PERTURBATIVE EQUATIONS OF MOTION
AND DIFFERENTIAL OPERATORS IN
NON-PLANAR CURVILINEAR COORDINATES

Kyoko Makino[1], Martin Berz[2]

Dept. of Physics and Astronomy
and
National Superconducting Cyclotron Laboratory
Michigan State University, East Lansing, MI 48824, USA
e-mails: [1]makino@msu.edu, [2]berz@msu.edu

Abstract: To analyze perturbative effects in the proximity of a reference orbit, it is often advantageous to describe the motion in terms of a set of relative coordinates. We study the relative motion in an attached moving Dreibein that has freedom of torsion in configuration space. The original motion is assumed to be due to the action of scalar or vector potentials, such as those that arise in gravitation or electromagnetic systems. Transformation rules for the potentials, common differential operators, and the resulting equations of motion are derived.

1. Introduction

In this paper, we will derive transformations of differential equations to so-called curvilinear coordinates. These coordinates are measured in a moving right-handed coordinate system that has one of its axes attached and parallel to a given reference curve in space.

While this approach at first sight perhaps seems to complicate the description of the motion, it has indeed several advantages. Firstly, if the chosen reference curve in space is itself a valid orbit, then the resulting flow of the differential equation will be origin preserving, because the origin just corresponds to the reference curve itself. This fact then opens the door to the use of perturbative techniques for the analysis of the motion in order to study how small deviations from the reference curve propagate, for example in the setting of perturbation theories popular in the field of beam physics [1], [2], [3]. In particular, if the system of interest is repetitive and the reference curve is closed, then the origin will be a fixed point of the motion. If the arc length is used as the independent variable, then after one revolution around the reference orbit, the system is repetitive, and perturbative techniques around fixed points [4] can be employed to study the one-turn flow of the differential equation, which here corresponds to a Poincare map of the motion [5].

The following sections describe in detail the derivation of the motion in curvilinear coordinates. We will describe the transformations between coordinates and the right hand sides, and will then derive the forms for standard differential operators including gradient, divergence, curl, and Laplacian in the new coordinates. In particular these will allow the transformation of differential equations of which parts are derived from scalar and vector potentials under preservation of the potential structure. As an application, we derive the transformation rule for relativistic motion in gravitational or electromagnetic fields. In a companion paper [6] we study the preservation of existing Lagrangian and Hamiltonian structure under the transformations.

2. Non-planar Curvilinear Coordinates

Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ denote a Dreibein, a right-handed set of fixed orthonormal basis vectors, which defines the so-called Cartesian coordinate systems. For any point in space, let (x_1, x_2, x_3) denote its Cartesian coordinates. In order to introduce the curvilinear coordinates, let $\vec{R}(s)$ be an infinitely often differentiable curve parameterized in terms of its arc length s, the so-called reference curve. For each value of s, let the vector $\vec{e}_s$ be parallel to the reference curve, i.e.

$$\vec{e}_s(s) = \frac{d\vec{R}}{ds}. \tag{1}$$

We now choose the infinitely often differentiable vectors $\vec{e}_x(s)$ and $\vec{e}_y(s)$ such that for any value of s, the three vectors $\{\vec{e}_s, \vec{e}_x, \vec{e}_y\}$ form a Dreibein, a right-handed orthonormal system. For notational simplicity, in the following we also sometimes denote the curvilinear basis vectors $\{\vec{e}_s, \vec{e}_x, \vec{e}_y\}$ by $\{\vec{e}_1^C, \vec{e}_2^C, \vec{e}_3^C\}$.
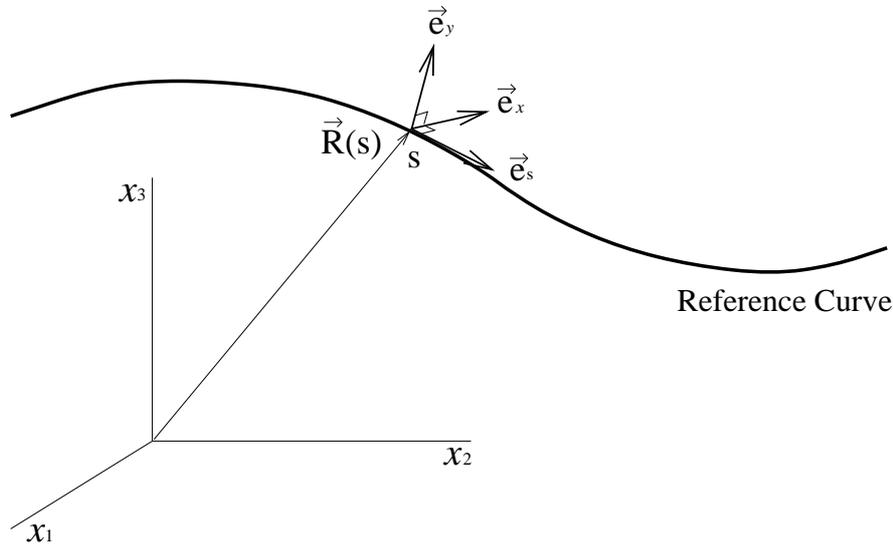
Figure 1: Reference curve and the locally attached Dreibeins.

