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Constructive Generation and Verification of Lyapunov Functions around Fixed Points of Nonlinear Dynamical Systems

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Abstract: An iterative method is developed that provides a transformation of coordinates of a dynamical system near a fixed point. In the new coordinates, Lyapunov functions and pseudo-Lyapunov functions can be determined. Using differential algebraic methods, the transformations can be chosen to belong to the same symmetry groups as the underlying motion. For area preserving stable motion, the method yields first integrals or near-first integrals of motion, and the convergence properties of the perturbative technique are tied to the question of integrability of the motion. For other stable motion, the method often yields Lyapunov functions that can be used to locally assert stability of the motion. Examples of the use of the method are given.

Key-Words: Nonlinear dynamical systems, stability, Lyapunov function, normal form method, verification.

1 Introduction

The study of nonlinear motion near a fixed point is a common problem of nonlinear dynamics, originating in the study of astronomical systems and nowadays playing an important role in the understanding of modern particle accelerators. There is a long history of the perturbative treatment of such motion, beginning with the work of Poincare[1].

The first step of the treatment consists in finding the flow of the $2v$ equations of motion $d/dt\,\vec{r} = \vec{f}(\vec{r}, t)$, i.e. the function representing the transformation of initial conditions $\vec{r}_i$ into final conditions $\vec{r}_f$ for the case the ODE has unique solutions. In the case the motion is autonomous or has a periodicity Δt, it is common to study the motion along discretized time steps Δt, resulting in the so-called Poincare map that transports coordinates by the period Δt :

$$\vec{r}_f = \mathcal{M}(\vec{r}_i).$$

The map $\mathcal{M}$ is usually studied in a perturbative way, and one common method is the iterative generation of its Taylor expansion $\mathcal{M}_n$ to a given order n. While the problem can be reduced to mere quadratures of functions made up of the solutions of the linear part, in practice the method is exceedingly complicated because of the usually tremendous growth of the complexity with order. An approach that could overcome these difficulties was the application of differential algebraic methods on spaces of truncated Taylor series[2], which allowed a very streamlined computation of Taylor coefficients to arbitrary order and is now probably the most commonly utilized technique for the study of Poincare maps in particle accelerator simulation codes [3, 4, 5]. An extension of the method even allows to obtain rigorous bounds for the remainder terms of the nth order Taylor expansion of $\mathcal{M}$ in a verified method [6].

After the Poincare map $\mathcal{M}$ of a given system is known, one of the important questions is to assess the stability of the system over extended times. This question can often be answered by finding a so-called Lyapunov function L that is non-decreasing along the orbit, i.e. it has the property that

$$L(\mathcal{M}(\vec{r})) \leq L(\vec{r}). \tag{1}$$

For a detailed discussion, see for example [7]. If each of the contour surfaces of the Lyapunov function L is bounded, this asserts boundedness of the motion since an orbit is always confined to the inside of the contour surface where it started. For many simple systems, Lyapunov functions can be established directly; but for complicated systems, their determination is often very difficult.

The special case of $L(\mathcal{M}(\vec{r})) = L(\vec{r})$ implies that the Lyapunov function is preserved and hence the dynamics restricted to its contour surfaces; it represents a first integral of the motion. The existence of first integrals often allows simplifications of the study of the motion, as they allow a reduction of the total number of free variables. Furthermore, similarly to the Lyapunov functions, if their contour surfaces are bounded, they can be utilized for stability estimates. Finally, even if the inequality in (1) can not be established outright, it can often be shown that

$$L(\mathcal{M}(\vec{r})) \leq L(\vec{r}) + d \tag{2}$$

for a sufficiently small value d over a domain A. In this case, L will be called a pseudo-Lyapunov function. Then estimates of the time of survival can be made; for if both the contour surfaces of $L = C_1$ and $L = C_2 > C_1$ lie in A, then an orbit beginning inside the contour surface C_1 will require at least $T = \Delta t \cdot (C_2 - C_1)/d$ to reach the contour surface C_2. Depending on the values of C_1, C_2 and d, often very favorable stability times can be obtained. More details about the practical use of the approach can be found for example in [8].

