# Studies on Performance of the *COSY Infinity* Optimizers on Constraint Satisfaction *

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**Abstract**

In this work we assess the performance of the built-in *COSY Infinity* optimizers (Nelder-Mead, Levenberg-Marquardt and Simulated Annealing) and their combinations on the constraint satisfaction problems formulated as optimization problem. For this study we used problems from the standard test suit for constrained optimization with Evolutionary Algorithms [20, 23]. Results of the simulations are presented and discussed.

## 1 Introduction

### 1.1 Optimization problems

Optimization problems form an important class of all problems in the field of the applied science and design. Many problems that are not originally formulated as optimization could be reformulated to become so. After a problem is formulated as a problem of optimization it could be studied and possibly solved using one of the many numerical optimization methods developed [24]. There exist many different types of those problems, e.g. combinatorial optimization, stochastic optimization and integer programming. In this work we restrict our consideration to the nonlinear real-valued optimization problems, i.e. problems that could be formulated in terms of the functions assuming real values with arguments from the real domain. Those arguments are typically some control parameters and the functions themselves determine certain measures of the performance that need to be optimized.

Real-valued optimization problems could be formulated as follows. Let $S \subseteq \mathbb{R}^v$ be a search domain, $\mathbf{x} \in S$ be a vector of $v$ control parameters assuming real values,

$$f : S \longmapsto \mathbb{R} \tag{1}$$

be an objective function. Then the unconstrained optimization problem is to find $f^* \in \mathbb{R}$ such that

$$f^* = \min_{\mathbf{x}\in S} f(\mathbf{x}) \tag{2}$$

and corresponding $\mathbf{x}^* \in S$:

$$f^* = f(\mathbf{x}^*)$$

which is usually written as

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in S} f(\mathbf{x}). \tag{3}$$

Some real-life problems could be formulated as unconstrained optimization problems, but we are mostly dealing with the situations where some constraints are imposed on control parameters. Usually they are

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enforced by certain physical limitations and time/cost considerations. Therefore constrained optimization methods form a very important subclass of all optimization methods.

Let $\mathbf{x}$ from the problem formulation also be subjected to equality and inequality constraints

$$g_i(\mathbf{x}) = 0,\ i = 1, \ldots, n \tag{4}$$

$$h_j(\mathbf{x}) \le 0,\ j = 1, \ldots, m\,, \tag{5}$$

then the set

$$F = \left\{\mathbf{x} \in S \subseteq \mathbb{R}^v \,\middle|\, g_i(\mathbf{x}) = 0,\ h_j(\mathbf{x}) \le 0,\ i = 1, \ldots, n\,,\ j = 1, \ldots, m\right\} \tag{6}$$

is called the *feasible set*. It contains all vectors from the search domain that simultaneously satisfy all constraints. Such vectors $\mathbf{x} \in F$ are called *feasible*, all other vectors are called *unfeasible*. If at some point $\mathbf{x} \in S$ inequality constraint $h_j(\mathbf{x})$ holds as equality ($h_j(\mathbf{x}) = 0$), it is called *active* at $\mathbf{x}$. Equality constraints are considered active on all $S$. Using those definitions, we can define constrained optimization problem based on (3) as

$$\mathbf{x}^* = \arg\min_{\mathbf{x} \in F} f(\mathbf{x}), \tag{7}$$

where a sought minimum is also called *feasible minimum*.

Inequality constraints (5) could be transformed into equality constraints by introducing "dummy" variables $\xi_j,\ j = 1, \ldots, m$. In this case each inequality constraint

$$h_j(\mathbf{x}) \le 0$$

is converted to an equivalent equality constraint

$$h_j(\mathbf{x}) + \xi_j^2 = 0.$$

Equality constraints (4) can in turn be transformed into two inequality constraints each

$$\begin{aligned} -g_i(\mathbf{x}) &\le 0,\ i = 1, \ldots, m \\ g_i(\mathbf{x}) &\le 0,\ i = 1, \ldots, m, \end{aligned} \tag{8}$$

or, for the methods that do not rely on smoothness of the constraint functions to one inequality constraint each

$$|g_i(\mathbf{x})| \le 0,\ i = 1, \ldots, m. \tag{9}$$

For practical purposes of non-rigorous optimization

$$|g_i(\mathbf{x})| - \varepsilon \le 0,\ i = 1, \ldots, m, \tag{10}$$

where $\varepsilon$ is an acceptable tolerance for equality constraint satisfaction is also frequently used. Using those transformation we can limit our consideration to the problems with either equality-only or inequality-only constraints without loss of generality. For simplicity we consider only inequality constraints, i.e. constraints of the type (5), treating $n$ as a total number of constraints. In this case, the feasible set (6) is defined as

$$F = \left\{\mathbf{x} \in S \subseteq \mathbb{R}^v \,\middle|\, h_j(\mathbf{x}) \le 0\,,\ j = 1, \ldots, n\right\}. \tag{11}$$

Constraints could frequently be incorporated into the objective function or treated as additional objective functions via penalty and barrier functions [24]. This way, constrained optimization problems could be explored using optimization methods designed for unconstrained problems. The penalty functions paradigm was proposed by Fiacco, McCormick and Zangwill [13], [19] as a general numerical method applicable to constrained optimization problems. Its basic idea is to transform the original constrained minimization problem (6), (7) into an equivalent unconstrained minimization problem (12) or (13). Here equivalence means that the feasible minimum of the original constrained problem is a minimum of the resulting unconstrained problem or at least is acceptably close to it.

This transformation is performed via a set of so called penalty functions $P_j(h_j(\mathbf{x})),\ j = 1, \ldots, n$ corresponding to a set of constraints. Here penalty function $P_j$ calculates the non-negative amount of penalty assigned to a vector $\mathbf{x}$ for violating $j$-th constraint. Utilizing those functions the problem of constrained minimization (6), (7) could be transformed into an unconstrained multi-objective minimization problem

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in S} \mathbf{\Phi}(\mathbf{x}), \tag{12}$$

where $\mathbf{\Phi}(\mathbf{x}) = \big(P_1(h_1(\mathbf{x})), P_2(h_2(\mathbf{x})), \ldots, P_n(h_n(\mathbf{x})), f(\mathbf{x})\big)^{\mathrm{T}}$, that could be solved by multi-objective optimization techniques. It could also be converted even further to an unconstrained single-objective minimization problem

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in S} \varphi(\mathbf{x}), \tag{13}$$

where $\varphi = \varphi\big(\mathbf{\Phi}(\mathbf{x})\big)$ is the function that combines the original objective function and penalty functions into a single objective function. Usually penalty functions are chosen such that $\|\varphi(\mathbf{x}) - f(\mathbf{x})\| \longrightarrow 0$ as $\mathbf{x} \to F$. The function $\varphi$ also has to be balanced to guide the search process to a feasible set $F$ and hold it there, but not to interfere with the search of the minimum inside $F$. Care must be taken to achieve this balance in terms of the influence of the original objective function and penalties to a combined function $\varphi$. In case penalties are dominant in a value of the $\varphi$, the pressure to produce feasible points might prevent the algorithm from finding an optimum. In the opposite situation, i.e. if the original objective function dominates in calculating the value of $\varphi$, the optimization result tends to be optimal but unfeasible and thus useless.

A variety of methods to define penalty functions for $\Phi$, to combine them with original objective function into function $\varphi(\mathbf{x})$, inspired a large number of different constrained minimization methods. Nevertheless, since different problems have different properties of the constraint functions sets, there seems to be no universally optimal penalty function definition strategy. Since multi-objective optimization problems are generally harder to solve due to an increased number of objectives to satisfy simultaneously, it is often more desirable to convert a constrained problem to a single-objective unconstrained problem (13) by choosing appropriate $P_1, P_2, \ldots, P_n, \varphi$.

The most frequently used method to define combining function $\varphi$ is via linear combination of the individual penalties:

$$\varphi(\mathbf{p}) = \sum_{k=1}^{n+1} w_j p_j,\ \ \mathbf{p} \in \mathbb{R}^{n+1}, \tag{14}$$

where $w_j$ are freely chosen weight constants. Under this choice of $\varphi$ the constrained optimization problem (6), (7) is transformed into an unconstrained optimization problem (13). Since $w_{n+1}$ is a weight coefficient of the objective function of the original constrained problem, for simplicity it is usually chosen to be unity. The objective function then assumes the form

$$\varphi(\mathbf{x}) = f(\mathbf{x}) + \sum_{j=1}^{n} w_j P_j(h_j(\mathbf{x})). \tag{15}$$

For a general-purpose optimizer, in cases where there is no information about a problem available, all weight coefficients for penalties are usually set to unity, at least initially. Since in practice constraints and thus penalty functions often have different ranges of values, weight coefficients $w_j$ can then be selected as to normalize penalty values in order to balance their influence on the combined objective function or to increase the relative impact of some constraints if they are known to be harder or more important to satisfy.

*Exterior penalty functions* allow unfeasible members to be considered during the search process but assign them a penalty that generally grows with their distance from the feasible set, while *interior penalty functions* prevent search methods from considering unfeasible points. Usually exterior penalty functions are such that $P_j = P_j(z) \geq 0$, $z \in R$, $j = 1, \ldots, n$ and defined in the following way

$$P(z) = \begin{cases} 0 & z \leq 0 \\ \text{penalty(z)} > 0 & \text{otherwise} \end{cases}. \tag{16}$$

Most frequently used penalty functions of this type are from the power penalty family:

$$P^a(z) = \begin{cases} 0 & z \le 0 \\ z^a & \text{otherwise} \end{cases} = (\max\{0, z\})^a, \tag{17}$$

from which $a = 0, 1, 2$ are most often selected.

If we then substitute the value of the constraint function into penalty function of the type (16)

$$P_j(h_j(\mathbf{x})),$$

we obtain a non-negative penalty assigned to a vector $\mathbf{x}$ for not satisfying $j$-th constraint or zero if $j$-th constraint is not violated. Here index $j$ of the penalty function is given because generally penalty functions could be selected separately for each constraint function. Power penalty functions (17) use a violated constraint function value at the unfeasible point raised to the $a$-th power as a penalty.

## 1.2 Evolutionary optimization methods

An interesting family of optimization methods is inspired by the process of evolution described by Darwin in his revolutionary work "Origin of Species" first published in 1859 [10]. The main driving forces of evolution according to it are variability in living organisms and natural selection implicitly performed on them by the environment. Over time those forces shape different species to be very sophisticated inhabitants of the environment, i.e. make them fit to it.

This familty of methods has a very broad field of real-life applications. Examples include control systems [12], image analysis [9], marketing [28] and economics [2], traffic control [6], manufacturing [15] and many others. While EAs do not guarantee to find even a local minimum, practical applications demonstrate that frequently they are able to find a global minimum or at least produce a practically acceptable solution. However, the problem is that Evolutionary Algorithms (EAs) were not originally created to handle constraints. Even though unconstrained EAs had already demonstrated themselves to be very efficient general-purpose optimizers, ability to handle constraints would significantly increase their range of applications and help in solving many important optimization problems.