Apparently, for a given curve $\vec{R}(s)$ there are a variety of choices for $\vec{e}_x(s)$ and $\vec{e}_y(s)$ that result in valid Dreibeins since $\vec{e}_x(s)$ and $\vec{e}_y(s)$ can be rotated around $\vec{e}_s$. A specific choice is often made such that additional requirements are satisfied; for example, if the curve $\vec{R}(s)$ is never parallel to the vertical Cartesian coordinate $\vec{e}_3$, one may demand that $\vec{e}_x(s)$ always lie in the horizontal plane spanned by $\vec{e}_1$ and $\vec{e}_2$.

The functions $\vec{R}(s)$, $\vec{e}_x(s)$, and $\vec{e}_y(s)$ describe the so-called curvilinear co-ordinate system, in which a position is described in terms of s, x and y via

$$\vec{r} = \vec{R}(s) + x\,\vec{e}_x + y\,\vec{e}_y.$$

Apparently the position $\vec{r}$ in Cartesian coordinates is uniquely determined for any choice of (s, x, y). The converse, however, is not generally true: a point with given Cartesian coordinates $\vec{r}$ may lie in several different planes that are perpendicular to the curve $\vec{R}(s)$, as shown in Figure 2.

The situation can be remedied if the curvature $\kappa(s)$ of the reference curve $\vec{R}(s)$ never grows beyond a threshold, i.e. if

$$r_1 = 1/\max_s |\kappa(s)| \tag{2}$$

is finite. As Figure 3 illustrates, if in this case we restrict ourselves to the inside of a tube of radius r_1 around $\vec{R}(s)$, for any vector within the tube, there is always one and only one set of coordinates (s, x, y) describing the point $\vec{r}$.

Let us now study the transformation matrix from the Cartesian basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ to the local basis of the curvilinear system $\{\vec{e}_s, \vec{e}_x, \vec{e}_y\} = \{\vec{e}_1^C, \vec{e}_2^C, \vec{e}_3^C\}$.
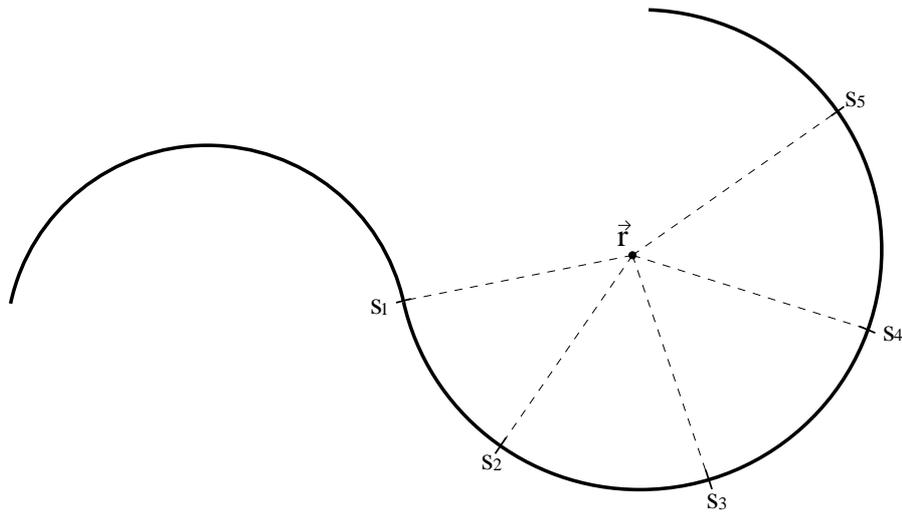
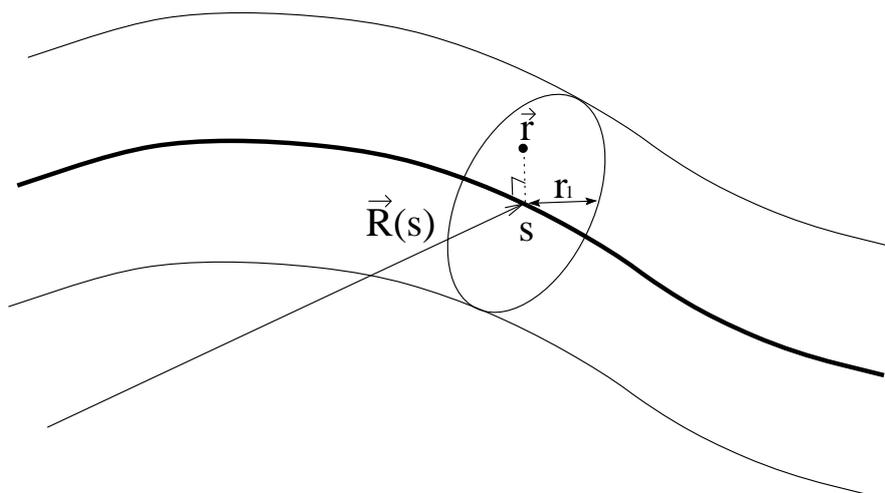Figure 2: Non-uniqueness of curvilinear coordinates.

Figure 3: Uniqueness of curvilinear coordinates within a tube.