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In the following sections we present an iterative method for the construction of Lyapunov functions and approximate first integrals in a perturbative way for many systems. It is based on a sequence of nonlinear changes of variables in such a way that depending on the eigenvalue spectrum of the linear part of the Poincare map, Lyapunov functions or approximate invariants can be read off.

2 The Perturbative Normal Form Algorithm

The normal form algorithm consists of a sequence of coordinate transformations $\mathcal{A}$ of the nth order Taylor expansion $\mathcal{M}_n$ of the flow:

$$\mathcal{A} \circ \mathcal{M}_n \circ \mathcal{A}^{-1}.$$

The first such coordinate transformation is a linear coordinate transformation that diagonalizes the linear part of the map. For this process, we have to assume that there are $2v$ pairwise distinct eigenvalues. This, together with the fact that no eigenvalue should be unity and that their product is positive are the only requirements we have to demand for the map.

The eigenvalues are now grouped in such a way that complex conjugate pairs are together; the remaining real eigenvalues are grouped in pairs of equal sign, which is possible because of the positive product of all eigenvalues.

Altogether, the resulting diagonalization is such that the linear map assumes the following form:

$$\begin{pmatrix} r_1 e^{+i\mu_1} & & & & & \\ & r_1 e^{-i\mu_1} & & & & \\ & & . & & 0 & \\ & 0 & & . & & \\ & & & & r_v e^{+i\mu_v} & \\ & & & & & r_v e^{-i\mu_v} \end{pmatrix}. \tag{3}$$

Here the phases μ_j are either purely real or purely imaginary. For systems with stable linearization, none of the $r_j e^{\pm i\mu_j}$ must exceed unity in modulus.

For area preserving systems the determinant is unity, which entails that the product of the r_j must be unity. This implies that for such systems, for any $r_j < 1$ there is another with $r_j > 1$. Thus stable area preserving systems have $r_j = 1$ for all j, because otherwise there would be one j for which r_j exceeds unity, and thus at least one of $r_j e^{\pm i\mu_j}$ would have modulus larger than unity. This would also happen if a μ_j were imaginary. So all μ_j are real, and they are even nonzero because we demanded distinct eigenvalues.

The eigenvectors $s_j^{\pm}$ belonging to the eigenvalue $r_j e^{\pm i\mu_j}$ now provide a basis. We consider another set of vectors $t_j^{\pm}$ associated to the $s_j^{\pm}$ as follows:

$$\begin{aligned} t_j^{+} &= (s_j^{+} + s_j^{-})/2 \\ t_j^{-} &= (s_j^{+} - s_j^{-})/2i. \end{aligned} \tag{4}$$

In case of complex $s_j^{\pm}$, the $t_j^{\pm}$ are just the real and imaginary parts and thus are real. In the unstable case, t_j^{+} is real and t_j^{-} is imaginary. Obviously the $s_j^{\pm}$ can be expressed in terms of the $t_j^{\pm}$ as

$$\begin{aligned} s_j^{+} &= t_j^{+} + i\ t_j^{-} \\ s_j^{-} &= t_j^{+} - i\ t_j^{-}. \end{aligned}$$

In the following, we will perform the manipulations in the $s_j^{\pm}$, while the results are most easily interpreted in the $t_j^{\pm}$.

We now show that a map in the $s_j^{\pm}$ can be subjected to nonlinear coordinate transformations that considerably simplify the nonlinear terms. The advertised transformation to the new coordinates is carried out in an iterative manner consisting of n steps, the first of which was the diagonalization of the linear part. All further steps are purely nonlinear and do not affect the linear part anymore. The mth step transforms only the mth order of the map and leaves the lower orders unaffected.