Those reasons served as a motivation for a large number of different approaches for constraints handling in EA that were invented and successfully applied to a number of different problems [8, 20, 21]. Such techniques could roughly be subdivided into several categories: killing, penalty functions, special genetic operators, selection rules, repair methods and other approaches. Repair algorithms are based on the idea of "repairing" the unfeasible members of the population to make them feasible and then either use the repaired version to evaluate the fitness of the original member or to replace it altogether. They seem particularly useful for problems where constraint satisfaction is particularly important. For example for problems where the number of generations is limited but the result is required to satisfy constraints even if it is not optimal. One of such problems is to quickly provide good cutoff values for a rigorous global optimizer [4].

We suggest a repair method called REPROPT (REpair by PROjecting through OPTimization). Its main idea is to perform projection of the unfeasible member to the feasible set by optimizing the penalty functions via some relatively inexpensive optimization method using unfeasible points as initial values for the optimizer. Note that by projection in this context we mean an element in the feasible set $F$ that is found in the optimization process, hence it depends on the method and method parameter. Moreover, if the method is stochastic (for example, Simulated Annealing), the results of the projection are not unique.

Parameters of REPROPT include the penalty functions method, projection algorithm, penalty satisfaction tolerance and maximum number of steps allowed. To select good default values of those parameters we performed a study on the performance of this method with different settings on a standard set of test problems for constrained optimization with Evolutionary Algorithms [20, 23]. Built-in *COSY Infinity* [3] unconstrained optimizers are used for this purpose. The list includes Nelder-Mead [16] (SIMPLEX). Levenberg-Marquardt [14] (LMDIF) and Simulated Annealing [18] paired with Random Walk (ANNEALING) algorithms, that proved themselves as versatile and robust optimizers frequently used as standard by many nonlinear optimization packages.

# 2 Problems

Test functions for Constrained Optimization single-objective EAs were suggested as a standard benchmark by Michalewicz [23], and later adopted to test performance of all new methods by the EA community [7, 11, 22, 25, 29]. This test bench includes various synthetic problems (G01-G13) with different properties of the constraints, feasible set, the sought minimum and several real-life design problems originally solved by constrained EAs (vess, tens). Problems listed using the notation from expressions (1), (4), (5), (2), (3) the search space $S$ is given as a set of allowed ranges for $x_i$, $i = 1, \ldots, v$, values for global minima are listed if known; best known values are given where the true minima are not known.

Rough empirical classification of the problem difficulty and estimates for $\rho = |F|/|S| \cdot 100$ parameter is taken from [20] and verified for correctness. Note that generally the most important factors that increase the difficulty of a constraint satisfaction problem include the presence of at least one nonlinear inequality and high dimensionality. Note also, that even though theoretically any feasible set where one of the constraints is equality has measure zero, the parameter $\rho$ obtained by a finite sampling of the feasible space might be non-zero. For practical purposes such estimation is more useful than purely theoretical measure. First, because, for the general set of constraints the problem of precise determination of $F$ could be extremely difficult. Second, for practical purposes $F$ that consists of a single point is harder to treat than $F$ that consists of the single line, which is, in turn harder to work with than $F$ that consists of the plane. Therefore those small deviations of $\rho$ from theoretical zero allow us to make such distinction even though only approximately. Values of $\rho$ in the problem descriptions are obtained by sampling the search space $S$ with $1,000,000$ random points.

**Listing 1:** *g01 Test problem*

DIFFICULT

$\rho \approx 0.0003$

$v = 13$

$n = 9$ (9 linear inequalities, $h_1$, $h_2$, $h_3$, $h_4$, $h_5$, $h_6$ are active)

quadratic objective function

$f(\mathbf{x}) = 5\sum_{i=1}^{4}(x_i - x_i^2) - \sum_{i=5}^{13} x_i$

$h_1(\mathbf{x}) = 2x_1 + 2x_2 + x_{10} + x_{11} - 10 \leq 0$
$h_2(\mathbf{x}) = 2x_1 + 2x_3 + x_{10} + x_{12} - 10 \leq 0$
$h_3(\mathbf{x}) = 2x_2 + 2x_3 + x_{11} + x_{12} - 10 \leq 0$
$h_4(\mathbf{x}) = -2x_4 - x_5 + x_{10} \leq 0$
$h_5(\mathbf{x}) = -2x_6 - x_7 + x_{11} \leq 0$
$h_6(\mathbf{x}) = -2x_8 - x_9 + x_{12} \leq 0$
$h_7(\mathbf{x}) = -8x_1 + x_{10} \leq 0$
$h_8(\mathbf{x}) = -8x_2 + x_{11} \leq 0$
$h_9(\mathbf{x}) = -8x_3 + x_{12} \leq 0$

$x_i \in [0, 1]$, $i = 1, \ldots, 9$
$x_i \in [0, 100]$, $i = 10, \ldots, 12$
$x_{13} \in [0, 1]$

$\mathbf{x}^* = (1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 1)$
$f(\mathbf{x}^*) = -15$

**Listing 2:** *g02 Test problem (best known value from [27])*

```
DIFFICULT
```
$\rho \approx 99.9973$

$v = 20$

$n = 2$ (1 linear inequality, 1 nonlinear inequality, $h_1$ almost active ($-10^{-8}$))

nonlinear objective function

$$f(\mathbf{x}) = -\left|\left(\sum_{i=1}^{v} \cos^4(x_i) - 2\prod_{i=1}^{v} \cos^2(x_i)\right)\left(\sum_{i=1}^{v} i x_i^2\right)^{-0.5}\right|$$

$$h_1(\mathbf{x}) = 0.75 - \prod_{i=1}^{v} x_i \leq 0$$
$$h_2(\mathbf{x}) = \sum_{i=1}^{v} x_i - 7.5v \leq 0$$

$x_i \in [0, 10],\ i = 1, \ldots, v$

best known $f(\mathbf{x}^*) = 0.803619$

**Listing 3:** *g03 Test problem*

```
DIFFICULT
```
$\rho \approx 0.0026$

$v = 10$

$n = 1$ (1 nonlinear equality, $g_1$ active)

nonlinear objective function

$$f(\mathbf{x}) = -v^{2/v} \prod_{i=1}^{v} x_i$$

$$g_1(\mathbf{x}) = \sum_{i=1}^{v} x_i^2 - 1 = 0$$

$x_i \in [0, 10],\ i = 1, \ldots, v$

$\mathbf{x}^* = 1/\sqrt{v}(1, 1, \ldots, 1)$, any combination of $\pm 1$'s such that their product is positive

$f(\mathbf{x}^*) = -1$

**Listing 4:** *g04 Test problem*

```
AVERAGE
```
$\rho \approx 27.0079$

$v = 5$

$n = 6$ (4 linear inequalities, 2 nonlinear inequalities, $h_1$, $h_6$ active)

quadratic objective function

$$f(\mathbf{x}) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141$$

$$h_1(\mathbf{x}) = 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 - 92 \leq 0$$
$$h_2(\mathbf{x}) = -85.334407 - 0.0056858x_2x_5 - 0.0006262x_1x_4 + 0.0022053x_3x_5 \leq 0$$
$$h_3(\mathbf{x}) = 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3^2 - 110 \leq 0$$
$$h_4(\mathbf{x}) = -80.51249 - 0.0071317x_2x_5 - 0.0029955x_1x_2 - 0.0021813x_3^2 + 90 \leq 0$$
$$h_5(\mathbf{x}) = 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 - 25 \leq 0$$
$$h_6(\mathbf{x}) = -9.300961 - 0.0047026x_3x_5 - 0.0012547x_1x_3 - 0.0019085x_3x_4 + 20 \leq 0$$

$x_1 \in [78, 102]$

$x_2 \in [33, 45]$

$x_i \in [27, 45]$, $i = 3, \ldots, 5$

$\mathbf{x}^* = (78, 33, 29.995256025682, 45, 36.775812905788)$

$f(\mathbf{x}^*) = -30665.539$

**Listing 5:** *g05 Test problem*

```
VERY DIFFICULT
```
$\rho \approx 0.0000$

$v = 4$

$n = 5$ (2 linear inequalities, 3 nonlinear equalities, $g_1$, $g_2$, $g_3$ are active)

nonlinear objective function

$$f(\mathbf{x}) = 3x_1 + 0.000001x_1^3 + 2x_2 + (0.000002/3)x_2^3$$

$$h_1(\mathbf{x}) = -x_4 + x_3 - 0.55 \leq 0$$
$$h_2(\mathbf{x}) = -x_3 + x_4 - 0.55 \leq 0$$
$$g_1(\mathbf{x}) = 1000\sin(-x_3 - 0.25) + 1000\sin(-x4 - 0.25) + 894.8 - x_1 = 0$$
$$g_2(\mathbf{x}) = 1000\sin(x_3 - 0.25) + 1000\sin(x_3 - x_4 - 0.25) + 894.8 - x_2 = 0$$
$$g_3(\mathbf{x}) = 1000\sin(x_4 - 0.25) + 1000\sin(x_4 - x_3 - 0.25) + 1294.8 = 0$$

$x_i \in [0, 1200]$, $i = 1, 2$

$x_i \in [-0.55, 0.55]$, $i = 3, 4$

best known $\mathbf{x}^* = (679.9453, 1026.067, 0.1188764, -0.3962336)$

$f(\mathbf{x}^*) = 5126.4981$

**Listing 6:** *g06 Test problem*

```
AVERAGE
```
$\rho \approx 0.0057$

$v = 2$

$n = 2$ (2 nonlinear inequalities, $h_1$, $h_2$ active)

nonlinear objective function

$$f(\mathbf{x}) = (x_1 - 10)^3 + (x_2 - 20)^3$$

$$h_1(\mathbf{x}) = -(x_1 - 5)^2 - (x_2 - 5)^2 + 100 \leq 0$$
$$h_2(\mathbf{x}) = (x_1 - 6)^2 + (x_2 - 5)^2 - 82.81 \leq 0$$

$x_1 \in [13, 100]$

$x_2 \in [0, 100]$

$\mathbf{x}^* = (14.095, 0.84296)$

$f(\mathbf{x}^*) = -6961.81388$

**Listing 7:** *g07 Test problem*

```
AVERAGE
```
$\rho \approx 0.0000$

$v = 10$

$n = 8$ (3 linear inequalities, 5 nonlinear inequalities $h_1$, $h_2$, $h_3$, $h_4$, $h_5$, $h_6$ active)

quadratic objective function

$$f(\mathbf{x}) = x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45$$

$$h_1(\mathbf{x}) = -105 + 4x_1 + 5x_2 - 3x_7 + 9x_8 \leq 0$$
$$h_2(\mathbf{x}) = 10x_1 - 8x_2 - 17x_7 + 2x_8 \leq 0$$
$$h_3(\mathbf{x}) = -8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12 \leq 0$$
$$h_4(\mathbf{x}) = 3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120 \leq 0$$
$$h_5(\mathbf{x}) = 5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40 \leq 0$$
$$h_6(\mathbf{x}) = x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6 \leq 0$$
$$h_7(\mathbf{x}) = 0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6 - 30 \leq 0$$
$$h_8(\mathbf{x}) = -3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10} \leq 0$$