The transformation between these basis vectors and the old ones is described by the matrix $\hat{O}(s)$ which has the form

$$\hat{O}(s) = \left(\begin{array}{c|c|c} \vec{e}_s(s) & \vec{e}_x(s) & \vec{e}_y(s) \end{array} \right) = \left(\begin{array}{ccc} (\vec{e}_s \cdot \vec{e}_1) & (\vec{e}_x \cdot \vec{e}_1) & (\vec{e}_y \cdot \vec{e}_1) \\ (\vec{e}_s \cdot \vec{e}_2) & (\vec{e}_x \cdot \vec{e}_2) & (\vec{e}_y \cdot \vec{e}_2) \\ (\vec{e}_s \cdot \vec{e}_3) & (\vec{e}_x \cdot \vec{e}_3) & (\vec{e}_y \cdot \vec{e}_3) \end{array} \right). \quad (3)$$

Because the system $\{\vec{e}_s, \vec{e}_x, \vec{e}_y\}$ is orthonormal, so is $\hat{O}(s)$, and hence it satisfies

$$\hat{O}(s) \cdot \hat{O}(s)^t = \hat{I} \quad \text{and} \quad \hat{O}(s)^t \cdot \hat{O}(s) = \hat{I}. \quad (4)$$

Since both the old and the new bases have the same handedness, we also have

$$\det(\hat{O}(s)) = 1, \quad (5)$$

and hence altogether, $\hat{O}(s)$ belongs to the group SO(3). We remind ourselves that elements of SO(3) preserve cross products, i.e. for $\hat{O} \in$ SO(3) and any vectors $\vec{a}, \vec{b}$, we have

$$(\hat{O}\vec{a}) \times (\hat{O}\vec{b}) = \hat{O}(\vec{a} \times \vec{b}). \quad (6)$$

One way to see this is to study the requirement of orthonormality on the matrix elements of $\hat{O}$. The elements of the matrix $\hat{O}$ describe the coordinates of the new parameter dependent basis vectors in terms of the original Cartesian basis; explicitly, we have

$$[\vec{e}_s]_k = O_{k1}, \quad [\vec{e}_x]_k = O_{k2}, \quad [\vec{e}_y]_k = O_{k3}. \quad (7)$$

The demand of the right-handedness then reads

$$\vec{e}_l^C \times \vec{e}_m^C = \sum_{n=1}^{3} \epsilon_{lmn} \vec{e}_n^C,$$

where ϵ_{ijk} is the common totally antisymmetric tensor of rank three defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for } (i,j,k) = (1,2,3) \text{ and any cyclic permutation thereof} \\ -1 & \text{for other permutations of } (1,2,3) \\ 0 & \text{for two or more equal indices} \end{cases},$$

and reduces to a condition on the elements of the matrix $\hat{O}$

$$\sum_{i,j=1}^{3} \epsilon_{ijk} O_{il} O_{jm} = \sum_{n=1}^{3} \epsilon_{lmn} O_{kn}. \quad (8)$$

We remind ourselves that the symbol ϵ_{ijk} is very useful for the calculation of vector cross products; for vectors $\vec{a}, \vec{b}$, we have

$$[\vec{a} \times \vec{b}]_k = \sum_{i,j=1}^{3} \epsilon_{ijk} a_i b_j.$$

Using condition (8), we readily obtain (6).

For the following discussion, it is useful to study how the transformation matrix $\hat{O}$ changes with s. Differentiating (4) with respect to the parameter s, we have

$$0 = \frac{d}{ds}(\hat{O}^t \cdot \hat{O}) = \frac{d\hat{O}^t}{ds}\hat{O} + \hat{O}^t\frac{d\hat{O}}{ds} = \left(\hat{O}^t\frac{d\hat{O}}{ds}\right)^t + \hat{O}^t\frac{d\hat{O}}{ds}.$$

So, the matrix $\hat{T} = \hat{O}^t \cdot d\hat{O}/ds$ is antisymmetric; we describe it in terms of its three free elements via

$$\hat{O}^t \cdot \frac{d\hat{O}}{ds} = \hat{T} = \begin{pmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0 \end{pmatrix}. \tag{9}$$

The three elements we group into the vector $\vec{\tau}$, which has the form

$$\vec{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}.$$

We observe that for any vector $\vec{a}$, we then have the relation

$$\hat{T} \cdot \vec{a} = \vec{\tau} \times \vec{a}.$$

The components of the vector $\vec{\tau}$, and hence the elements of the matrix $\hat{T}$, can be computed as

$$\tau_1 = \vec{e}_y \cdot \frac{d\vec{e}_x}{ds} = -\vec{e}_x \cdot \frac{d\vec{e}_y}{ds},$$
$$\tau_2 = \vec{e}_s \cdot \frac{d\vec{e}_y}{ds} = -\vec{e}_y \cdot \frac{d\vec{e}_s}{ds},$$
$$\tau_3 = \vec{e}_x \cdot \frac{d\vec{e}_s}{ds} = -\vec{e}_s \cdot \frac{d\vec{e}_x}{ds}. \tag{10}$$

These relationships give some practical meaning to the components of the vector $\vec{\tau}$: Apparently, τ_1 describes the current rate of rotation of the Dreibein around the reference curve $\vec{R}(s)$; τ_2 describes the current amount curvature of $\vec{R}(s)$ in the plane spanned by $\vec{e}_y$ and $\vec{e}_s$; and τ_3 similarly describes the curvature of $\vec{R}(s)$ in the plane spanned by $\vec{e}_x$ and $\vec{e}_s$. In more mathematical terms, because of

$$\vec{e}_s \cdot \frac{d\vec{e}_s}{ds} = 0, \quad \vec{e}_x \cdot \frac{d\vec{e}_x}{ds} = 0, \quad \vec{e}_y \cdot \frac{d\vec{e}_y}{ds} = 0, \tag{11}$$

we have

$$\frac{d\vec{e}_s}{ds} = \tau_3\vec{e}_x - \tau_2\vec{e}_y$$

$$\frac{d\vec{e}_x}{ds} = -\tau_3\vec{e}_s + \tau_1\vec{e}_y$$

$$\frac{d\vec{e}_y}{ds} = \tau_2\vec{e}_s - \tau_1\vec{e}_x, \tag{12}$$

as successive multiplication with $\vec{e}_s$, $\vec{e}_x$ and $\vec{e}_y$ and comparison with (10) reveals.