We begin the mth step by splitting the momentary map $\mathcal{M}_n$ into its linear and nonlinear parts $\mathcal{R}$ and $\mathcal{S}_m$, i.e. $\mathcal{M}_n = \mathcal{R} + \mathcal{S}_m$. The linear part $\mathcal{R}$ has the form of (3). Then we perform a transformation using a map $\mathcal{A}_m$ of which we demand that to mth order has the form

$$\mathcal{A}_m = \mathcal{I} + \mathcal{T}_m,$$

where $\mathcal{T}_m$ vanishes to order $m-1$. Then, whatever the detailed higher order behavior of $\mathcal{A}_m$, up to order m, we have

$$\mathcal{A}_m^{-1} =_m \mathcal{I} - \mathcal{T}_m.$$

Indeed we have

$$(\mathcal{I} + \mathcal{T}_m) \circ (\mathcal{I} - \mathcal{T}_m) \;=_m\; (\mathcal{I} - \mathcal{T}_m) + \mathcal{T}_m \circ (\mathcal{I} - \mathcal{T}_m) \;=_m\; (\mathcal{I} - \mathcal{T}_m) + \mathcal{T}_m \;=\; \mathcal{I},$$

where in the second step use has been made of the fact that $\mathcal{T}_m$ has no parts of order lower than m, and hence only the linear part of anything that is inserted into it contributes to order m. To study the effect on the transformation, we first consider the terms of $\mathcal{T}_m$ of exact order m. To this end, we study the effect of the transformation, and infer that up to order m, we have

$$\begin{aligned}
\mathcal{A} \circ \mathcal{M}_n \circ \mathcal{A}^{-1} &=_m (\mathcal{I} + \mathcal{T}_m) \circ (\mathcal{R} + \mathcal{S}_m) \circ (\mathcal{I} - \mathcal{T}_m) \\
&=_m (\mathcal{I} + \mathcal{T}_m) \circ (\mathcal{R} + \mathcal{S}_m - \mathcal{R} \circ \mathcal{T}_m) \\
&=_m \mathcal{R} + \mathcal{S}_m + (\mathcal{T}_m \circ \mathcal{R} - \mathcal{R} \circ \mathcal{T}_m)
\end{aligned}$$

For the first step, we have used $\mathcal{S}_m \circ (\mathcal{I} - \mathcal{T}_m) =_m \mathcal{S}_m$ which holds because $\mathcal{S}_m$ is nonlinear and $\mathcal{T}_m$ is of order m. In the second step we used $\mathcal{T}_m \circ (\mathcal{R} + \mathcal{S}_m - \mathcal{R} \circ \mathcal{T}_m) =_m \mathcal{T}_m \circ \mathcal{R}$ which holds because $\mathcal{T}_m$ is of exact order m and everything in the second term is nonlinear except $\mathcal{R}$.

A closer inspection of the last line reveals that $\mathcal{S}_m$ can be simplified by choosing the commutator $\mathcal{C}_m = \{\mathcal{T}_m, \mathcal{R}\} = (\mathcal{T}_m \circ \mathcal{R} - \mathcal{R} \circ \mathcal{T}_m)$ appropriately. Indeed, if the range of $\mathcal{C}_m$ is the full space, then $\mathcal{S}_m$ can be removed entirely. However, as we shall see, most of the time this is not the case.

Let $(\mathcal{T}_{mj}^{\pm}|k_1^+, k_1^-, ..., k_n^+, k_n^-)$ be the Taylor expansion coefficient of $\mathcal{T}_{mj}^{\pm}$ in the j-th component pair of $\mathcal{T}_m$ with respect to $(s_1^+)^{k_1^+}(s_1^-)^{k_1^-} \cdot \ldots \cdot (s_n^+)^{k_n^+}(s_n^-)^{k_n^-}$. So $\mathcal{T}_{mj}^{\pm}$ is written as

$$\mathcal{T}_{mj}^{\pm} = \sum \; (\mathcal{T}_{mj}^{\pm}|k_1^+, k_1^-, ..., k_n^+, k_n^-) \; \cdot (s_1^+)^{k_1^+}(s_1^-)^{k_1^-} \cdot \ldots \cdot (s_n^+)^{k_n^+}(s_n^-)^{k_n^-}.$$