$x_i \in [-10, 10], \ i = 1, \ldots, 10$

$\mathbf{x}^* = (2.171996, 2.363683, 8.773926, 5.095984, 0.9906548, 1.430574, 1.321644, 9.828726, 8.280092, 8.375927)$

$f(\mathbf{x}^*) = 24.3062091$

**Listing 8:** *g08 Test problem*

```
EASY
```
$\rho \approx 0.8581$

$v = 2$

$n = 2$ (2 nonlinear inequalities)

nonlinear objective function

$$f(\mathbf{x}) = -\sin^3(2\pi x_1)\sin(2\pi x_2)\big(x_1^3(x_1 + x_2)\big)^{-1}$$

$$h_1(\mathbf{x}) = x_1^2 - x_2 + 1 \leq 0$$
$$h_2(\mathbf{x}) = 1 - x_1 + (x_2 - 4)^2 \leq 0$$

$x_i \in [0, 10], \ i = 1, 2$

$\mathbf{x}^* = (1.2279713, 4.2453733)$

$f(\mathbf{x}^*) = -0.095825$

**Listing 9:** *g09 Test problem*

```
AVERAGE
```
$\rho \approx 0.5199$

$v = 7$

$n = 4$ (4 nonlinear inequalities, $h_1$, $h_4$ active)

nonlinear objective function

$$f(\mathbf{x}) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7$$

$$h_1(\mathbf{x}) = -127 + 2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 \leq 0$$
$$h_2(\mathbf{x}) = -282 + 7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 \leq 0$$
$$h_3(\mathbf{x}) = -196 + 23x_1 + x_2^2 + 6x_6^2 - 8x_7 \leq 0$$
$$h_4(\mathbf{x}) = 4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7 \leq 0$$

$x_i \in [-10, 10], \ i = 1, \ldots, 7$

$\mathbf{x}^* = (2.330499, 1.951372, -0.4775414, 4.365726, -0.6244870, 1.038131, 1.594227)$

$f(\mathbf{x}^*) = 680.6300573$

**Listing 10:** *g10 Test problem*

```
DIFFICULT
```
$\rho \approx 0.0020$

$v = 8$

$n = 6$ (3 linear inequalities, 3 nonlinear inequalities, $h_1$, $h_2$, $h_3$ active)

linear objective function

$f(\mathbf{x}) = x_1 + x_2 + x_3$

$h_1(\mathbf{x}) = -1 + 0.0025(x_4 + x_6) \leq 0$

$h_2(\mathbf{x}) = -1 + 0.0025(x_5 + x_7 - x_4) \leq 0$

$h_3(\mathbf{x}) = -1 + 0.01(x_8 - x_5) \leq 0$

$h_4(\mathbf{x}) = -x_1 x_6 + 833.33252 x_4 + 100 x_1 - 83333.333 \leq 0$

$h_5(\mathbf{x}) = -x_2 x_7 + 1250 x_5 + x_2 x_4 - 1250 x_4 \leq 0$

$h_6(\mathbf{x}) = -x_3 x_8 + 1250000 + x_3 x_5 - 2500 x_5 \leq 0$

$x_1 \in [100, 10000]$

$x_i \in [1000, 10000],\ i = 2, \ldots, 3$

$x_i \in [10, 1000],\ i = 4, \ldots, 8$

$\mathbf{x}^* = (579.3167, 1359.943, 5110.071, 182.0174, 295.5985, 217.9799, 286.4162, 395.5979)$

$f(\mathbf{x}^*) = 7049.3307$

**Listing 11:** *g11 Test problem*

EASY

$\rho \approx 0.0973$

$v = 2$

$n = 1$ (1 nonlinear equality, $g_1$ active)

linear objective function

$f(\mathbf{x}) = x_1^2 + (x_2 - 1)^2$

$g_1(\mathbf{x}) = x_2 - x_1^2 = 0$

$x_i \in [-1, 1],\ i = 1, 2$

$\mathbf{x}^* = (\pm 1/\sqrt{2}, 1/2)$

$f(\mathbf{x}^*) = 0.75$

**Listing 12:** *g12 Test problem*

```
EASY
```
$\rho \approx 4.7697$
$v = 3$
$n = 1$ ($9^3$ nonlinear inequalities joined by logical OR instead of usual AND, disjoint $F$)
quadratic objective function

$f(\mathbf{x}) = -100^{-1}(100 - (x_1 - 5)^2 - (x_2 - 5)^2 - (x_3 - 5)^2)$

$h_i(\mathbf{x}) = (x_1 - p)^2 + (x_2 - q)^2 + (x_3 - r)^2 - 0.0625 \leq 0$, $i = 1, \ldots, 9^3$, $p, q, r = 1, \ldots, 9$
$\mathbf{x}$ is feasible if it satisfies one of $h_i$

$x_i \in [0, 10]$, $i = 1, 2, 3$

$\mathbf{x}^* = (5, 5, 5)$
$f(\mathbf{x}^*) = -1$

**Listing 13:** *g13 Test problem*

VERY DIFFICULT
$\rho \approx 0.0000$
$v = 5$
$n = 3$ (1 linear equality, 2 nonlinear equalities, $g1$, $g2$, $g3$ active)
nonlinear objective function

$f(\mathbf{x}) = e^{x_1 x_2 x_3 x_4 x_5}$

$g_1(\mathbf{x}) = \sum_{i=1}^{5} x_i^2 - 10 = 0$
$g_2(\mathbf{x}) = x_2 x_3 - 5 x_4 x_5 = 0$
$g_3(\mathbf{x}) = x_1^3 + x_2^3 + 1 = 0$

$x_i \in [-2.3, 2.3]$, $i = 1, 2$
$x_i \in [-3.2, 3.2]$, $i = 3, 4, 5$

$\mathbf{x}^* = (-1.717143, 1.595709, 1.827247, -0.7636413, -0.763645)$
$f(\mathbf{x}^*) = 0.0539498$

**Listing 14:** *Design of a Pressure Vessel (vess) [17] (best known value from [7])*

```
AVERAGE
```
$\rho \approx 39.6762$

$v = 4$

$n = 4$ (3 linear inequalities, 1 nonlinear inequality)

quadratic objective function

$f(\mathbf{x}) = 0.6224x_1x_3x_4 + 1.7781x_2x_3^2 + 3.1661x_1^2x_4 + 19.84x_1^2x_3$

$h_1(\mathbf{x}) = -x_1 + 0.0193x_3 \leq 0$
$h_2(\mathbf{x}) = -x_2 + 0.00954x_3 \leq 0$
$h_3(\mathbf{x}) = -\pi x_3^2x_4 - 4/3\pi x_3^3 + 1296000 \leq 0$
$h_4(\mathbf{x}) = x_4 - 240 \leq 0$

$x_i \in [1, 99],\ i = 1, 2$
$x_i \in [10, 200],\ i = 3, 4$

best known: $f(\mathbf{x}^*) = 6059.946341$

**Listing 15:** *Design of a Tension/Compression Spring (tens) [1] (best known value from [7])*

```
EASY
```
$\rho \approx 0.7537$

$v = 3$

$n = 4$ (1 linear inequality, 3 nonlinear inequalities)

quadratic objective function

$f(\mathbf{x}) = (x_3 + 2)x_2x_1^2$

$h_1(\mathbf{x}) = 1 - x_2^3x_3(71785x_1^4)^{-1} \leq 0$
$h_2(\mathbf{x}) = (4x_2^2 - x_1x_2)(12566(x_2x_1^3 - x_1^4))^{-1} + (5108x_1^2)^{-1} - 1 \leq 0$
$h_3(\mathbf{x}) = 1 - 140.45x_1x_2^{-2}x_3^{-1} \leq 0$
$h_4(\mathbf{x}) = (x_2 + x_1)1.5^{-1} - 1 \leq 0$

$x_1 \in [0.05, 2]$
$x_2 \in [0.25, 1.3]$
$x_3 \in [2, 15]$

best known: $f(\mathbf{x}^*) = 0.012681$

# 3 Methodology

For all test problems certain transformations and conventions were used.

All equality constraints of the type (4) were converted into equivalent inequality constraints (5) using transformation (8) or (9) so that the feasible set is given by (11).

All constraints in the test set are known to be satisfiable, i.e. feasible set is known to be non-empty. Since we were not interested in the global minima of the constraint functions, but rather in the simultaneous satisfactions of all constraints, a set of constraint functions was converted to a set of penalties using power penalties (17) with $a = 0, 1, 2$. Using the property (16), that power penalty functions satisfy, the problem of projecting the point $\mathbf{x}_0$ onto $F$ via a chosen optimizer could be formulated as follows: using $\mathbf{x}_0$ as a starting value, find $\mathbf{x}_f$ such that

$$oP_i\big(h_i(\mathbf{x}_f)\big) = \min_{\mathbf{x}\in S} P_i\big(h_i(\mathbf{x})\big) = 0,\ i = 1, \ldots, n. \tag{18}$$

Such $\mathbf{x}_f$ would then be feasible automatically. Note that this method is equivalent to approach (12) that allow to convert single-objective constrained optimization problems to multi-objective unconstrained problems via penalty functions. The difference is that in our case we do not have an objective function to minimize. Note that in (18) for practical purposes we might be satisfied with non-zero penalty values if they are within the desired tolerance from zero. This is particularly applicable to converted equality constraints because they might be non-zero simply due to the limited precision of the computer arithmetic and floating-point errors in computations.

Three types of the objective functions were tested:

- *all combined:* the multi-objective problem (18) was converted to a single-objective problem (15) via the combining function (14) with all $w_i = 1$.

- *equality combined + inequality combined:* the multi-objective problem (18) was converted to a two-objective optimization problem with inequality constraints and equality constraints (transformed to inequality constraints using (9) but still more difficult to satisfy than true inequalities) converted to 2 separate objective functions using the same method as for *all combined* approach. This distinction was made because equality constraints are usually harder to satisfy; thus, they might require more severe penalties to be satisfied.

- *separate:* the multi-objective optimization problem (18) was treated as-is. It must be noted, however, that for ANNEALING and SIMPLEX methods it was internally converted into the single-objective optimization problem by optimizing the sum of the squares of the objective functions, i.e. equivalent to the *all combined* method for $a = 2$. LMDIF has the ability to solve multidimensional problems directly.

The following abbreviations for the search methods are used: S — SIMPLEX, L — LMDIF, A — ANNEALING optimization methods. Combined methods were implemented by making several steps using one method and then making several steps using another method with the hope to combine the strengths of both methods and to compensate for their weaknesses. Combinations of methods and their abbreviations are: S+A — SIMPLEX + ANNEALING, S+L — SIMPLEX + LMDIF, L+A — LMDIF + ANNEALING.

Each combination of the penalty function ($a = 0, 1, 2$, selected separately for equality and inequality constraints) and optimization problem formulation (all combined, equality combined + inequality combined, separate) was tested for each of the simple (S, L, A) and combined (S+A, S+L, L+A) methods. For problems without equality constraints, optimization problems *all combined* and *equality combined + inequality combined* are equivalent, hence only *all combined* was tested. For problems with only one constraint all formulations of optimization problems are equivalent. Therefore for problems with both types of constraints the total number of tested approaches is $3 \times 3 \times 3 \times 6 = 162$, for problems with inequality constraints only $2 \times 3 \times 6 = 36$ and for problems with one inequality constraint the number of tested cases was 12.

