	0.4440158403262133	1	1	0	0
2	0.6335810656653997	1	0	1	0
3	−1.029322282408272	2	2	0	0
4	0.4452197131126671	2	1	1	0
5	0.9767302144879608E−01	2	0	2	0
6	−.9310536195454117	3	3	0	0
7	−.7814348139394898	3	2	1	0
8	−.7214969045085790	3	1	2	0
9	0.1172683765076182	3	0	3	0

```
A_f
```

I	COEFFICIENT	ORDER	EXPONENTS		
1	−1.267162131330799	1	1	0	0
2	0.4440158403262133	1	0	1	0
3	−.3987403747459333	2	2	0	0
4	−.3499052358016756	2	1	1	0
5	−.7510014111251326	2	0	2	0
6	−.6758776475462280	3	3	0	0
7	−.2919765941781459	3	2	1	0
8	−1.233526213798173	3	1	2	0
9	−.2301781799921575	3	0	3	0

The deviations g_1, g_2, and g_3 from the conditions of symplecticity listed in eq. 33 were as follows:

$$g_1 = 0.4440892098500626 \times 10^{-15},$$
$$g_2 = 0.2220446049250313 \times 10^{-15},$$
$$g_3 = 0.2498001805406602 \times 10^{-15}.$$

The *COSY INFINITY* code for calculation of this transfer map of the `ECL` element is listed in App. D. We note that for simplicity, essentially the same result can be obtained using the relativistic equations of motion that are by default used in *COSY INFINITY* by simply using a very low kinetic energy for the calculation. For example, the transfer map computed using the relativistic equations with the kinetic energy 10^{-7} agrees with the above listed transfer map to about 10^{-10}. In this case, the values of the deviations g_1, g_2, and g_3 are $\sim 10^{-15}$.

5.3 Transfer Map Computed Using the Electrostatic Cylindrical Deflector Element in the Program *GIOS*

The 2nd order transfer map of the electrostatic cylindrical deflector computed using the code sequence E S in the program *GIOS* is as follows.

```
TRANSFER MAP COMPUTED USING THE PROGRAM GIOS

X_f
      I    COEFFICIENT              ORDER EXPONENTS
      1    0.4440158574              1     1 0  0
      2    0.6335810705              1     0 1  0
      3   −1.029322250               2     2 0  0
      4   −.1883613764               2     1 1  0
      5    0.9767302679E−01          2     0 2  0




      _____

A_f
      I    COEFFICIENT              ORDER EXPONENTS
      1   −1.267162098               1     1 0  0
      2    0.4440158574              1     0 1  0
      3   −.9613804652               2     2 0  0
      4   −1.399620967               2     1 1  0
      5   −.4696813690               2     0 2  0
      _____
```

The deviations g_1, g_2, and g_3 from the conditions of symplecticity listed in eq. 33 were as follows:

$$g_1 = 0.1705231511550664 \times 10^{-9},$$
$$g_2 = -0.5559841747496004,$$
$$g_3 = 0.6335810760905867.$$

We note that the deviations g_2 and g_3 are significant in magnitude and indicate error(s) in the program *GIOS*.

The *GIOS* input for calculation of this transfer map is listed in App. E. We also note that these differences are not due to the fact that *GIOS* uses momentum-like coordinates that differ from those of *COSY INFINITY*. The respective effects manifest themselves only in order three in x and a terms, which we are not comparing here.

6 Calculation Results, Comparison, and Conclusion

The electrostatic spherical deflector transfer maps calculated in laboratory coordinates by integration of the ODEs and using the Lagrange coefficients transition matrix are in very high agreement with the transfer map of *COSY INFINITY*'s built-in electrostatic spherical deflector element `ESP`. The deviations from the conditions of symplecticity g_1, g_2, and g_3 were the highest at $\sim 10^{-13}$ in case of integration of the ODEs in polar laboratory coordinates, the lowest at $\sim 10^{-16}$ in case of calculation using the Lagrange coefficients, and at $\sim 10^{-16}$ in case of the built-in element `ESP`.