Similarly we identify the coefficients of $\mathcal{C}$ by $(\mathcal{C}_j^{\pm}|k_1^+, k_1^-, ..., k_n^+, k_n^-)$. Because $\mathcal{R}$ is diagonal, it is easily possible to express the coefficients of $\mathcal{C}$ in terms of the ones of $\mathcal{T}$. One obtains

$$\begin{aligned}
(\mathcal{C}_j^{\pm}|k_1^+, k_1^-, ..., k_n^+, k_n^-) &= \left((\prod_{l=1}^{n} r_l^{(k_l^+ + k_l^-)}) \cdot e^{i\vec{\mu}\cdot(\vec{k}^+ - \vec{k}^-)} - r_j \cdot e^{\pm i\mu_j} \right) \cdot (\mathcal{T}_j^{\pm}|k_1^+, k_1^-, ..., k_n^+, k_n^-) \\
&= C_j^{\pm}(\vec{k}^+, \vec{k}^-) \;\cdot\; (\mathcal{T}_j^{\pm}|k_1^+, k_1^-, ..., k_n^+, k_n^-)
\end{aligned} \tag{5}$$

Now it is apparent that a term in $\mathcal{S}_j^{\pm}$ can be removed if and only if the factor $C(\vec{k}^+, \vec{k}^-)$ is nonzero; if it is nonzero, then the required term in $\mathcal{T}_j^{\pm}$ is just the negative of the respective term in $\mathcal{S}_j^{\pm}$ divided by $C(\vec{k}^+, \vec{k}^-)$.

So the outcome of the whole normal form transformation depends upon the conditions under which the term $C(\vec{k}^+, \vec{k}^-)$ vanishes. This is obviously the case if and only if the moduli and the arguments of $r_j \cdot e^{\pm i\mu_j}$ and $(\prod_{l=1}^{n} r_l^{(k_l^+ + k_l^-)}) \cdot e^{i\vec{\mu}\cdot(\vec{k}^+ - \vec{k}^-)}$ are identical. Below we will discuss the conditions of this for various special cases and draw conclusions.

3 Differential Algebraic Determination of The Transformation Functions

The mth step of the normal form method shows how to choose the terms of exact order m of the map $\mathcal{T}_m$; the terms of higher order are in principle free, but they have to be chosen in such a way that the necessary inversion of the map $\mathcal{A}_m$ is possible. We choose all the transformations $\mathcal{A}_m$ such that they themselves are flows of dynamical systems. In many cases, this also has the additional benefit that all of the transformations belong to the same symmetry group as the original Poincare map. Let T_m equal to the mth order part of the nonlinear part $\mathcal{T}_m$. Then we claim that the autonomous dynamical system

$$\frac{d}{dt}\vec{r} = T_m(\vec{r}) \tag{6}$$

has a flow that for $\Delta t = 1$ agrees with $\mathcal{A}_m = \mathcal{I} + \mathcal{T}_m$ up to order m; moreover, the flow for $\Delta t = -1$ is the inverse of the flow for $\Delta t = 1$. To see this, we employ the so-called Lie derivative $L_{T_m} = T_m \cdot \vec{\nabla}$. This has the property that for any function on phase space $g(\vec{r})$, the change of g along a solution curve is given by $L_{T_m}\, g$. In fact,

$$\frac{d}{dt}g\,(\vec{r}) = \frac{d}{dt}\vec{r}\cdot\vec{\nabla}g = T_m(\vec{r})\cdot\vec{\nabla}g = L_{T_m}g.$$