3. Differential Operators in Non-planar Curvilinear Coordinates

As the first step in the transformation of the differential equation to the curvilinear coordinates, it is necessary to study the form of common differential operators in the new coordinates. From (7), which has the form

$$\vec{r} = \sum_{k=1}^{3} x_k\vec{e}_k = \sum_{k=1}^{3} \left\{ \vec{R} \cdot \vec{e}_k + xO_{k2} + yO_{k3} \right\} \vec{e}_k,$$

we see that the Cartesian components of $\vec{r}$ are

$$x_k = \vec{R} \cdot \vec{e}_k + xO_{k2} + yO_{k3}, \text{ for } k = 1, 2, 3. \tag{13}$$

Hence the partial derivatives of x_k with respect to s, x and y are

$$\frac{\partial x_k}{\partial s} = \frac{d\vec{R}(s)}{ds} \cdot \vec{e}_k + x\frac{dO_{k2}}{ds} + y\frac{dO_{k3}}{ds} = O_{k1} + x\frac{dO_{k2}}{ds} + y\frac{dO_{k3}}{ds}$$

$$\frac{\partial x_k}{\partial x} = O_{k2}, \quad \text{and} \quad \frac{\partial x_k}{\partial y} = O_{k3},$$

where (1) and (7) have been used. Thus, the Jacobian matrix $\hat{C}$ is

$$\hat{C} = \left(\frac{\partial(x_1, x_2, x_3)}{\partial(s, x, y)}\right) = \begin{pmatrix} \dfrac{\partial x_1}{\partial s} & \dfrac{\partial x_2}{\partial s} & \dfrac{\partial x_3}{\partial s} \\ \dfrac{\partial x_1}{\partial x} & \dfrac{\partial x_2}{\partial x} & \dfrac{\partial x_3}{\partial x} \\ \dfrac{\partial x_1}{\partial y} & \dfrac{\partial x_2}{\partial y} & \dfrac{\partial x_3}{\partial y} \end{pmatrix} = \begin{pmatrix} O_{11} & O_{21} & O_{31} \\ O_{12} & O_{22} & O_{32} \\ O_{13} & O_{23} & O_{33} \end{pmatrix}$$

$$+ \begin{pmatrix} x\dfrac{dO_{12}}{ds} + y\dfrac{dO_{13}}{ds} & x\dfrac{dO_{22}}{ds} + y\dfrac{dO_{23}}{ds} & x\dfrac{dO_{32}}{ds} + y\dfrac{dO_{33}}{ds} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \hat{O}^t + \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \frac{d\hat{O}^t}{ds} = \hat{O}^t + \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \frac{d\hat{O}^t}{ds} \cdot (\hat{O} \cdot \hat{O}^t)$$

$$= \left\{ \hat{I} + \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \hat{T}^t \right\} \cdot \hat{O}^t = \begin{pmatrix} 1 - \tau_3 x + \tau_2 y & -\tau_1 y & \tau_1 x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \hat{O}^t.$$

It is convenient to denote the first part of the Jacobian matrix $\hat{C}$ by $\hat{A}$, i.e.

$$\hat{A} = \begin{pmatrix} 1 - \tau_3 x + \tau_2 y & -\tau_1 y & \tau_1 x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ;$$

then the Jacobian matrix can be written as

$$\hat{C} = \hat{A} \cdot \hat{O}^t.$$

The inverse matrix of $\hat{A}$ is found easily; we obtain

$$\hat{A}^{-1} = \begin{pmatrix} \dfrac{1}{1 - \tau_3 x + \tau_2 y} & \dfrac{\tau_1 y}{1 - \tau_3 x + \tau_2 y} & \dfrac{-\tau_1 x}{1 - \tau_3 x + \tau_2 y} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For later convenience, it is advantageous to introduce the abbreviation

$$\alpha = 1 - \tau_3 x + \tau_2 y. \tag{14}$$

We note that for x and y sufficiently close to zero, α does not vanish and is positive. Hence besides the restriction for the motion to be inside a tube of radius r_1 imposed by the need for uniqueness of the transformation to curvilinear coordinates in (2), there is another condition; defining

$$r_2 = \frac{1}{2} \min_s (|\frac{1}{\tau_3}|, |\frac{1}{\tau_2}|), \tag{15}$$

then if we restrict x, y to satisfy $|x|, |y| < r_2$, the quantity α never vanishes.

In this case, for the inverse matrix of the Jacobian matrix $\hat{C}$ we have

$$\hat{C}^{-1} = \hat{O} \cdot \hat{A}^{-1} = \hat{O} \cdot \begin{pmatrix} \dfrac{1}{\alpha} & \dfrac{\tau_1 y}{\alpha} & \dfrac{-\tau_1 x}{\alpha} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Now all the necessary preparations are made for the calculation of partial differential operators such as gradient, divergence, curl and Laplacian in the curvilinear coordinate system.