The transfer map of the electrostatic spherical deflector computed using the program *GIOS* significantly disagrees with the other three transfer maps. However, the deviations g_2 and g_3 in the *GIOS* case were also significant in magnitude and indicate that the disagreement is due to error(s) in *GIOS*.

We arrive at an equivalent conclusion for the electrostatic cylindrical deflector transfer maps. The transfer map obtained by integrating the ODEs in laboratory coordinates agrees well with those obtained by *COSY INFINITY*'s built-in electrostatic cylindrical deflector element `ECL`, both satisfying the symplectic conditions. On the other hand, the transfer map obtained using *GIOS* significantly disagrees with the other two transfer maps, and also the symplectic condition was significantly deviated.

References

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A COSY INFINITY Code for Transfer Map Calculation Using ODEs in Laboratory Coordinates

The following is a *COSY INFINITY* code that calculates the transfer map of a $45°$ sector of the electrostatic spherical deflector in (x, a) beamline coordinates by integrating the ODEs of motion in polar laboratory coordinates. This version of the code uses a 4th order Runge-Kutta integrator with fixed step size. The inner sphere of the electrostatic spherical deflector is charged to result in a circular reference orbit of radius $r_0 = 1$ m. The DA computation order 3 is used.

```
INCLUDE 'COSY' ;

PROCEDURE RUN ;

VARIABLE Y1 4000 8 ; VARIABLE YT 4000 8 ;
VARIABLE XX0 4000 ; {Initial DA-valued x coordinate}
VARIABLE XX1 4000 ; {Final DA-valued x coordinate}
VARIABLE AA0 4000 ; {Initial DA-valued a coordinate}
VARIABLE AA1 4000 ; {Final DA-valued a coordinate}
VARIABLE NM 1 ; {DA variable size}
VARIABLE MU 1 ; {mu}
VARIABLE HPAR 4000 ; {h}

PROCEDURE FNC F X T ;
    {FUNCTION F(X,T) DEFINING THE ODEs OF MOTION
        X   COORDINATES VECTOR
        T   TIME (OR ARC LENGTH) }
    F(1) := X(2)/X(4) ; {r'}
    F(2) := -MU/X(1)/X(1)/X(4) + X(1)*X(4) ; {v_r'}
    F(3) := 1 ; {theta'}
    F(4) := -2*HPAR/X(1)/X(1)/X(1) * X(2)/X(4) ; {omega'}
ENDPROCEDURE ;

PROCEDURE RK4A N X0 X1 Y0 NS Y1 ;
    {FOURTH ORDER RUNGE KUTTA INTEGRATOR}
    VARIABLE I 1 ; VARIABLE J 1 ; VARIABLE T 1 ;
    VARIABLE Z1 NM 8 ; VARIABLE Z2 NM 8 ;
    VARIABLE Z3 NM 8 ; VARIABLE Z4 NM 8 ;
    VARIABLE Z5 NM 8 ; VARIABLE F NM 8 ;
    VARIABLE HS1 1 ;
    T := X0 ;
    HS1 := (X1-X0)/NS ;
    LOOP J 1 N ; Y1(J) := Y0(J) ; ENDLOOP ;
    LOOP I 1 NS ;
        FNC F Y1 T ;
        LOOP J 1 N ; Z1(J) := HS1*F(J) ; ENDLOOP ;
        LOOP J 1 N ; Z5(J) := Y1(J) + Z1(J)/2 ; ENDLOOP ;
        FNC F Z5 T+HS1/2 ;