It is then also clear that $d^n/dt^n = L_{T_m}^n g$. For the special choice of choosing g as the components of the vector $\vec{r}$, and in case the flow of the ODE can be Taylor expanded in t, we have $\vec{r}_f(\Delta t = 1) = \exp(L_{T_m})\,\vec{r}$ and hence choose

$$\mathcal{A}_m = \exp(L_{T_m})\,\vec{r}.$$

The first application of the Lie derivative on $\vec{r}$ yields T_m. But since T_m has no terms of order lower than m, the second and higher applications of the Lie derivative in the exponential series do not contribute to order m, and indeed $\mathcal{A}_m = \mathcal{I} + \mathcal{T}_m$ up to order m as needed. Furthermore, for this choice of $\mathcal{A}_m$ it is easy to compute the inverse, which is given by the flow for $\Delta t = -1$ and hence

$$\mathcal{A}_m^{-1} = \exp(-L_{T_m})\,\vec{r}.$$

Finally we observe that in order to determine the transformation map $\mathcal{A}_m$ and its inverse within the truncated Taylor series of order n utilized in our perturbative approach, only finitely many terms of the exponential series have to be kept. Indeed, the first term has contributions of exact order m, the second term contributions of order $2m-1$, and the kth term of order $k(m-1)+1$, which for sufficiently large k will exceed n.

The Lie derivative occurring in the exponential series represents a derivation on the space of infinitely often differentiable functions. So does its projection onto the space of truncated Taylor series, where together with the other operations, it induces a differential algebra. Expressing the Lie series via the derivation of this differential algebra allows the determination of the solution of ODE (6) in finitely many steps.

4 Stable Volume Preserving Systems

In this sections, we study the normal form transformation algorithm for the special case of linearly stable area preserving systems, which also include all symplectic systems. In this case all the r_j are equal to one, and the μ_j are purely real. So the moduli of the first and second terms in $C_j^{\pm}(\vec{k}^+, \vec{k}^-)$ are equal if and only if their phases agree modulo 2π. This is obviously the case if

$$\vec{\mu} \cdot (\vec{k}^+ - \vec{k}^-) = \pm \mu_j \ (\bmod 2\pi \ ),$$

where the different signs apply for $C_j^+(\vec{k}^+, \vec{k}^-)$ and $C_j^-(\vec{k}^+, \vec{k}^-)$, respectively. This can occur in two possible ways:

1. $k_l^+ = k_l^- \quad \forall\, l \neq j$, and $k_j^+ = k_j^- \pm 1$.
2. $\vec{\mu} \cdot \vec{n} = 0 \ (\bmod 2\pi \ )$ has nontrivial solutions.

The first case is of mathematical nature and lies at the heart of the normal form algorithm, and we will discuss its consequences below. The second case is equivalent to the system lying on a higher order resonance and is of more physical nature. In case the second condition is satisfied, there will be resonance driven terms that cannot be removed and that prevent the generation of Lyapunov functions or perturbative invariants.

Before proceeding in the discussion, we note that the second condition entails complications even if it is almost, but not exactly, satisfied. In this case, the removal of the respective term produces a small denominator that generates terms that become larger and larger, depending on the proximity to the resonance. In the removal process, this resonance proximity factor is multiplied by the respective expansion coefficient, and so this product obviously is an excellent characteristic of the significance of the resonance.

With higher and higher orders, i.e. larger k^+ and k^-, the number of relevant resonances increases. Since the resonances lie dense in tune space, eventually the growth of terms is almost inevitable and hence produces a map that is much more nonlinear than the underlying one. As we shall see in the next section, this problem is alleviated by damping.

We now discuss the form of the map if no resonances occur. In this case, the transformed map will have the form

$$\begin{aligned} \mathcal{M}_j^+ &= s_j^+ \cdot f_j(s_1^+ s_1^-, ..., s_v^+ s_v^-) \\ \mathcal{M}_j^- &= s_j^- \cdot \bar{f}_j(s_1^+ s_1^-, ..., s_v^+ s_v^-). \end{aligned}$$