Special abbreviations for each variant of the problem formulation and optimization strategy is employed. The description starts with the abbreviation of the optimization method (S, L, A, S+A, S+L, L+A) followed by the type of the penalty function used for the constraints in parentheses. For problems with equality and inequality constraints both types are separated by a comma, the first type corresponds to equality constraints. Types are: 1 for power 0, $z$ for power 1, $z^2$ for the power 2. For problems with inequality or equality constraints only, one type denotes the type of the penalty used for the corresponding constraints. For optimization problems of the *all combined* type ":c" is added after the method abbreviation before parenthesis. For problems with both equality and inequality constraints type *equality combined + inequality combined* is marked with ":c", types of the penalties are separated by "+" instead of comma. Examples: S+L:c($z^2$) denotes SIMPLEX+LMDIF combined method, problem with inequality constraints only, *all combined* objective function, penalty power is 2. L($z$) denotes LMDIF method, *separate* objective functions, penalty power 1. L+A:c($z$ + $z^2$) denotes combined LMDIF+ANNEALING optimization method, *equality combined + inequality combined* optimization problem with penalty power 1 for equality constraints and 2 for inequality constraints.

Test problems were built by taking constraints from the standard constrained optimization test bench for EAs [20, 23] (see section 2. Since it mostly consists of inequality constrained problems only, a simple 2-dimensional problem (19) with one equality and four inequality constraints was suggested [5].

$$
\begin{aligned}
g_1(\mathbf{x}) &= x_1^2 + x_2^2 - 1.1^2 = 0 \\
h_1(\mathbf{x}) &= x_1 - 1 \leq 0 \\
h_2(\mathbf{x}) &= -x_1 - 1 \leq 0 \\
h_3(\mathbf{x}) &= x_2 - 1 \leq 0 \\
h_4(\mathbf{x}) &= -x_2 - 1 \leq 0
\end{aligned}
\tag{19}
$$

Initial points were generated randomly uniformly distributed over

$$S = [-100, 100]^v$$

and

$$S = [-1000, 1000]^v.$$

Total number of different points tested for each combination: 1000.

For all methods the maximum number of steps is 1000, precision is $10^{-5}$. For combined methods, the maximum number of steps with the first and second methods in one step of the combined algorithm, was 10, the total maximum number of steps was counted by summing steps made by both methods and was 1000. The projection was considered successful if all objective functions were within tolerance from the global minimum of zero. Projection was considered failed if the desired tolerance was not reached and method either converged or reached a maximum allowed number of steps.

# 4 Results

With all conventions from section 3 a series of tests was performed. Output is summarized in the tables; where, for every combination of the method, penalty functions and the objective function construction method, the percentage of the successful runs and average number of steps (including the failed runs) are listed. The best methods in terms of the number of the successful runs are listed in **boldface**, the number of steps of those methods is also marked for convenience. Note that for methods with similar success rates the one with smaller average number of steps is preferred. Headers of the columns represent powers of the penalty functions as described in methodology.

For each method three rows contain results for *all combined*, *equality combined + inequality combined* and *separate* objective function construction methods. In case there are no equality constraints or no inequality constraints, *equality combined + inequality combined* method is equivalent to *all combined* and is not tested, therefore the number of rows for each method in this case is two. Problems G03 and G11 have one equality constraint each, hence the number of rows in this case is one.

## 4.1 Problem G00 from (19)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $1,1$ | $1,z$ | $1,z^2$ | $z,1$ | $z,z$ | $z,z^2$ | $z^2,1$ | $z^2,z$ | $z^2,z^2$ |
| SIMPLEX | 0.00 | 0.00 | 0.00 | 60.20 | 57.10 | 31.70 | 78.50 | 72.50 | 74.50 |
| | 0.00 | 0.00 | 0.00 | 29.10 | 29.10 | 29.70 | 37.00 | 37.00 | 37.70 |
| | 0.00 | 0.00 | 0.00 | 29.10 | 29.10 | 29.70 | 37.00 | 37.00 | 37.70 |
| LMDIF | 0.00 | 0.00 | 0.00 | 50.60 | 60.90 | 81.90 | 64.70 | 92.70 | 94.10 |
| | 0.00 | 0.00 | 0.00 | 66.80 | 91.80 | **95.50** | 80.10 | 85.90 | 89.80 |
| | 0.00 | 0.00 | 0.00 | 66.80 | **98.10** | **99.60** | 80.00 | **98.10** | 94.00 |
| ANNEALING | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.20 | 0.10 | 0.20 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.30 | 0.00 | 0.20 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.50 | 0.20 |
| SMP+LMD | 0.00 | 0.00 | 0.00 | 91.00 | 64.90 | 85.50 | **96.10** | **100.0** | **99.40** |
| | 0.00 | 0.00 | 0.00 | 84.50 | **100.0** | **100.0** | **98.80** | **99.90** | **100.0** |
| | 0.00 | 0.00 | 0.00 | 84.50 | **100.0** | **100.0** | **98.80** | **100.0** | **100.0** |
| SMP+ANN | 0.00 | 0.00 | 0.00 | 0.10 | 0.30 | 0.100 | 77.30 | 76.30 | 71.60 |
| | 0.00 | 0.00 | 0.00 | 0.20 | 0.20 | 0.200 | 76.60 | 73.30 | 74.10 |
| | 0.00 | 0.00 | 0.00 | 0.20 | 0.20 | 0.00 | 77.70 | 75.90 | 75.80 |
| LMD+ANN | 0.00 | 0.00 | 0.00 | **99.60** | 91.20 | 73.10 | **98.20** | 93.00 | **95.50** |
| | 0.00 | 0.00 | 0.00 | **99.70** | **100.0** | **100.0** | **98.50** | 75.00 | **97.30** |
| | 0.00 | 0.00 | 0.00 | **99.70** | **100.0** | **100.0** | **98.50** | **100.0** | **96.60** |

Average number of steps:

| Method | % avg.steps | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $1,1$ | $1,z$ | $1,z^2$ | $z,1$ | $z,z$ | $z,z^2$ | $z^2,1$ | $z^2,z$ | $z^2,z^2$ |
| SIMPLEX | 4. | 37. | 40. | 338. | 409. | 422. | 262. | 338. | 329. |
| | 4. | 4. | 4. | 452. | 452. | 452. | 359. | 359. | 359. |
| | 4. | 4. | 4. | 452. | 452. | 452. | 359. | 359. | 359. |
| LMDIF | 6. | 26. | 34. | 113. | 94. | 72. | 134. | 128. | 166. |
| | 6. | 23. | 73. | 56. | 79. | **91**. | 112. | 203. | 176. |
| | 6. | 10. | 62. | 56. | **50**. | **65**. | 112. | **92**. | 145. |
| ANNEALING | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| SMP+LMD | 1000. | 1000. | 1000. | 236. | 405. | 206. | **169**. | **118**. | **125**. |
| | 1000. | 1000. | 1000. | 248. | **140**. | **271**. | **129**. | **233**. | **179**. |
| | 1000. | 1000. | 1000. | 248. | **70**. | **201**. | **129**. | **113**. | **126**. |
| SMP+ANN | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 428. | 435. | 476. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 431. | 458. | 476. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 410. | 439. | 468. |
| LMD+ANN | 1000. | 1000. | 1000. | **118**. | 240. | 383. | **225**. | 279. | **273**. |
| | 1000. | 1000. | 1000. | **104**. | **170**. | **232**. | **217**. | 741. | **382**. |
| | 1000. | 1000. | 1000. | **101**. | **82**. | **147**. | **220**. | **156**. | **226**. |

- 1000 random points from $[-1000, 1000]^v$

  Success rate:

| Method | % success | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $1, 1$ | $1, z$ | $1, z^2$ | $z, 1$ | $z, z$ | $z, z^2$ | $z^2, 1$ | $z^2, z$ | $z^2, z^2$ |
| SIMPLEX | 0.00 | 0.00 | 0.00 | 56.40 | 58.30 | 31.80 | 70.40 | 73.90 | 74.80 |
| | 0.00 | 0.00 | 0.00 | 29.00 | 29.00 | 29.30 | 40.60 | 40.60 | 41.10 |
| | 0.00 | 0.00 | 0.00 | 29.00 | 29.00 | 29.30 | 40.60 | 40.60 | 41.10 |
| LMDIF | 0.00 | 0.00 | 0.00 | 45.80 | 56.10 | 81.30 | 64.70 | 86.90 | 92.00 |
| | 0.00 | 0.00 | 0.00 | 64.90 | 92.80 | **97.70** | 81.20 | 80.60 | 88.80 |
| | 0.00 | 0.00 | 0.00 | 65.10 | **99.60** | **99.10** | 81.10 | **99.60** | 92.80 |
| ANNEALING | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| SMP+LMD | 0.00 | 0.00 | 0.00 | 90.80 | 63.20 | 88.50 | **98.20** | **100.0** | **99.70** |
| | 0.00 | 0.00 | 0.00 | 84.50 | **100.0** | **100.0** | **99.60** | **100.0** | **100.0** |
| | 0.00 | 0.00 | 0.00 | 84.50 | **100.0** | **100.0** | **99.60** | **100.0** | **100.0** |
| SMP+ANN | 0.00 | 0.00 | 0.00 | 0.20 | 0.20 | 0.10 | 76.60 | 73.90 | 70.20 |
| | 0.00 | 0.00 | 0.00 | 0.30 | 0.20 | 0.40 | 76.20 | 71.80 | 69.10 |
| | 0.00 | 0.00 | 0.00 | 0.20 | 0.10 | 0.10 | 76.20 | 74.10 | 71.60 |
| LMD+ANN | 0.00 | 0.00 | 0.00 | **99.40** | 93.60 | 70.60 | **98.30** | 92.70 | 91.50 |
| | 0.00 | 0.00 | 0.00 | **99.50** | **100.0** | **100.0** | **97.70** | 0.700 | 51.00 |
| | 0.00 | 0.00 | 0.00 | **99.60** | **100.0** | **100.0** | **97.70** | **100.0** | 87.40 |

Average number of steps:

| Method | % avg.steps | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $1, 1$ | $1, z$ | $1, z^2$ | $z, 1$ | $z, z$ | $z, z^2$ | $z^2, 1$ | $z^2, z$ | $z^2, z^2$ |
| SIMPLEX | 4. | 51. | 53. | 386. | 395. | 455. | 351. | 343. | 344. |
| | 4. | 4. | 4. | 447. | 447. | 447. | 366. | 366. | 366. |
| | 4. | 4. | 4. | 447. | 447. | 447. | 366. | 366. | 366. |
| LMDIF | 6. | 33. | 48. | 187. | 168. | 121. | 210. | 207. | 209. |
| | 6. | 30. | 87. | 131. | 92. | **114**. | 191. | 271. | 227. |
| | 6. | 11. | 72. | 131. | **45**. | **79**. | 192. | **101**. | 167. |
| ANNEALING | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| SMP+LMD | 1000. | 1000. | 1000. | 261. | 442. | 201. | 189. | 161. | 163. |
| | 1000. | 1000. | 1000. | 274. | 192. | 332. | 164. | 342. | 281. |
| | 1000. | 1000. | 1000. | 274. | 90. | 233. | 164. | 154. | 171. |
| SMP+ANN | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 512. | 531. | 573. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 508. | 541. | 599. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 513. | 537. | 573. |
| LMD+ANN | 1000. | 1000. | 1000. | **151**. | 223. | 437. | **297**. | 341. | 366. |
| | 1000. | 1000. | 1000. | **135**. | **271**. | **332**. | **299**. | 1000. | 788. |
| | 1000. | 1000. | 1000. | **133**. | **106**. | **171**. | **292**. | **218**. | 318. |