        LOOP J 1 N ; Z2(J) := HS1*F(J) ; ENDLOOP ;
        LOOP J 1 N ; Z5(J) := Y1(J) + Z2(J)/2 ; ENDLOOP ;
        FNC F Z5 T+HS1/2 ;
        LOOP J 1 N ; Z3(J) := HS1*F(J) ; ENDLOOP ;
        LOOP J 1 N ; Z5(J) := Y1(J) + Z3(J) ; ENDLOOP ;
```

```
        FNC F Z5 T+HS1 ;
        LOOP J 1 N ; Z4(J) := HS1*F(J) ; ENDLOOP ;
        LOOP J 1 N ;
            Y1(J) := Y1(J) + (1/6)*(Z1(J)+2*Z2(J)+2*Z3(J)+Z4(J)) ;
        ENDLOOP ;
        T := T + HS1 ;
    ENDLOOP ;
ENDPROCEDURE ;

PROCEDURE KEPLERODEPOLAR R00 PHI X0 A0 X1 A1 ;
    {This procedure calculates the transfer map of the electrostatic
    spherical deflector in (x,a) coordinates by integrating the
    ODEs of motion in laboratory coordinates using a Runge−Kutta
    integrator.
    Input parameters:
        R00     Radius of the circular reference orbit
        PHI     Central angle of tracked sector of the deflector
        X0      Initial x coordinate
        A0      Initial a coordinate
        X1      Final x coordinate
        A1      Final a coordinate }
    VARIABLE R0 NM 2 ; {Initial radius vector}
    VARIABLE W0 NM 2 ; {Initial velocity vector}
    VARIABLE R1 NM 2 ; {Final radius vector}
    VARIABLE W1 NM 2 ; {Final velocity vector}
    VARIABLE DOT NM ; {A scalar product}
    VARIABLE R0S NM ; {Initial radius magnitude}
    VARIABLE RS NM ; {Final radius magnitude}
    VARIABLE SIGMA0 NM ; {sigma_0}
    VARIABLE P NM ; {Focal parameter}
    VARIABLE F NM ; VARIABLE G NM ; {Lagrange coefficients F, G}
    VARIABLE FT NM ; VARIABLE GT NM ; {Lagrange coefficients FT, GT}
    VARIABLE V0 NM ; {Reference velocity}
    VARIABLE V1 NM ; {Initial velocity magnitude}
    VARIABLE V2 NM ; {Final velocity magnitude}
    VARIABLE CHIE 1 ; {Electric rigidity}
    PHI := PHI*DEGRAD ;
    V0 := CONS(SQRT(2*ETA)*CLIGHT) ;
    R0(1) := X0 + R00 ; R0(2) := 0 ;
    CHIE := AMU*RE(M0)*V0*V0 / Z0/EZERO ; {R00*CHIE}
    V1 := SQRT(V0^2−2*Z0*EZERO*CHIE*(1−R00/(R0(1)))/AMU/RE(M0)) ;
    W0(1) := A0*V0 ; W0(2) := SQRT(V1*V1–W0(1)*W0(1)) ;
    MU := R00*CONS(V1)^2 ;
    HPAR := R0(1)*W0(2) ;
    YT(1) := R0(1) ;
    YT(2) := W0(1) ;
    YT(3) := 0 ;
    YT(4) := W0(2)/YT(1) ;
    {RK8A 4 0 PHI YT HS Y1 1e−5 RESCODE ;}
    RK4A 4 0 PHI YT 1e6 Y1;
    X1 := Y1(1) − R00 ;
    V2 := SQRT(Y1(2)*Y1(2)+Y1(1)*Y1(1)*Y1(4)*Y1(4)) ;
    A1 := Y1(2) / CONS(V2) ;
ENDPROCEDURE ;

OV 3 1 0 ;
RP 1 1 1 ;
```

```
NM :=  4000  ;
XX0  :=  DA( 1 )  ;  AA0  :=  DA( 2 )  ;

KEPLERODEPOLAR 1  45  XX0 AA0 XX1 AA1  ;
WRITE  6  'TRANSFER MAP OBTAINED IN LAB COORDINATES'  ;
WRITE  6  'BY INTEGRATION OF THE ODEs'  ;
WRITE  6  'X_f' XX1 'A_f' AA1  ;

ENDPROCEDURE ; RUN ; END ;
```