3.1. The Gradient

Let f be a scalar function, expressed either in the Cartesian coordinates (x_1, x_2, x_3), or the curvilinear coordinates (s, x, y). From the chain rule, we have

$$
\begin{pmatrix} \dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} f = \begin{pmatrix} \dfrac{\partial x_1}{\partial s} & \dfrac{\partial x_2}{\partial s} & \dfrac{\partial x_3}{\partial s} \\ \dfrac{\partial x_1}{\partial x} & \dfrac{\partial x_2}{\partial x} & \dfrac{\partial x_3}{\partial x} \\ \dfrac{\partial x_1}{\partial y} & \dfrac{\partial x_2}{\partial y} & \dfrac{\partial x_3}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{pmatrix} f = \hat{C} \cdot \vec{\nabla}^{ct} f,
$$

where $\vec{\nabla}^{ct}$ is the Cartesian differential operator vector. Multiplying with $\hat{C}^{-1}$, we obtain the expression of this vector in terms of partial derivatives with respect to the particle optical coordinates as

$$
\begin{aligned}
\vec{\nabla}^{ct} f &= \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{pmatrix} f = \hat{C}^{-1} \cdot \begin{pmatrix} \dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} f \\
&= \hat{O} \cdot \begin{pmatrix} \dfrac{1}{\alpha}\left(\dfrac{\partial}{\partial s} + \tau_1 y \dfrac{\partial}{\partial x} - \tau_1 x \dfrac{\partial}{\partial y} \right) \\ \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} f,
\end{aligned}
$$

where (16) was used. We now define the vector differential operator $\vec{\nabla}^C$ as

$$
\vec{\nabla}^C f = \begin{pmatrix} \nabla_1^C \\ \nabla_2^C \\ \nabla_3^C \end{pmatrix} f = \begin{pmatrix} \nabla_s \\ \nabla_x \\ \nabla_y \end{pmatrix} f = \begin{pmatrix} \dfrac{1}{\alpha}\left(\dfrac{\partial}{\partial s} + \tau_1 y \dfrac{\partial}{\partial x} - \tau_1 x \dfrac{\partial}{\partial y} \right) \\ \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} f. \tag{17}
$$

Then we have

$$
\vec{\nabla}^{ct} f = \hat{O} \cdot \vec{\nabla}^C f \quad \text{and} \quad \vec{\nabla}^C f = \hat{O}^t \cdot \vec{\nabla}^{ct} f, \tag{18}
$$

or, for later use, in components,

$$
\nabla_k^{ct} f = \sum_{l=1}^{3} O_{kl} \nabla_l^C f \quad \text{and} \quad \nabla_k^C f = \sum_{l=1}^{3} O_{lk} \nabla_l^{ct} f. \tag{19}
$$

Let us consider a vector function $\vec{A}$; we express it in both Cartesian and curvilinear coordinates:

$$\vec{A} = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3 = A_s\vec{e}_s + A_x\vec{e}_x + A_y\vec{e}_y.$$

We denote the component vectors in Cartesian and curvilinear coordinates with $\vec{A}^{ct}$ and $\vec{A}^C$, respectively, and have

$$\vec{A}^{ct} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \quad \vec{A}^C = \begin{pmatrix} A_1^C \\ A_2^C \\ A_3^C \end{pmatrix} = \begin{pmatrix} A_s \\ A_x \\ A_y \end{pmatrix}.$$

Then, because of (3)

$$\begin{aligned} \vec{A} &= A_s\vec{e}_s + A_x\vec{e}_x + A_y\vec{e}_y = A_s(\hat{O}\vec{e}_1) + A_x(\hat{O}\vec{e}_2) + A_y(\hat{O}\vec{e}_3) \\ &= \hat{O} \cdot (A_s\vec{e}_1 + A_x\vec{e}_2 + A_y\vec{e}_3), \end{aligned}$$

and so we have

$$\vec{A}^{ct} = \hat{O} \cdot \vec{A}^C \text{ as well as } \vec{A}^C = \hat{O}^t \cdot \vec{A}^{ct} . \tag{20}$$

As a first step, we now want to determine the form of the gradient operator in curvilinear coordinates. In the Cartesian system, the gradient operation is

$$\text{grad}^{ct} f = \vec{\nabla}^{ct} f = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial f}{\partial x_3} \end{pmatrix} = \frac{\partial f}{\partial x_1}\vec{e}_1 + \frac{\partial f}{\partial x_2}\vec{e}_2 + \frac{\partial f}{\partial x_3}\vec{e}_3.$$

As in the situation with the more common coordinate systems, the gradient operator in the curvilinear system should determine the Cartesian gradient of a function, and then expresses it in terms of curvilinear coordinates; so we must have

$$\text{grad}^C f = \hat{O}^t \cdot \text{grad}^{ct} f.$$

We find that $\vec{\nabla}^C f$ defined in (18) satisfies this demand; so the gradient operation in the curvilinear system is

$$\text{grad}^C f = \vec{\nabla}^C f = \begin{pmatrix} \dfrac{1}{\alpha}\left(\dfrac{\partial}{\partial s} + \tau_1 y\dfrac{\partial}{\partial x} - \tau_1 x\dfrac{\partial}{\partial y}\right) f \\ \dfrac{\partial}{\partial x} f \\ \dfrac{\partial}{\partial y} f \end{pmatrix},$$

that is

$$\text{grad}^C f = \left[\frac{1}{\alpha}\left(\frac{\partial}{\partial s} + \tau_1 y\frac{\partial}{\partial x} - \tau_1 x\frac{\partial}{\partial y}\right) f\right]\vec{e}_s + \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y. \tag{21}$$