## 4.2 Problem G01 (Listing 1)

- 1000 random points from $[-100, 100]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | **90.00** | **98.67** |
| | 0.00 | 2.00 | 2.00 |
| LMDIF | 0.00 | **98.67** | **96.67** |
| | 0.00 | **100.0** | **100.0** |
| ANNEALING | 0.33 | 2.33 | 0.66 |
| | 0.33 | 0.33 | 0.33 |
| SIMPLEX+LMDIF | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+ANNEALING | 0.00 | 1.66 | 4.00 |
| | 0.00 | 2.00 | 2.66 |
| LMDIF+ANNEALING | 0.33 | 3.33 | 3.33 |
| | 0.00 | 2.33 | 2.33 |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 14. | **445**. | **391**. |
| | 14. | 62. | 62. |
| LMDIF | 16. | **94**. | **374**. |
| | 16. | **45**. | **234**. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+ANNEALING | 1000. | 998. | 995. |
| | 1000. | 999. | 1000. |
| LMDIF+ANNEALING | 1000. | 997. | 996. |
| | 1000. | 999. | 1000. |

- 1000 random points from $[-1000, 1000]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 85.50 | **99.00** |
| | 0.00 | 1.50 | 1.50 |
| LMDIF | 0.00 | **98.50** | **95.50** |
| | 0.00 | **100.0** | **100.0** |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF+ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |

Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 15. | 466. | **411.** |
| | 14. | 67. | 67. |
| LMDIF | 16. | **97.** | **435.** |
| | 16. | **45.** | **278.** |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |

### 4.3 Problem G02 (Listing 2)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 36.00 | 77.80 | 81.60 |
| | 36.00 | 70.00 | 70.00 |
| LMDIF | 35.80 | 84.40 | **91.40** |
| | 35.80 | **97.80** | 82.60 |
| ANNEALING | 73.80 | 81.40 | 81.20 |
| | 73.60 | 75.60 | 76.80 |
| SIMPLEX+LMDIF | 38.20 | 81.20 | 81.20 |
| | 38.20 | 81.20 | 81.20 |
| SIMPLEX+ANNEALING | 55.20 | **97.00** | **97.60** |
| | 55.20 | **96.20** | **97.00** |
| LMDIF+ANNEALING | 56.20 | **95.00** | **95.60** |
| | 56.60 | **92.40** | **93.60** |

Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 24. | 622. | 638. |
| | 22. | 275. | 275. |
| LMDIF | 24. | 95. | **574**. |
| | 24. | **65**. | 463. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 630. | 243. | 243. |
| | 630. | 243. | 243. |
| SIMPLEX+ANNEALING | 478. | **94**. | **93**. |
| | 481. | **107**. | **100**. |
| LMDIF+ANNEALING | 471. | **118**. | **116**. |
| | 470. | **140**. | **127**. |

- 1000 random points from $[-1000, 1000]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 27.40 | 57.20 | 75.40 |
| | 27.40 | 48.80 | 53.20 |
| LMDIF | 27.00 | 73.00 | 80.60 |
| | 27.00 | **94.00** | 76.00 |
| ANNEALING | 31.20 | 50.80 | 50.00 |
| | 31.60 | 50.60 | 52.20 |
| SIMPLEX+LMDIF | 32.60 | 77.00 | 77.00 |
| | 32.60 | 77.00 | 77.00 |
| SIMPLEX+ANNEALING | 34.20 | 78.80 | 78.60 |
| | 34.40 | 78.60 | 79.20 |
| LMDIF+ANNEALING | 28.20 | 39.20 | 39.40 |
| | 28.40 | 38.00 | 38.20 |

Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 32. | 719. | 732. |
| | 22. | 267. | 329. |
| LMDIF | 24. | 145. | 635. |
| | 24. | **103**. | 481. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 701. | 317. | 317. |
| | 701. | 317. | 317. |
| SIMPLEX+ANNEALING | 687. | 302. | 301. |
| | 686. | 304. | 299. |
| LMDIF+ANNEALING | 726. | 639. | 638. |
| | 724. | 648. | 645. |

### 4.4 Problem G03 (Listing 3)

- 1000 random points from $[-100, 100]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | **100.0** | **99.90** |
| LMDIF | 0.00 | 83.30 | 77.70 |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | **100.0** | **100.**0 |
| SIMPLEX+ANNEALING | 0.00 | 46.80 | 55.20 |
| LMDIF+ANNEALING | 0.00 | **100.0** | **100.0** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 7. | **258.** | **257.** |
| LMDIF | 9. | 338. | 430. |
| ANNEALING | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | **333.** | **488.** |
| SIMPLEX+ANNEALING | 1000. | 924. | 901. |
| LMDIF+ANNEALING | 1000. | **270.** | **361.** |

- 1000 random points from $[-1000, 1000]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | **99.50** | **99.50** |
| LMDIF | 0.00 | 69.40 | 68.80 |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | **99.90** | **100.0** |
| SIMPLEX+ANNEALING | 0.00 | 0.00 | 0.00 |
| LMDIF+ANNEALING | 0.00 | **100.0** | **100.**0 |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 7. | **343.** | **343.** |
| LMDIF | 9. | 475. | 526. |
| ANNEALING | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | **466.** | **691.** |
| SIMPLEX+ANNEALING | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | **419.** | **614.** |

## 4.5 Problem G04 (Listing 4)

- 1000 random points from $[-100, 100]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 2.20 | **97.80** | **99.60** |
| | 2.20 | 2.20 | 2.20 |
| LMDIF | 1.50 | 96.60 | 98.70 |
| | 1.50 | **100.0** | **100.0** |
| ANNEALING | 9.40 | 23.30 | 20.20 |
| | 6.90 | 6.30 | 7.80 |
| SIMPLEX+LMDIF | 10.90 | **93.80** | **99.00** |
| | 8.80 | **99.20** | **99.30** |
| SIMPLEX+ANNEALING | 10.70 | 86.00 | **93.80** |
| | 9.30 | 62.00 | 60.20 |
| LMDIF+ANNEALING | 3.30 | **98.00** | **99.90** |
| | 3.30 | **100.0** | **100.0** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 8. | **123.** | **100.** |
| | 7. | 7. | 7. |
| LMDIF | 9. | 46. | 112. |
| | 9. | **19.** | **80.** |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 932. | **137.** | **168.** |
| | 938. | **53.** | **165.** |
| SIMPLEX+ANNEALING | 935. | 499. | 463. |
| | 936. | 680. | 689. |
| LMDIF+ANNEALING | 975. | **110.** | **139.** |
| | 977. | **53.** | **146.** |

- 1000 random points from $[-1000, 1000]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | **95.20** | **97.10** |
| | 0.00 | 1.20 | 1.20 |
| LMDIF | 0.00 | 58.90 | 79.50 |
| | 0.00 | **99.90** | **100.0** |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | 79.00 | 86.80 |
| | 0.00 | **98.40** | **99.50** |
| SIMPLEX+ANNEALING | 0.00 | 31.40 | 35.10 |
| | 0.00 | 3.40 | 3.50 |
| LMDIF+ANNEALING | 0.00 | 64.60 | 77.50 |
| | 0.00 | **100.0** | **100.0** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 12. | **159.** | **143.** |
| | 7. | 22. | 22. |
| LMDIF | 9. | 348. | 430. |
| | 9. | **41.** | **125.** |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 400. | 491. |
| | 1000. | **134.** | **374.** |
| SIMPLEX+ANNEALING | 1000. | 920. | 904. |
| | 1000. | 996. | 997. |
| LMDIF+ANNEALING | 1000. | 568. | 604. |
| | 1000. | **130.** | **297.** |

## 4.6 Problem G05 (Listing 5)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success eq+ineq/eq,ineq/separate | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $1,1$ | $1,z$ | $1,z^2$ | $z,1$ | $z,z$ | $z,z^2$ | $z^2,1$ | $z^2,z$ | $z^2,z^2$ |
| | 0.00 | 0.00 | 0.00 | 0.20 | 0.10 | 0.50 | 0.90 | 0.90 | 0.80 |
| SIMPLEX | 0.00 | 0.00 | 0.00 | 0.10 | 0.10 | 0.10 | 0.90 | 0.90 | 0.90 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.30 | 0.30 | 0.30 |
| LMDIF | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.30 | 10.10 | 1.40 |
| | 0.00 | 0.00 | 0.00 | 0.70 | 1.20 | 1.00 | 1.00 | 3.30 | 3.80 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| ANNEALING | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| SMP+LMD | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.20 | 1.30 | 0.30 | 2.10 | 3.10 | 3.60 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| SMP+ANN | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| LMD+ANN | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.80 | 0.90 | 0.70 | 1.60 | 2.20 | 3.30 |

Average number of steps:

| Method | % success eq+ineq/eq,ineq/separate | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $1,1$ | $1,z$ | $1,z^2$ | $z,1$ | $z,z$ | $z,z^2$ | $z^2,1$ | $z^2,z$ | $z^2,z^2$ |
| | 12. | 52. | 52. | 661. | 663. | 699. | 660. | 679. | 717. |
| SIMPLEX | 6. | 6. | 6. | 661. | 661. | 661. | 660. | 660. | 660. |
| | 6. | 6. | 6. | 506. | 506. | 506. | 458. | 458. | 458. |
| | 8. | 26. | 38. | 534. | 523. | 502. | 801. | 646. | 979. |
| LMDIF | 8. | 15. | 95. | 531. | 542. | 454. | 456. | 938. | 991. |
| | 8. | 15. | 98. | 153. | 741. | 456. | 177. | 336. | 211. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| ANNEALING | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| SMP+LMD | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 997. | 1000. | 997. | 992. | 989. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| SMP+ANN | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| LMD+ANN | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 999. | 996. | 990. |

- 1000 random points from $[-1000, 1000]^v$
  Success rate:

| Method | % success eq+ineq/eq,ineq/separate | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $1,1$ | $1,z$ | $1,z^2$ | $z,1$ | $z,z$ | $z,z^2$ | $z^2,1$ | $z^2,z$ | $z^2,z^2$ |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 1.30 | 0.10 | 0.10 | 0.10 |
| SIMPLEX | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.10 | 0.10 | 0.10 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| LMDIF | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 1.90 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.10 | 0.00 | 0.00 | 0.20 | 0.10 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| ANNEALING | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| SMP+LMD | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.10 | 0.10 | 0.20 | 0.00 | 0.10 | 0.20 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| SMP+ANN | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| LMD+ANN | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.30 | 0.30 |