To integrate the ODEs of motion in polar laboratory coordinates to calculate the transfer map of a 45° sector of the electrostatic cylindrical deflector in (x, a), the code has to be adjusted according to the changes in equations as described in Sub-Sec. 2.2. The change in eq. 18 is to be reflected in the F(2) line in PROCEDURE FNC as listed below.

```
PROCEDURE FNC F X T  ;
    {FUNCTION F(X,T) DEFINING THE ODEs OF MOTION
        X   COORDINATES VECTOR
        T   TIME (OR ARC LENGTH)  }
    F( 1 )  :=  X( 2 )/X( 4 )  ;  { r ' }
    F( 2 )  :=  −MU/X( 1 )/X( 4 ) + X( 1 )*X( 4 )  ;  { v_r ' }
    F( 3 )  :=  1  ;  { theta ' }
    F( 4 )  :=  −2*HPAR/X( 1 )/X( 1 )/X( 1 )  *  X( 2 )/X( 4 )  ;  { omega ' }
ENDPROCEDURE  ;
```

The change in v_i in eq. 15 is to be reflected in the V1 line in PROCEDURE KEPLERODEPOLAR as listed below. The rest of the code can stay the same.

```
PROCEDURE KEPLERODEPOLAR R00 PHI X0 A0 X1 A1  ;
    {This procedure calculates the transfer map of the electrostatic
    cylindrical deflector in (x,a) coordinates by integrating the
    ODEs of motion in laboratory coordinates using a Runge−Kutta
    integrator.
    ...( continue )...
    CHIE  :=  ...( the same as above )...
    V1 :=  SQRT( V0^2−2*Z0*EZERO*CHIE*R00*LOG( R0( 1 )/R00 )/AMU/RE( M0 ) )  ;
    W0( 1 )  :=  ...( the same as above )...
    ...( continue )...
ENDPROCEDURE  ;
```

B *Mathematica* Code for Integration of the ODEs in Laboratory Coordinates

The following is a *Mathematica* notebook that integrates the ODEs of motion of a particle with specified initial beamline coordinates (x, a) through the electrostatic spherical deflector in polar laboratory coordinates, plots one turn of the orbit, and outputs the final beamline coordinates at true anomaly difference $\theta = 45°$. The inner sphere of the electrostatic spherical deflector is charged to result in a circular reference orbit of radius $r_0 = 1$ m.

```
(* Integration of the ODEs of motion for a charged \
particle of kinetic energy 1 MeV, mass 1 amu, and \
charge 1 e in an electrostatic spherical deflector \
with the inner sphere charge such that the particle \
would have a circular reference orbit of radius 1 m. *)
v00 = 13891388.79714028; (* Reference velocity *)
r00 = 1; (* Reference orbit radius *)
x = -0.5; (* Initial x beamline coordinate *)
a = 0.5; (* Initial a beamline coordinate *)
r0 = {x + r00, 0, 0}; (* Initial radius vector *)
v0 = {a v00, v00 Sqrt[1 - a^2], 0}; (* Initial velocity vector *)
mu = r00  v00^2; (* mu *)
vr0 = v0[[1]];
rr0 = r0[[1]];
omega0 = v0[[2]]/rr0; (* omega_0 *)
theta0 = 0; (* Initial polar angle *)
system = { (* ODEs and the initial condition *)
   vr'[t] == -mu/(omega[t] (rr[t])^2) + rr[t] omega[t],
   rr'[t] == vr[t]/omega[t],
   omega'[t] == -2 rr0 v0[[2]] vr[t]/(omega[t] (rr[t])^3),
   theta'[t] == 1,
   vr[0] == vr0,
   rr[0] == rr0,
   omega[0] == omega0,
   theta[0] == theta0
   };
sol = NDSolve[system, {vr, rr, omega, theta}, {t, 0, 2 \[Pi]}] ;
rsol[t_] = First[Evaluate[rr[t] /. sol]];
 (* Radius magnitude as a function of polar angle theta *)
vsol[t_] = First[Evaluate[vr[t] /. sol]];
 (* Velocity magnitude as a function of polar angle theta *)
Print[PolarPlot[
   rsol[t], {t, 0, 2 \[Pi]}]]; (* Plot one turn of the orbit *)
x1[t_] = rsol[t] - r00; (* x coordinate at polar angle theta *)
a1[t_] = vsol[t]/v00; (* a coordinate at polar angle theta *)
Print["x_f=", x1[Pi/4]];
Print["a_f=", a1[Pi/4]];
```