The variables $s_j^{\pm}$ are not particularly well suited for the discussion of the result, and we express the map in terms of the adjoined variables $t_j^{\pm}$ introduced in (4). Simple arithmetic shows that

$$s_j^+ \cdot s_j^- = (t_j^+)^2 + (t_j^-)^2.$$

It is now advantageous to write f_j in terms of amplitude and phase as $f_j = a_j \cdot e^{i\phi_j}$. Performing the transformation to the coordinates $t_j^{\pm}$, we thus obtain

$$\mathcal{M}_j^{\pm} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2i & -1/2i \end{pmatrix} \cdot \begin{pmatrix} (t^+ + it^-) \cdot f_j \\ (t^+ - it^-) \cdot \bar{f}_j \end{pmatrix} = a_j \cdot \begin{pmatrix} \cos(\phi_j) & -\sin(\phi_j) \\ \sin(\phi_j) & \cos(\phi_j) \end{pmatrix} \cdot \begin{pmatrix} t^+ \\ t^- \end{pmatrix},$$

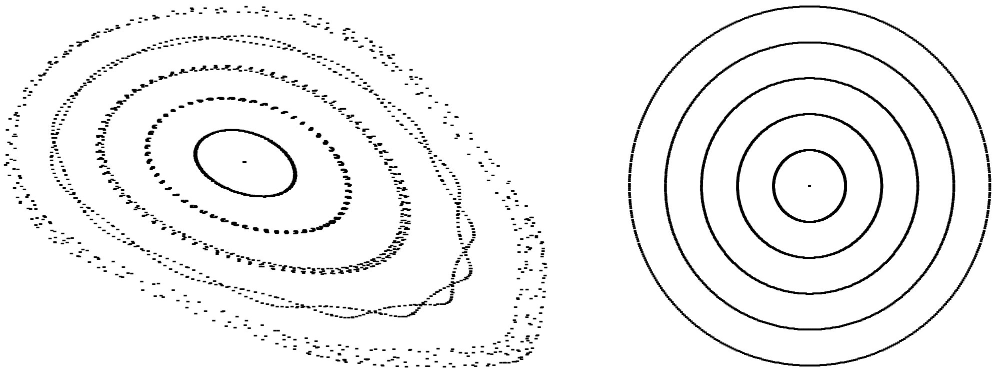

Fig.1: Motion of five particles of a dynamical system around a fixed point. The linearized motion of the system has eigenvalues of unity modulus. The left is displayed in conventional coordinates, and the right is in normal form coordinates. The distance to the origin represents a perturbative first integral of the motion.

where

$$\begin{aligned} f_j &= f_j[(t_1^+)^2 + (t_1^-)^2, ..., (t_v^+)^2 + (t_v^-)^2] \\ \bar{f}_j &= \bar{f}_j[(t_1^+)^2 + (t_1^-)^2, ..., (t_v^+)^2 + (t_v^-)^2]. \end{aligned}$$

Here, $\phi_j = \phi_j[(t_1^+)^2 + (t_1^-)^2, ..., (t_v^+)^2 + (t_v^-)^2]$ depends on a rotationally invariant quantity.

So in these coordinates, the motion is now given by a rotation, the frequency of which depends only on the amplitudes $(t_j^+)^2 + (t_j^-)^2$ and thus does not vary from turn to turn.

As an example and for purposes of illustration, we study the stability of such a four dimensional nonlinear system around a fixed point. Because of nonlinear effects as well as coupling between the various degrees of freedom, the motion is very complicated, and it is very hard to assess stability by studying the motion of individual points. The left picture in Fig.1 follows the motion of an ensemble of five particles around the fixed point for five hundred iterations of the Poincare map $\mathcal{M}$. After each of the time steps, the first two coordinates are recorded, resulting in a so-called Poincare plot. The right picture in Fig.1 shows the motion displayed in normal