3.2. The Divergence

The divergence in the curvilinear system is calculated as follows. In the Cartesian system,

$$\text{div}\vec{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}.$$

Our goal is to express it in terms of the curvilinear system. For this purpose, we apply (19) to the three components $A_k^{ct} = \sum_m O_{km} A_m^C$,

$$\text{div}\vec{A} = (\vec{\nabla}^{ct} \cdot \vec{A}^{ct}) = \sum_k \nabla_k^{ct} \cdot A_k^{ct} = \sum_k \sum_l O_{kl} \nabla_l^C \left(\sum_m O_{km} A_m^C \right)$$

$$= \sum_k \sum_{l,m} O_{kl} O_{km} \nabla_l^C A_m^C + \sum_k \sum_{l,m} O_{kl} (\nabla_l^C O_{km}) A_m^C.$$

Since $\hat{O} = \hat{O}(s)$, we have

$$\nabla_l^C O_{km} = \delta_{ls} \nabla_s O_{km} = \delta_{ls} \frac{1}{\alpha} \frac{dO_{km}}{ds}. \tag{22}$$

Using this relationship, we obtain

$$\text{div}\vec{A} = \sum_{l,m} \sum_k O_{kl} O_{km} \nabla_l^C A_m^C + \sum_k \sum_{l,m} O_{kl} \delta_{ls} \frac{1}{\alpha} \frac{dO_{km}}{ds} A_m^C$$

$$= \sum_{l,m} [\hat{O}^t \hat{O}]_{lm} \nabla_l^C A_m^C + \sum_{l,m} \delta_{ls} \frac{1}{\alpha} [\hat{O}^t \frac{d\hat{O}}{ds}]_{lm} A_m^C$$

$$= \sum_{l,m} \delta_{lm} \nabla_l^C A_m^C + \sum_m \frac{1}{\alpha} T_{sm} A_m^C = \sum_m \nabla_m^C A_m^C + \frac{1}{\alpha}(-\tau_3 A_x + \tau_2 A_y)$$

$$= \frac{1}{\alpha} \left(\frac{\partial}{\partial s} + \tau_1 y \frac{\partial}{\partial x} - \tau_1 x \frac{\partial}{\partial y} \right) A_s + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{1}{\alpha}(-\tau_3 A_x + \tau_2 A_y).$$

Thus, the divergence expressed in the curvilinear system is obtained as

$$\text{div}\vec{A} = \frac{1}{\alpha} \left\{ \left(\frac{\partial}{\partial s} + \tau_1 y \frac{\partial}{\partial x} - \tau_1 x \frac{\partial}{\partial y} \right) A_s + \frac{\partial}{\partial x}(\alpha A_x) + \frac{\partial}{\partial y}(\alpha A_y) \right\}. \tag{23}$$

3.3. The Curl

The derivation of the curl in the curvilinear coordinates is a little more involved. In the Cartesian system,

$$\text{curl}^{ct}\vec{A} = \vec{\nabla}^{ct} \times \vec{A}^{ct} = [\vec{\nabla}^{ct} \times \vec{A}^{ct}]_1 \vec{e}_1 + [\vec{\nabla}^{ct} \times \vec{A}^{ct}]_2 \vec{e}_2 + [\vec{\nabla}^{ct} \times \vec{A}^{ct}]_3 \vec{e}_3$$

$$= \begin{pmatrix} \dfrac{\partial A_3}{\partial x_2} - \dfrac{\partial A_2}{\partial x_3} \\ \dfrac{\partial A_1}{\partial x_3} - \dfrac{\partial A_3}{\partial x_1} \\ \dfrac{\partial A_2}{\partial x_1} - \dfrac{\partial A_1}{\partial x_2} \end{pmatrix}.$$

The curl in the curvilinear coordinates, which we denote by $\text{curl}^C\vec{A}$ and which has the components

$$\text{curl}^C\vec{A} = \begin{pmatrix} \left[\text{curl}^C\vec{A}\right]_s \\ \left[\text{curl}^C\vec{A}\right]_x \\ \left[\text{curl}^C\vec{A}\right]_y \end{pmatrix},$$

has to satisfy the condition

$$\text{curl}^{ct}\vec{A} = \hat{O} \cdot \text{curl}^C\vec{A}. \tag{24}$$

First, let us express each component of $\text{curl}^{ct}\vec{A}$ in terms of the curvilinear system, again using the transformation rules for derivative components (19). We obtain

$$[\text{curl}^{ct}\vec{A}]_k = [\vec{\nabla}^{ct} \times \vec{A}]_k = \sum_{i,j} \epsilon_{ijk}\nabla_i^{ct}A_j^{ct} = \sum_{i,j}\sum_{l,m} \epsilon_{ijk}O_{il}\nabla_l^C(O_{jm}A_m^C)$$

$$= \sum_{l,m}\sum_{i,j} \epsilon_{ijk}O_{il}O_{jm}\nabla_l^C A_m^C + \sum_{l,m}\sum_{i,j} \epsilon_{ijk}O_{il}(\nabla_l^C O_{jm})A_m^C$$

$$= \sum_{l,m}\left(\sum_n \epsilon_{lmn}O_{kn}\right)\nabla_l^C A_m^C + \sum_{l,m}\sum_{i,j} \epsilon_{ijk}O_{il}\delta_{ls}\frac{1}{\alpha}\frac{dO_{jm}}{ds}A_m^C$$