C *COSY INFINITY* Code for Transfer Map Calculation Using Lagrange Coefficients in Laboratory Coordinates

The following is a *COSY INFINITY* code that calculates the transfer map of a $45°$ sector of the electrostatic spherical deflector in (x, a) beamline coordinates using Lagrange coefficients in polar laboratory coordinates as described in Sec. 3. The DA computation order 3 is used.

```
INCLUDE 'COSY' ;

PROCEDURE RUN ;

VARIABLE XX0 4000 ; {Initial DA-valued x coordinate}
VARIABLE XX1 4000 ; {Final DA-valued x coordinate}
VARIABLE AA0 4000 ; {Initial DA-valued a coordinate}
VARIABLE AA1 4000 ; {Final DA-valued a coordinate}
VARIABLE NM 1 ; {DA variable size}

PROCEDURE KEPLERANALYTICPOLAR R00 PHI X0 A0 X1 A1 ;
    VARIABLE R0 NM 2 ; {Initial radius vector}
    VARIABLE W0 NM 2 ; {Initial velocity vector}
    VARIABLE R1 NM 2 ; {Final radius vector}
    VARIABLE W1 NM 2 ; {Final velocity vector}
    VARIABLE MU NM ; {mu}
    VARIABLE DOT NM ; {A scalar product}
    VARIABLE R0S NM ; {Initial radius}
    VARIABLE RS NM ; {Final radius}
    VARIABLE SIGMA0 NM ; {sigma_0}
    VARIABLE P NM ; {Focal parameter p}
    VARIABLE F NM ; VARIABLE G NM ; {Lagrange coefficients F and G}
    VARIABLE FT NM ;
    VARIABLE GT NM ; {Lagrange coefficients F_t and G_t}
    VARIABLE V0 NM ; {Initial velocity at zero potential}
    VARIABLE V1 NM ; {Initial velocity}
    VARIABLE V2 NM ; {Final velocity}
    VARIABLE CHIE 1 ; {Electric rigidity}
    PHI := PHI*DEGRAD ;
    V0 := CONS(SQRT(2*ETA)*CLIGHT) ;
    R0(1) := X0 + R00 ; R0(2) := 0 ;
    CHIE := AMU*RE(M0)*V0*V0 / Z0/EZERO ; {R00*CHIE}
    V1 := SQRT(V0^2 - 2*Z0*EZERO*CHIE*(1-R00/(R0(1)))/AMU/RE(M0)) ;
    W0(1) := A0*V0 ;
    W0(2) := SQRT(V1*V1-W0(1)*W0(1)) ;
    MU := R00*CONS(V1)^2 ;
    DOT := R0(1)*W0(1) + R0(2)*W0(2) ;
    SIGMA0 := DOT/SQRT(MU) ;
    P := ((R0(1)*R0(1)+R0(2)*R0(2))*(W0(1)*W0(1)+W0(2)*W0(2))
        -DOT*DOT) / MU ;
    R0S := R0(1) ;
    RS := P*R0S / (R0S+(P-R0S)*COS(PHI)-SQRT(P)*SIGMA0*SIN(PHI)) ;
    F := 1 - (RS/P)*(1-COS(PHI)) ;
    G := (RS*R0S/SQRT(MU*P)) * SIN(PHI) ;
    FT := (SQRT(MU)/R0S/P) * (SIGMA0*(1-COS(PHI))-SQRT(P)*SIN(PHI)) ;
    GT := 1 - (R0S/P)*(1-COS(PHI)) ;
```

```
      R1(1) := F*R0(1) + G*W0(1)  ; R1(2) := F*R0(2) + G*W0(2)  ;
      W1(1) := FT*R0(1) + GT*W0(1)  ; W1(2) := FT*R0(2) + GT*W0(2)  ;
      X1 := (COS(PHI)*R1(1)+SIN(PHI)*R1(2)) − R00 ;
      V2 := SQRT(W1(1)*W1(1)+W1(2)*W1(2))  ;
      A1 := (COS(PHI)*W1(1)+SIN(PHI)*W1(2)) / CONS(V2) ;
ENDPROCEDURE ;

OV 3 1 0 ;
RP 1 1 1 ;

NM := 4000 ;
XX0 := DA(1) ; AA0 := DA(2) ;

KEPLERANALYTICPOLAR 1 45 XX0 AA0 XX1 AA1 ;
WRITE 6 'TRANSFER MAP OBTAINED IN LAB COORDINATES' ;
WRITE 6 'USING LAGRANGE COEFFICIENTS' ;
WRITE 6 'X_f' XX1 'A_f' AA1 ;

ENDPROCEDURE ; RUN ; END ;
```