Average number of steps:

| Method | % success eq+ineq/eq,ineq/separate | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $1,1$ | $1,z$ | $1,z^2$ | $z,1$ | $z,z$ | $z,z^2$ | $z^2,1$ | $z^2,z$ | $z^2,z^2$ |
| | 6. | 71. | 71. | 866. | 873. | 867. | 860. | 870. | 897. |
| SIMPLEX | 6. | 6. | 6. | 866. | 866. | 866. | 860. | 860. | 860. |
| | 6. | 6. | 6. | 854. | 854. | 854. | 841. | 841. | 841. |
| | 8. | 26. | 55. | 569. | 564. | 437. | 857. | 748. | 985. |
| LMDIF | 8. | 16. | 113. | 585. | 568. | 420. | 633. | 973. | 994. |
| | 8. | 16. | 115. | 411. | 946. | 525. | 221. | 338. | 184. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| ANNEALING | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| SMP+LMD | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| SMP+ANN | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| LMD+ANN | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |

- 1000 random points from $[0, 1200] \times [0, 1200] \times [-0.55, 0.55] \times [-0.55, 0.55]$ (from problem formulation on Listing 5)

  Success rate:

| Method | % success eq+ineq/eq,ineq/separate | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $1, 1$ | $1, z$ | $1, z^2$ | $z, 1$ | $z, z$ | $z, z^2$ | $z^2, 1$ | $z^2, z$ | $z^2, z^2$ |
| | 0.00 | 0.00 | 0.00 | 12.60 | 13.10 | 12.90 | 56.80 | 56.60 | 57.00 |
| SIMPLEX | 0.00 | 0.00 | 0.00 | 12.90 | 12.90 | 13.10 | 56.60 | 56.60 | 57.20 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 74.90 | 74.70 | 75.30 |
| LMDIF | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.10 | 74.80 | 73.30 | 75.60 |
| | 0.00 | 0.00 | 0.00 | 80.70 | 87.00 | **91.40** | **93.10** | **100.0** | **100.0** |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| ANNEALING | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.10 | 0.00 | 0.00 |
| SMP+LMD | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 79.70 | 83.10 | 84.50 | **90.00** | **100.0** | **100.0** |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| SMP+ANN | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| LMD+ANN | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 | 84.60 | 80.90 | 84.80 | **93.10** | **99.90** | **100.0** |

  Average number of steps:

| Method | % success eq+ineq/eq,ineq/separate | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $1, 1$ | $1, z$ | $1, z^2$ | $z, 1$ | $z, z$ | $z, z^2$ | $z^2, 1$ | $z^2, z$ | $z^2, z^2$ |
| | 6. | 10. | 10. | 484. | 487. | 485. | 230. | 265. | 266. |
| SIMPLEX | 6. | 6. | 6. | 481. | 481. | 481. | 230. | 230. | 230. |
| | 6. | 6. | 6. | 158. | 158. | 158. | 111. | 111. | 111. |
| | 8. | 10. | 9. | 224. | 224. | 229. | 429. | 496. | 466. |
| LMDIF | 8. | 9. | 20. | 221. | 218. | 239. | 403. | 500. | 478. |
| | 8. | 9. | 20. | 37. | 68. | 75. | **99.** | **97.** | **98.** |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| ANNEALING | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| SMP+LMD | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 282. | 259. | 278. | **465.** | **407.** | **411.** |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| SMP+ANN | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| LMD+ANN | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. | 242. | 288. | 279. | **477.** | **438.** | **436.** |

### 4.7 Problem G06 (Listing 6)

- 1000 random points from $[-100, 100]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 80.30 | 80.60 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF | 0.00 | 7.10 | 2.90 |
| | 0.00 | **99.90** | **99.60** |
| ANNEALING | 0.90 | 3.20 | 2.40 |
| | 0.70 | 0.20 | 0.40 |
| SIMPLEX+LMDIF | 0.00 | 20.20 | 5.50 |
| | 0.10 | **100.0** | **99.80** |
| SIMPLEX+ANNEALING | 0.20 | 73.70 | 47.60 |
| | 0.00 | 9.90 | 9.80 |
| LMDIF+ANNEALING | 0.00 | 89.50 | 56.50 |
| | 0.00 | **100.0** | **100.0** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 4. | 292. | 314. |
| | 4. | 4. | 4. |
| LMDIF | 6. | 175. | 974. |
| | 6. | **83**. | **121**. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 814. | 958. |
| | 999. | **191**. | **270**. |
| SIMPLEX+ANNEALING | 1000. | 701. | 812. |
| | 1000. | 981. | 979. |
| LMDIF+ANNEALING | 1000. | 506. | 803. |
| | 1000. | **228**. | **309**. |

- 1000 random points from $[-1000, 1000]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 77.90 | 77.10 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF | 0.00 | 8.20 | 4.20 |
| | 0.00 | **99.50** | **98.40** |
| ANNEALING | 0.00 | 0.10 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | 20.80 | 1.70 |
| | 0.00 | **100.0** | **99.90** |
| SIMPLEX+ANNEALING | 0.00 | 63.00 | 39.90 |
| | 0.00 | 0.10 | 0.10 |
| LMDIF+ANNEALING | 0.00 | 87.00 | 52.80 |
| | 0.00 | **100.0** | **100.0** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 4. | 321. | 346. |
| | 4. | 4. | 4. |
| LMDIF | 6. | 216. | 971. |
| | 6. | **116.** | **183.** |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 813. | 993. |
| | 1000. | **209.** | **310.** |
| SIMPLEX+ANNEALING | 1000. | 772. | 841. |
| | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | 528. | 845. |
| | 1000. | **255.** | **375.** |

## 4.8 Problem G07 (Listing 7)

- 1000 random points from $[-100, 100]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 48.20 | 64.70 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF | 0.00 | **100.0** | **90.10** |
| | 0.00 | **100.0** | **99.30** |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+ANNEALING | 0.00 | 0.10 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF+ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 12. | 782. | 741. |
| | 12. | 44. | 44. |
| LMDIF | 14. | **130**. | **441**. |
| | 14. | **122**. | **342**. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |

- 1000 random points from $[-1000, 1000]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 25.20 | 40.40 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF | 0.00 | **99.80** | **92.90** |
| | 0.00 | **100.0** | **97.20** |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF+ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 12. | 908. | 869. |
| | 12. | 50. | 50. |
| LMDIF | 14. | **139.** | **499.** |
| | 14. | **129.** | **514.** |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |

## 4.9 Problem G08 (Listing 8)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | **100.0** | **100.0** |
| | 0.00 | 1.60 | 1.70 |
| LMDIF | 0.00 | 86.80 | 57.90 |
| | 0.00 | **99.50** | 89.20 |
| ANNEALING | 0.80 | 5.70 | 2.80 |
| | 1.10 | 2.20 | 2.40 |
| SIMPLEX+LMDIF | 0.00 | **100.0** | **100.0** |
| | 0.00 | **100.0** | **100.0** |
| SIMPLEX+ANNEALING | 0.10 | **100.0** | **100.0** |
| | 0.10 | 87.50 | 90.20 |
| LMDIF+ANNEALING | 0.20 | **100.0** | **100.0** |
| | 0.10 | **100.0** | **100.0** |

Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 4. | **47**. | **50**. |
| | 4. | 35. | 35. |
| LMDIF | 6. | 75. | 502. |
| | 6. | **56**. | 194. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | **62**. | **119**. |
| | 1000. | **67**. | **140**. |
| SIMPLEX+ANNEALING | 1000. | **185**. | **179**. |
| | 1000. | 450. | 437. |
| LMDIF+ANNEALING | 1000. | **64**. | **126**. |
| | 1000. | **66**. | **132**. |

- 1000 random points from $[-1000, 1000]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | **100.0** | **100.0** |
| | 0.00 | 0.10 | 0.10 |
| LMDIF | 0.00 | 82.60 | 54.60 |
| | 0.00 | **98.10** | 88.50 |
| ANNEALING | 0.00 | 0.10 | 0.10 |
| | 0.10 | 0.10 | 0.10 |
| SIMPLEX+LMDIF | 0.00 | **100.0** | **100.0** |
| | 0.00 | **100.0** | **100.0** |
| SIMPLEX+ANNEALING | 0.00 | **100.0** | **100.0** |
| | 0.00 | 57.80 | 59.10 |
| LMDIF+ANNEALING | 0.00 | **100.0** | **100.0** |
| | 0.00 | **100.0** | **100.0** |

Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 4. | **60**. | **62**. |
| | 4. | 43. | 43. |
| LMDIF | 6. | 119. | 545. |
| | 6. | **80**. | 217. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | **87**. | **167**. |
| | 1000. | **91**. | **192**. |
| SIMPLEX+ANNEALING | 1000. | **303**. | **300**. |
| | 1000. | 682. | 670. |
| LMDIF+ANNEALING | 1000. | **96**. | **207**. |
| | 1000. | **92**. | **193**. |

## 4.10 Problem G09 (Listing 9)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 78.20 | **96.10** |
| | 0.00 | 4.60 | 4.60 |
| LMDIF | 0.00 | 28.40 | 1.50 |
| | 0.00 | 87.20 | 54.60 |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | 42.70 | 41.20 |
| | 0.00 | 69.00 | 52.50 |
| SIMPLEX+ANNEALING | 0.00 | 2.00 | 2.30 |
| | 0.00 | 0.60 | 0.50 |
| LMDIF+ANNEALING | 0.00 | 62.50 | 81.90 |
| | 0.00 | 89.60 | **97.50** |

Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 9. | 423. | **327.** |
| | 9. | 120. | 120. |
| LMDIF | 11. | 291. | 991. |
| | 11. | 241. | 626. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 787. | 930. |
| | 1000. | 579. | 912. |
| SIMPLEX+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | 637. | 672. |
| | 1000. | 373. | **513.** |

- 1000 random points from $[-1000, 1000]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 43.10 | 59.30 |
| | 0.00 | 0.20 | 0.20 |
| LMDIF | 0.00 | 0.70 | 0.00 |
| | 0.00 | 33.60 | 7.50 |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | 5.10 | 21.80 |
| | 0.00 | 7.80 | 10.90 |
| SIMPLEX+ANNEALING | 0.00 | 0.00 | 9.40 |
| | 0.00 | 0.00 | 6.10 |
| LMDIF+ANNEALING | 0.00 | 9.20 | 28.10 |
| | 0.00 | 25.40 | 47.40 |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 9. | 772. | 715. |
| | 9. | 265. | 265. |
| LMDIF | 11. | 390. | 1000. |
| | 11. | 733. | 936. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 991. | 803. |
| | 1000. | 978. | 906. |
| SIMPLEX+ANNEALING | 1000. | 1000. | 921. |
| | 1000. | 1000. | 952. |
| LMDIF+ANNEALING | 1000. | 984. | 745. |
| | 1000. | 913. | 572. |