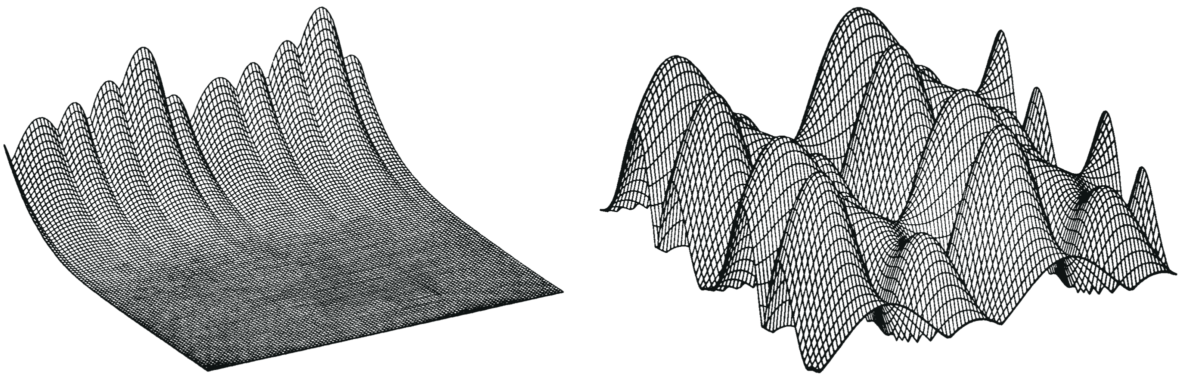

Fig.2: The deviation from invariance of the perturbative first integrals generated by DA normal form theory. The scale of the function is $\pm 10^{-8}$.

form coordinates of order 7. The normal form algorithm produces rotationally invariant motion, and because of the unity modulus of the eigenvalues of the linear motion, the motion follows a set of nearly perfect circles.

The resulting radius in the system is a pseudo-Lyapunov function with an apparently very small deviation d from perfect circles, which is here in the range of about 10^{-8}. The pictures in Fig.2 show some examples of the deviation function.

5 Non-Volume Preserving Systems

In the case of stable, non-symplectic maps, all r_j must satisfy $r_j \leq 1$, because otherwise at least one of the $r_j e^{\pm i\mu_j}$ is larger than unity in modulus. Since in the normal form transformation, terms can be removed if and only if the phases or amplitudes for the two contributions in $C(k^+, k^-)$ are different and the amplitudes contribute, more terms can be removed.

Of particular practical interest is the totally damped case in which $r_j < 1$ for all j and all μ_j are real. In this case an inspection of (5) reveals that now every nonlinear term can be removed. Then a similar argument as in the previous section shows that now the motion assumes the form

$$\mathcal{M}_j^{\pm} = r_j \cdot \begin{pmatrix} \cos(\phi_j) & -\sin(\phi_j) \\ \sin(\phi_j) & \cos(\phi_j) \end{pmatrix} \cdot \begin{pmatrix} t_j^+ \\ t_j^- \end{pmatrix},$$

where now the angle ϕ_j does not depend on the phase space variables anymore but only on the parameters. This means that the normal form transformation of a totally damped system leads to exponential spirals with constant frequency ϕ_j. In particular this entails that totally damped systems do not have any amplitude dependent tune shifts, and that they eventually collapse into the origin. If in practice the damping is only very small, over the short term these effects may be masked by the proximity of resonances.

It is quite illuminating to consider the small denominator problem in the case of totally damped systems. Clearly the denominator can never fall below $1 - \max(r_j)$ in magnitude. This puts a limit on the influence of any low order resonance on the dynamics; in fact, even sitting exactly on a low order resonance does not have any serious consequences if the damping is strong enough. In general, the influence of a resonance now depends on two quantities: the distance in tune space and the contraction strength r_j. High order resonances are suppressed particularly strongly because of the contribution of additional powers of r_j.

Clearly the normal form algorithm also works for unstable maps. The number of terms that can be removed will be at least the same as in the symplectic case, and sometimes it is possible to remove all terms. Among the many possible combinations of r_j and μ_j, the most common case in which the μ_j are real is worth studying in more detail. In this case, all terms can be removed unless the logarithms of the r_j and the tunes satisfy the same resonance condition, i.e.

$$\vec{n} \cdot (\log(r_1), ..., \log(r_v)) = 0$$
$$\vec{n} \cdot \vec{\mu} = 0 \ (\bmod 2\pi)$$

have simultaneous nontrivial solutions. This situation characterizes a special type of resonance, the coupled phase-amplitude resonance. They can never occur if all r_j are greater than unity in magnitude. This case corresponds to a totally unbound motion, and the motion in normal form coordinates moves along growing exponential spirals.