D   *COSY INFINITY* Codes for Transfer Map Calculation of the Built-in Electrostatic Deflector Elements

The following is a *COSY INFINITY* code to compute the transfer map of a 45° sector of the electrostatic spherical deflector. The voltages on the plates of the electrostatic deflectors are picked to result in a circular reference orbit of radius $r_0 = 1\,\mathrm{m}$. The DA computation order 3 is used.

```
INCLUDE 'COSY' ;

PROCEDURE RUN ;

OV 3 1 0 ;
RP 1e−7 1 1 ;

UM ;
ESP 1 45 0.1 ; {Electrostatic spherical deflector}

WRITE 6 'TRANSFER MAP OF COSY INFINITY''S ESP ELEMENT' ;
WRITE 6 'X_f' MAP(1) 'A_f' MAP(2) ;

ENDPROCEDURE ; RUN ; END ;
```

To compute the transfer map of a 45° sector of the electrostatic cylindrical deflector, the `ESP` call line above is to be replaced by a `ECL` call.

```
...(the same as above)...
ECL 1 45 0.1 ; {Electrostatic cylindrical deflector}
...(the same as above)...
```

E *GIOS* Input for Transfer Map Calculation of the Electrostatic Deflector Elements

The following is a *GIOS* input to compute the transfer map of a 45° sector of the electrostatic spherical deflector. The computation order 2 is used. We note that the SC command is not essential for our purpose to compare with the transfer maps computed by *COSY INFINITY*.

```
R P  1  1  1  ;  Reference  Particle
C O  2  2  ;      Calculation  Order
S C  ;           Symplectic  Coordinate  system  for  map  printout
;
E S  1  45  0.01  1  −1  1  ;  Electrostatic  Sector ,  spherical
;
END  ;
END  ;
```

To compute the transfer map of a 45° sector of the electrostatic cylindrical deflector, the description of the ES command is modified, and a *GIOS* input is as follows. We note again that the SC command is not essential for our purpose.

```
R P  1  1  1  ;  Reference  Particle
C O  2  2  ;      Calculation  Order
S C  ;           Symplectic  Coordinate  system  for  map  printout
;
E S  1  45  0.01  0  0  0  ;  Electrostatic  Sector ,  cylindrical
;
END  ;
END  ;
```