### 4.11 Problem G10 (Listing 10)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 26.40 | 48.60 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF | 0.00 | 52.40 | 73.00 |
| | 0.00 | 65.50 | **81.90** |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.40 | 1.70 | 23.50 |
| | 0.10 | 69.00 | **76.00** |
| SIMPLEX+ANNEALING | 0.40 | 1.60 | 1.10 |
| | 0.10 | 0.50 | 0.60 |
| LMDIF+ANNEALING | 0.00 | 0.80 | 0.50 |
| | 0.00 | **74.10** | **72.80** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 10. | 877. | 740. |
| | 10. | 10. | 10. |
| LMDIF | 12. | 328. | 478. |
| | 12. | 287. | **386**. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 1000. | 951. |
| | 1000. | 427. | **501**. |
| SIMPLEX+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | **379**. | **515**. |

- 1000 random points from $[-1000, 1000]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.30 | 47.30 | 55.70 |
| | 0.30 | 0.30 | 0.30 |
| LMDIF | 0.30 | 27.30 | 33.30 |
| | 0.30 | 52.00 | **77.80** |
| ANNEALING | 0.30 | 0.40 | 0.40 |
| | 0.30 | 0.30 | 0.40 |
| SIMPLEX+LMDIF | 2.10 | 38.00 | 45.10 |
| | 2.00 | 64.70 | **74.30** |
| SIMPLEX+ANNEALING | 2.10 | 7.10 | 7.80 |
| | 2.00 | 3.80 | 3.80 |
| LMDIF+ANNEALING | 0.30 | 12.90 | 22.40 |
| | 0.30 | 66.60 | 65.70 |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 11. | 678. | 603. |
| | 10. | 24. | 24. |
| LMDIF | 12. | 364. | 591. |
| | 12. | 343. | **445.** |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 750. | 718. |
| | 1000. | 472. | **540.** |
| SIMPLEX+ANNEALING | 1000. | 977. | 975. |
| | 1000. | 993. | 993. |
| LMDIF+ANNEALING | 1000. | 893. | 878. |
| | 1000. | 453. | 599. |

## 4.12 Problem G11 (Listing 11)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 66.90 | 83.60 |
| LMDIF | 0.00 | **100.0** | **100.0** |
| ANNEALING | 0.00 | 0.00 | 1.40 |
| SIMPLEX+LMDIF | 0.00 | **100.0** | **100.0** |
| SIMPLEX+ANNEALING | 0.00 | 2.50 | 55.00 |
| LMDIF+ANNEALING | 0.00 | **100.0** | **100.0** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 4. | 322. | 250. |
| LMDIF | 6. | **20**. | **66**. |
| ANNEALING | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | **50**. | **122**. |
| SIMPLEX+ANNEALING | 1000. | 992. | 615. |
| LMDIF+ANNEALING | 1000. | **56**. | **147**. |

- 1000 random points from $[-1000, 1000]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 70.60 | 85.80 |
| LMDIF | 0.00 | **99.90** | **100.0** |
| ANNEALING | 0.00 | 0.00 | 0.30 |
| SIMPLEX+LMDIF | 0.00 | **99.90** | **100.0** |
| SIMPLEX+ANNEALING | 0.00 | 2.70 | 50.80 |
| LMDIF+ANNEALING | 0.00 | **100.0** | **100.0** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 4. | 295. | 224. |
| LMDIF | 6. | **25**. | **87**. |
| ANNEALING | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | **69**. | **169**. |
| SIMPLEX+ANNEALING | 1000. | 995. | 706. |
| LMDIF+ANNEALING | 1000. | **79**. | **217**. |

## 4.13 Problem G12 (Listing 12)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 90.60 | 90.80 |
| LMDIF | 0.00 | 90.40 | 85.20 |
| ANNEALING | 0.00 | 0.60 | 0.20 |
| SIMPLEX+LMDIF | 0.00 | **100.0** | **100.0** |
| SIMPLEX+ANNEALING | 0.00 | **100.0** | **100.0** |
| LMDIF+ANNEALING | 0.00 | **100.0** | **100.0** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 5. | 340. | 338. |
| LMDIF | 7. | 191. | 287. |
| ANNEALING | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | **125.** | **210.** |
| SIMPLEX+ANNEALING | 1000. | **368.** | **352.** |
| LMDIF+ANNEALING | 1000. | **132.** | **221.** |

- 1000 random points from $[-1000, 1000]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 91.60 | 92.80 |
| LMDIF | 0.00 | 84.40 | 83.20 |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | **100.0** | **100.0** |
| SIMPLEX+ANNEALING | 0.00 | **99.60** | **99.80** |
| LMDIF+ANNEALING | 0.00 | **100.0** | **100.0** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 5. | 431. | 429. |
| LMDIF | 7. | 266. | 326. |
| ANNEALING | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | **171.** | **295.** |
| SIMPLEX+ANNEALING | 1000. | **744.** | **727.** |
| LMDIF+ANNEALING | 1000. | **187.** | **335.** |

## 4.14 Problem G13 (Listing 13)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 13.50 | 73.90 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF | 0.00 | 0.00 | 1.80 |
| | 0.00 | **75.60** | **73.80** |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | 1.10 | 26.00 |
| | 0.00 | **98.30** | **98.90** |
| SIMPLEX+ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF+ANNEALING | 0.00 | 0.90 | 25.80 |
| | 0.00 | **99.00** | **99.60** |

Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 7. | 756. | 621. |
| | 7. | 605. | 522. |
| LMDIF | 9. | 862. | 990. |
| | 9. | **342**. | **421**. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 1000. | 967. |
| | 1000. | **361**. | **684**. |
| SIMPLEX+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | 1000. | 974. |
| | 1000. | **327**. | **580**. |

- 1000 random points from $[-1000, 1000]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 1.40 | 21.30 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF | 0.00 | 0.00 | 0.30 |
| | 0.00 | 57.80 | **66.60** |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | 0.00 | 7.60 |
| | 0.00 | **98.10** | **86.00** |
| SIMPLEX+ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF+ANNEALING | 0.00 | 0.00 | 2.70 |
| | 0.00 | **98.30** | **96.50** |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 7. | 904. | 931. |
| | 7. | 714. | 660. |
| LMDIF | 9. | 834. | 999. |
| | 9. | 530. | **544**. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 1000. | 956. |
| | 1000. | **502**. | **595**. |
| SIMPLEX+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | 1000. | 994. |
| | 1000. | **472**. | **693**. |

## 4.15 Problem of the Design of a Pressure Vessel (vess) (Listing 14)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 3.70 | 87.30 | **89.40** |
| | 3.70 | 3.80 | 3.80 |
| LMDIF | 3.40 | **87.40** | **87.10** |
| | 3.40 | **87.40** | 82.40 |
| ANNEALING | 7.00 | 11.30 | 11.50 |
| | 6.50 | 7.70 | 8.90 |
| SIMPLEX+LMDIF | 10.40 | **96.20** | **98.30** |
| | 10.90 | 93.40 | 79.30 |
| SIMPLEX+ANNEALING | 11.00 | 59.40 | 60.00 |
| | 11.30 | 31.60 | 30.20 |
| LMDIF+ANNEALING | 4.10 | **93.30** | 92.80 |
| | 4.20 | **91.60** | 80.60 |

Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 6. | 170. | **179**. |
| | 6. | 21. | 21. |
| LMDIF | 8. | **37**. | **177**. |
| | 8. | **90**. | 150. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 919. | **123**. | **242**. |
| | 918. | 112. | 386. |
| SIMPLEX+ANNEALING | 921. | 731. | 723. |
| | 919. | 830. | 829. |
| LMDIF+ANNEALING | 959. | **142**. | 231. |
| | 958. | **141**. | 323. |

- 1000 random points from $[-1000, 1000]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 4.40 | **92.10** | **93.90** |
| | 4.40 | 8.30 | 9.60 |
| LMDIF | 4.30 | 51.10 | 52.00 |
| | 4.30 | 68.10 | 66.90 |
| ANNEALING | 4.80 | 5.40 | 6.80 |
| | 4.70 | 4.70 | 5.40 |
| SIMPLEX+LMDIF | 4.60 | **91.30** | **93.50** |
| | 4.70 | 71.30 | 75.70 |
| SIMPLEX+ANNEALING | 4.60 | 32.70 | 47.80 |
| | 4.80 | 8.30 | 12.50 |
| LMDIF+ANNEALING | 4.40 | 72.10 | 75.80 |
| | 4.40 | 78.00 | 77.50 |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 6. | **121**. | **123**. |
| | 6. | 21. | 21. |
| LMDIF | 8. | 80. | 190. |
| | 8. | 85. | 177. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 961. | **207**. | **268**. |
| | 961. | 347. | 408. |
| SIMPLEX+ANNEALING | 961. | 836. | 714. |
| | 960. | 944. | 913. |
| LMDIF+ANNEALING | 956. | 367. | 406. |
| | 956. | 286. | 396. |

## 4.16 Problem of the Design of a Tension/Compression Spring (tens) (Listing 15)

- 1000 random points from $[-100, 100]^v$
  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 13.30 | 18.00 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF | 0.00 | 0.10 | 0.70 |
| | 0.00 | 20.70 | 22.80 |
| ANNEALING | 0.00 | 0.10 | 0.30 |
| | 0.20 | 0.00 | 0.10 |
| SIMPLEX+LMDIF | 0.00 | 0.20 | 5.20 |
| | 0.00 | 0.70 | 8.40 |
| SIMPLEX+ANNEALING | 0.10 | 22.50 | 25.10 |
| | 0.00 | 1.00 | 1.00 |
| LMDIF+ANNEALING | 0.00 | 2.70 | 6.00 |
| | 0.00 | 1.30 | 11.30 |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 5. | 119. | 117. |
| | 5. | 12. | 14. |
| LMDIF | 7. | 183. | 566. |
| | 7. | 329. | 202. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 1000. | 976. |
| | 1000. | 1000. | 974. |
| SIMPLEX+ANNEALING | 1000. | 916. | 902. |
| | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | 995. | 985. |
| | 1000. | 1000. | 971. |

- 1000 random points from $[-1000, 1000]^v$

  Success rate:

| Method | % success | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 0.00 | 5.10 | 8.80 |
| | 0.00 | 0.00 | 0.00 |
| LMDIF | 0.00 | 0.00 | 0.10 |
| | 0.00 | 2.50 | 4.60 |
| ANNEALING | 0.00 | 0.00 | 0.00 |
| | 0.00 | 0.00 | 0.00 |
| SIMPLEX+LMDIF | 0.00 | 0.00 | 1.50 |
| | 0.00 | 0.00 | 0.60 |
| SIMPLEX+ANNEALING | 0.00 | 12.10 | 15.30 |
| | 0.00 | 0.10 | 0.00 |
| LMDIF+ANNEALING | 0.00 | 0.20 | 1.30 |
| | 0.00 | 0.20 | 0.40 |

  Average number of steps:

| Method | avg # of steps | | |
|---|---|---|---|
| | 1 | $z$ | $z^2$ |
| SIMPLEX | 5. | 111. | 111. |
| | 5. | 12. | 14. |
| LMDIF | 7. | 187. | 332. |
| | 7. | 168. | 196. |
| ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+LMDIF | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |
| SIMPLEX+ANNEALING | 1000. | 990. | 984. |
| | 1000. | 1000. | 1000. |
| LMDIF+ANNEALING | 1000. | 1000. | 1000. |
| | 1000. | 1000. | 1000. |