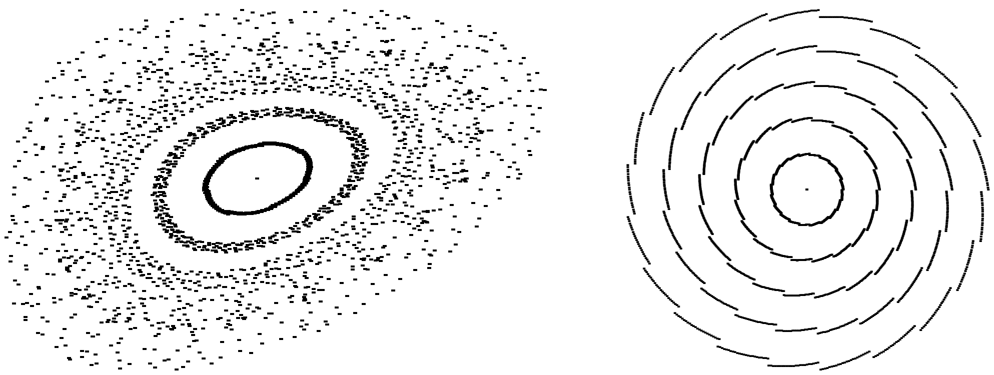

Fig.3: Motion of five particles of a dynamical system. Different from the system shown in Fig. 1, here the linearized motion is damped. The left is displayed in conventional coordinates, and the right is in normal form coordinates. The distance to the origin represents a Lyapunov function which is apparently decreasing for each time step, suggesting stability of the system.

The special case of area preserving systems already discussed satisfies $\prod_{l=1}^{n} r_l = 1$. So if there are r_j with both signs of the logarithm, and thus the possibility for amplitude resonances exists. In fact, any symplectic system lies on the fundamental amplitude resonance characterized by $\vec{n} = (1, 1, ..., 1)$. In this light, the stable symplectic case is a degeneracy in which all logarithms vanish and so the system lies on every amplitude resonances and is susceptible to every possible phase resonance.

We study the stability of a four dimensional system around a fixed point for which linear motion is damped. The magnitudes of the complex conjugate eigenvalue pairs were found to be 0.99991774 and 0.99999842, respectively. The left picture in Fig.3 shows the Poincare plot of the first two coordinates of an ensemble of five particles around the fixed point for five hundred iterations. Again it is very difficult to attempt to draw conclusions about the longer term behavior of the motion.

The right picture in Fig.3 shows the motion displayed in normal form coordinates of order 7. In these coordinates, the motion is very regular, and the damping is clearly apparent. The radius of the motion in normal form coordinates represents a Lyapunov function that is strictly decreasing with time, suggesting permanent stability of the system.

6 Verification of the Stability Estimates

In order to assert that the radius function L in normal form coordinates is a true Lyapunov function, or in case it is a pseudo-Lyapunov function for the purpose of establishing the magnitude of d in (2), it is necessary to find a sharp and rigorous global bound of the function

$$D(\vec{r}) = L(\mathcal{M}(\vec{r})) - L(\vec{r})$$

over the domain of interest A. As shown in [6], it is possible to obtain a Taylor model of the map $\mathcal{M}(\vec{r})$ over any part of the domain A, and the functions L are high-dimensional polynomials. Thus using Taylor model methods[9] it is possible to determine rigorous Taylor model representations of the function D over any subdomain of A. In this case, it is of crucial importance that the function D is subject to a severe cancellation problem that cannot be dealt with using regular verified interval methods, but benefits from the ability of the Taylor model approach to suppress the dependency problem which is achieved by the method in [10].

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