$$= \sum_n O_{kn}\left(\sum_{l,m} \epsilon_{lmn}\nabla_l^C A_m^C\right) + \frac{1}{\alpha}\sum_m\left(\sum_{i,j} \epsilon_{ijk}O_{is}\frac{dO_{jm}}{ds}\right)A_m^C,$$

where (8) and (22) are used to obtain the third line. Making use of the fact that

$$\sum_{i,j} \epsilon_{ijk}O_{is}\frac{dO_{jm}}{ds} = [\vec{e}_s \times \frac{d\vec{e}_m^C}{ds}]_k$$

as well as the relationships in (12) which entail

$$\vec{e}_s \times \frac{d\vec{e}_s}{ds} = \tau_3 \vec{e}_y + \tau_2 \vec{e}_x, \ \vec{e}_s \times \frac{d\vec{e}_x}{ds} = -\tau_1 \vec{e}_x, \ \text{and} \ \vec{e}_s \times \frac{d\vec{e}_y}{ds} = -\tau_1 \vec{e}_y,$$

we obtain

$$[\text{curl}^{ct} \vec{A}]_k$$
$$= \sum_n O_{kn}[\vec{\nabla}^C \times \vec{A}^C]_n + \frac{1}{\alpha}\{(\tau_3 O_{k3} + \tau_2 O_{k2})A_s - \tau_1 O_{k2} A_x - \tau_1 O_{k3} A_y\}$$
$$= \sum_n O_{kn}[\vec{\nabla}^C \times \vec{A}^C]_n + \frac{1}{\alpha}\{O_{k2}(\tau_2 A_s - \tau_1 A_x) + O_{k3}(\tau_3 A_s - \tau_1 A_y)\}.$$

So

$$\text{curl}^{ct} \vec{A} = \hat{O} \cdot \left\{ \left(\begin{bmatrix} \vec{\nabla}^C \times \vec{A}^C \end{bmatrix}_1 \\ \begin{bmatrix} \vec{\nabla}^C \times \vec{A}^C \end{bmatrix}_2 \\ \begin{bmatrix} \vec{\nabla}^C \times \vec{A}^C \end{bmatrix}_3 \right) + \frac{1}{\alpha} \begin{pmatrix} 0 \\ \tau_2 A_s - \tau_1 A_x \\ \tau_3 A_s - \tau_1 A_y \end{pmatrix} \right\}.$$

Now transforming the curl vector to curvilinear coordinates according to (24), we have

$$\text{curl}^C \vec{A} = \hat{O}^t \cdot \text{curl}^{ct} \vec{A} = \left(\begin{bmatrix} \vec{\nabla}^C \times \vec{A}^C \end{bmatrix}_1 \\ \begin{bmatrix} \vec{\nabla}^C \times \vec{A}^C \end{bmatrix}_2 \\ \begin{bmatrix} \vec{\nabla}^C \times \vec{A}^C \end{bmatrix}_3 \right) + \frac{1}{\alpha} \begin{pmatrix} 0 \\ \tau_2 A_s - \tau_1 A_x \\ \tau_3 A_s - \tau_1 A_y \end{pmatrix}$$
$$= \begin{pmatrix} \nabla_x A_y - \nabla_y A_x \\ \nabla_y A_s - \nabla_s A_y + \frac{1}{\alpha}(\tau_2 A_s - \tau_1 A_x) \\ \nabla_s A_x - \nabla_x A_s + \frac{1}{\alpha}(\tau_3 A_s - \tau_1 A_y) \end{pmatrix}.$$

So altogether, expressed in terms of partial derivatives with respect to the curvilinear coordinates, we have

$$\text{curl}^C \vec{A} = \begin{pmatrix} \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \\ \frac{\partial A_s}{\partial y} - \frac{1}{\alpha}\left(\frac{\partial}{\partial s} + \tau_1 y \frac{\partial}{\partial x} - \tau_1 x \frac{\partial}{\partial y}\right) A_y + \frac{1}{\alpha}(\tau_2 A_s - \tau_1 A_x) \\ \frac{1}{\alpha}\left(\frac{\partial}{\partial s} + \tau_1 y \frac{\partial}{\partial x} - \tau_1 x \frac{\partial}{\partial y}\right) A_x - \frac{\partial A_s}{\partial x} + \frac{1}{\alpha}(\tau_3 A_s - \tau_1 A_y) \end{pmatrix}$$

$$= \frac{1}{\alpha} \begin{pmatrix} \alpha \frac{\partial A_y}{\partial x} - \alpha \frac{\partial A_x}{\partial y} \\ \frac{\partial}{\partial y}(\alpha A_s) - \tau_1 A_x - \left(\frac{\partial}{\partial s} + \tau_1 y \frac{\partial}{\partial x} - \tau_1 x \frac{\partial}{\partial y} \right) A_y \\ -\frac{\partial}{\partial x}(\alpha A_s) + \left(\frac{\partial}{\partial s} + \tau_1 y \frac{\partial}{\partial x} - \tau_1 x \frac{\partial}{\partial y} \right) A_x - \tau_1 A_y \end{pmatrix}. \tag{25}$$

3.4. The Laplacian

The Laplacian operator in the Cartesian system is

$$\triangle^{ct} f = (\vec{\nabla}^{ct} \cdot \vec{\nabla}^{ct}) f = (\vec{\nabla}^{ct})^t \cdot \vec{\nabla}^{ct} f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}.$$