# 5 Conclusions

The test results from section 4 are summarized in the following two performance tables. For every problem the three best approaches to constraint satisfaction from section 3 are listed for every initial point sampling range. Comparison is based on the percentage of successful runs and average number of steps made in search process (including failed ones):

- On 1000 random initial points from $[-100, 100]^v$

| problem | I | | | II | | | III | | |
|---|---|---|---|---|---|---|---|---|---|
| | name | succ | st | name | succ | st | name | succ | st |
| 0 | L+A$(z,z)$ | 100.0 | 82 | L$(z,z)$ | 98.1 | 50 | S+L$(z,z)$ | 100.0 | 70 |
| 1 | L$(z)$ | 100.0 | 45 | L:c$(z)$ | 98.67 | 94 | L$(z^2)$ | 100.0 | 234 |
| 2 | L$(z)$ | 97.8 | 65 | S+A:c$(z^2)$ | 97.6 | 93 | S+A:c$(z)$ | 97.0 | 94 |
| 3 | S$(z)$ | 100.0 | 258 | L+A$(z)$ | 100.0 | 270 | S+L$(z)$ | 100.0 | 333 |
| 4 | L$(z)$ | 100.0 | 19 | L+A$(z)$ | 100.0 | 53 | L$(z^2)$ | 100.0 | 80 |
| 5 | L$(z^2+z)$ | 10.1 | 938 | - | - | - | - | - | - |
| 6 | L$(z)$ | 99.9 | 83 | L$(z^2)$ | 99.6 | 121 | S+L$(z)$ | 100.0 | 191 |
| 7 | L$(z)$ | 100.0 | 122 | L$(z^2)$ | 99.3 | 342 | - | - | - |
| 8 | L+A$(z)$ | 100.0 | 66 | S+L$(z)$ | 100.0 | 67 | L$(z)$ | 99.5 | 56 |
| 9 | S:c$(z^2)$ | 96.1 | 327 | L+A$(z^2)$ | 97.5 | 513 | L+A$(z)$ | 89.6 | 373 |
| 10 | L$(z^2)$ | 81.9 | 386 | S+L$(z^2)$ | 76.0 | 501 | L+A$(z)$ | 74.1 | 379 |
| 11 | L$(z)$ | 100.0 | 20 | S+L$(z)$ | 100.0 | 50 | L+A$(z)$ | 100.0 | 56 |
| 12 | S+L$(z)$ | 100.0 | 125 | L+A$(z)$ | 100.0 | 132 | S+L$(z^2)$ | 100.0 | 210 |
| 13 | L+A$(z)$ | 99.9 | 361 | S+L$(z)$ | 98.3 | 327 | L$(z)$ | 75.6 | 342 |
| pres | S+L:c$(z^2)$ | 98.3 | 242 | L+A$(z)$ | 91.6 | 141 | L$(z)$ | 89.4 | 90 |
| tens | L$(z^2)$ | 22.8 | 202 | L$(z)$ | 20.7 | 329 | S+A:c$(z^2)$ | 25.1 | 902 |

- On 1000 random initial points from $[-1000, 1000]^v$

| problem | I | | | II | | | III | | |
|---|---|---|---|---|---|---|---|---|---|
| | name | succ | st | name | succ | st | name | succ | st |
| 0 | L+A$(z,z)$ | 100.0 | 106 | L$(z,z)$ | 99.6 | 45 | S+L$(z,z)$ | 100.0 | 90 |
| 1 | L$(z)$ | 100.0 | 45 | L:c$(z)$ | 98.5 | 97 | L$(z^2)$ | 100.0 | 278 |
| 2 | L$(z)$ | 94.0 | 103 | S+A:c$(z)$ | 78.8 | 302 | S+A:c$(z^2)$ | 78.6 | 301 |
| 3 | S$(z)$ | 99.5 | 343 | S+L$(z)$ | 99.9 | 466 | L+A$(z)$ | 100.0 | 419 |
| 4 | L$(z)$ | 99.9 | 41 | L+A$(z)$ | 100.0 | 130 | L$(z^2)$ | 100.0 | 125 |
| 5 | - | - | - | - | - | - | - | - | - |
| 6 | L$(z)$ | 99.5 | 116 | L$(z^2)$ | 98.4 | 183 | S+L$(z)$ | 100.0 | 209 |
| 7 | L$(z)$ | 100.0 | 129 | L$(z^2)$ | 97.2 | 514 | - | - | - |
| 8 | L+A$(z)$ | 100.0 | 92 | S+L$(z)$ | 100.0 | 91 | L$(z)$ | 98.1 | 80 |
| 9 | S:c$(z^2)$ | 59.3 | 715 | L+A$(z^2)$ | 47.4 | 572 | L+A$(z)$ | 25.4 | 913 |
| 10 | L$(z^2)$ | 77.8 | 445 | S+L$(z^2)$ | 74.3 | 540 | L+A$(z)$ | 66.6 | 453 |
| 11 | L$(z)$ | 99.9 | 25 | S+L$(z)$ | 99.9 | 69 | L+A$(z)$ | 100.0 | 79 |
| 12 | S+L$(z)$ | 100.0 | 171 | L+A$(z)$ | 100.0 | 187 | S+L$(z^2)$ | 100.0 | 295 |
| 13 | L+A$(z)$ | 98.3 | 472 | S+L$(z)$ | 98.1 | 502 | L$(z)$ | 66.6 | 542 |
| pres | S+L:c$(z^2)$ | 93.5 | 268 | S:c$(z^2)$ | 93.3 | 123 | S:c$(z)$ | 92.1 | 121 |
| tens | L$(z^2)$ | 4.6 | 196 | L$(z)$ | 2.5 | 168 | S+A:c$(z^2)$ | 15.3 | 984 |

From those tables it could be clearly seen that the optimal approach to constraint satisfaction on the selected set of problems is:

- optimizer: LMDIF
- objective function type: separate, i.e. penalties for individual constraints are treated as separate objectives in a multi-objective optimization problem (12)
- power for the penalty function: $a = 1$ for both equality and inequality constraints

This approach is the first best for problems G01, G02, G04, G06, G07 and G11, second best for G00 and tens, third best for G08, G13 and pres. Combined LMDIF+ANNEALING search method used with the same penalty function and objective function type is a second best approach with a slightly larger number of steps. However, for some problems (G03, G13), it demonstrated significantly better performance; and, for most of them it does not perform significantly worse than the leader. We believe that this is caused by the fact that the random and very heuristic ANNEALING method helps the deterministic and analytic LMDIF method to avoid getting stuck on difficult landscapes in the search space of the complicated problems. We also believe that a good performance of the next best SIMPLEX+LMDIF combined method is also due to the LMDIF while the heuristic SIMPLEX method helps LMDIF to not get stuck. Therefore we consider LMDIF (possibly paired with heuristic "helper method") as a best selection for the constraint satisfaction on the presented set of problems. ANNEALING method alone demonstrated the worse results and SIMPLEX showed generally average performance.

In view of the "No Free Lunch Theorems for Search and Optimization" [30] such a superior performance of one optimization method over others could be explained by the fact that it uses the largest amount of information about the problem under consideration to guide the search process. While SIMPLEX and ANNEALING are purely heuristic methods and do not use any information about a problem apart from function values, LMDIF uses both first derivative and approximation of the second derivative [14] to determine the direction to the minimum. As one can see (section 2), most of the constraints in the presented set of the problems are given in a form of nice, twice continuously differentiable functions. Hence it is possible to use this extra available information to run the specialized method. We speculate that for general constraint functions that do not possess such nice properties, results in terms of the best constraint satisfaction method might be quite different. Other optimization methods exploiting certain properties of the considered classes of the problems should be more efficient for those problems.

Data in the summary tables could also be used to select an optimal number of steps for guaranteed constraint satisfaction. However, we are generally interested not only in performance but also in the computational price as well; hence, a different set of tests might be needed in order to determine a minimal maximum number of steps allowed to reach a desired rate of successful runs to all runs. Here we can only conclude that this level would depend on the maximum allowed number of steps. Setting it to values less than the average from the tables would most likely lead to degraded performance.

We also note that problems with equality constraints (G03, G05, G13) and the high-dimensional problems (G03, G07, G09, G10) have indeed demonstrated themselves as being harder to solve. However, the high-dimensional problem G02 and problem G11 with equality constraint only did not obey this empirical rule. Hence we suggest the estimation of the difficulty of the problem based on this rule to be taken with care and always verified by simulations.

We see that for those problems power penalty functions (17) with $a = 1$ are the best choice, while $a = 2$ are significantly inferior. However, this result is not only problem-dependent but also also optimizer-dependent hence we could not conclude that those functions would be a best choice for any combination of the problem and optimizer. We believe that step penalty functions, i.e. $a = 0$, that demonstrated near zero percent successful runs in our test (see tables in section 4), should generally be avoided as they do not provide any information about the direction in which penalty is increasing and decreasing. Since they only indicate if the point is feasible or not, the search landscape for such penalties is flat which leads most optimization methods to fail because of the inability to make a move to a point better than the initial. This conclusion is in accordance with the previous studies on penalty functions [26].

Wherever it applies (problems G00, G05) our studies do not demonstrate a significant difference in performance between the *all combined* and *equality combined + inequality combined* optimization problem formulation methods except for the G05 tested on 1000 random points from $[-100, 100]^v$ where it demonstrated 2.5 better performance than ony other method. However, those results were not verified by the test performed on the search domain from the problem formulation (see Listing 5). Both those objective function types were outperformed by the *separate* method and thus are not recommended.

Poor results for the problem G05 for both test ranges is observed to be due to a difference in 3 orders of magnitude between search domains for $x_1, x_2 \in [0, 1200]$ and $x_3, x_4 \in [-0.55, 0.55]$ from the problem formulation (see Listing 5) that is inconsistent with the search domains of $[-100, 100]^4$ and $[-1000, 1000]^4$ used in testing. Additional testing on the suggested search domain supported all observations about the best method and objective function construction method presented earlier. It must be noted, however, that the best results were obtained when quadratic power penalties were used for equality constraints, i.e. when penalties for violating inequality constraints were steeper than the ones for violating equality constraints.

We should finally note that the tolerance used for constraint satisfaction definitely influences the overall performance, especially in case of equality constraints. In our tests we used tolerance of $10^{-5}$ but this value is generally problem-dependent and might have to be either increased or softened.

Based on our tests we conclude that the transformation of the constraint satisfaction problem into a multi-objective unconstrained optimization problem (12) via power penalty functions (17) with $a = 1$ and successive treatment of the resulting optimization problem with the LMDIF *COSY Infinity* optimizer is a reasonable choice of the default parameters for REPROPT. However, we should note that the problem set is not very large and is not covering all possible cases hence the results are not universal and thus might not be universally applicable. In case of the poor performance of the REPROPT method we suggest tuning of parameters based on the information about the problem, possibly after studies similar to the ones performed for this work.

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