In terms of the curvilinear system, utilizing (19), we have

$$\triangle^C f = (\vec{\nabla}^{ct})^t \cdot \vec{\nabla}^{ct} f = \sum_k \sum_{l,m} O_{kl} \nabla_l^C (O_{km} \nabla_m^C) f$$

$$= \sum_{l,m} \sum_k O_{kl} O_{km} \nabla_l^C (\nabla_m^C f) + \sum_k \sum_{l,m} O_{kl} (\nabla_l^C O_{km})(\nabla_m^C f)$$

$$= \sum_{l,m} \delta_{lm} \nabla_l^C (\nabla_m^C f) + \sum_k \sum_m \frac{1}{\alpha} O_{ks} \frac{dO_{km}}{ds} (\nabla_m^C f)$$

$$= \sum_m \left(\nabla_m^C + \frac{1}{\alpha} T_{sm} \right) (\nabla_m^C f)$$

$$= \nabla_s(\nabla_s f) + \left(\nabla_x - \frac{\tau_3}{\alpha} \right) (\nabla_x f) + \left(\nabla_y + \frac{\tau_2}{\alpha} \right) (\nabla_y f),$$

where (4) and (22) are used from the second to the third line. Thus, the Laplacian operator in the curvilinear system, expressed in partials of curvilinear coordinates, has the form

$$\triangle^C f = \frac{1}{\alpha} \left(\frac{\partial}{\partial s} + \tau_1 y \frac{\partial}{\partial x} - \tau_1 x \frac{\partial}{\partial y} \right) \left\{ \frac{1}{\alpha} \left(\frac{\partial f}{\partial s} + \tau_1 y \frac{\partial f}{\partial x} - \tau_1 x \frac{\partial f}{\partial y} \right) \right\}$$

$$+ \frac{1}{\alpha} \frac{\partial}{\partial x} \left(\alpha \frac{\partial f}{\partial x} \right) + \frac{1}{\alpha} \frac{\partial}{\partial y} \left(\alpha \frac{\partial f}{\partial y} \right). \tag{26}$$

3.5. The Velocity Vector

The final differential quantity we want to express in terms of curvilinear coordinates is the velocity vector $\vec{v}$. It is expressed as

$$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3 = v_s \vec{e}_s + v_x \vec{e}_x + v_y \vec{e}_y,$$

and similarly before, we define

$$\vec{v}^{ct} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \vec{v}^C = \begin{pmatrix} v_s \\ v_x \\ v_y \end{pmatrix},$$

and we have $\vec{v}^{ct} = \hat{O} \cdot \vec{v}^C$. To determine the velocity expressed in curvilinear coordinates, we differentiate the position vector $\vec{r}$ with respect to time t; from (13), we have

$$\vec{v}^{ct} = \frac{d\vec{r}}{dt} = \sum_{k=1}^{3} \frac{d}{dt} \{\vec{R} \cdot \vec{e}_k + xO_{k2} + yO_{k3}\} \vec{e}_k$$

$$= \sum_{k=1}^{3} \left\{ O_{k1}\dot{s} + O_{k2}\dot{x} + O_{k3}\dot{y} + \dot{s}\frac{dO_{k2}}{ds}x + \dot{s}\frac{dO_{k3}}{ds}y \right\} \vec{e}_k$$

$$= \hat{O} \cdot \begin{pmatrix} \dot{s} \\ \dot{x} \\ \dot{y} \end{pmatrix} + \dot{s}\frac{d\hat{O}}{ds} \cdot \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} = \hat{O} \cdot \left\{ \begin{pmatrix} \dot{s} \\ \dot{x} \\ \dot{y} \end{pmatrix} + \dot{s}\hat{O}^t \cdot \frac{d\hat{O}}{ds} \cdot \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} \right\}$$

$$= \hat{O} \cdot \left\{ \begin{pmatrix} \dot{s} \\ \dot{x} \\ \dot{y} \end{pmatrix} + \dot{s}\hat{T} \cdot \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} \right\} = \hat{O} \cdot \begin{pmatrix} \dot{s}\,(1 - \tau_3 x + \tau_2 y) \\ \dot{x} - \dot{s}\tau_1 y \\ \dot{y} + \dot{s}\tau_1 x \end{pmatrix},$$

where (1) is used from the first line to the second line. Comparing with the previous equation, we see that the velocity expressed in terms of curvilinear coordinates is given by

$$\vec{v}^C = \begin{pmatrix} v_s \\ v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \dot{s} \cdot (1 - \tau_3 x + \tau_2 y) \\ \dot{x} - \dot{s}\tau_1 y \\ \dot{y} + \dot{s}\tau_1 x \end{pmatrix} = \begin{pmatrix} \dot{s}\alpha \\ \dot{x} - \dot{s}\tau_1 y \\ \dot{y} + \dot{s}\tau_1 x \end{pmatrix}, \qquad (27)$$

where $\alpha = 1 - \tau_3 x + \tau_2 y$ as (14). For future reference, we note that because of the orthonormality of $\hat{O}$, we also have the relationships

$$v^2 = \vec{v}^{ct} \cdot \vec{v}^{ct} = \vec{v}^C \cdot \vec{v}^C \qquad (28)$$

$$\vec{v}^{ct} \cdot \vec{A}^{ct} = \vec{v}^C \cdot \vec{A}^C. \qquad (29)$$

4. Transformation of Fields

In this section, we study the transformation rules for fields derived from scalar and vector potentials. In particular, this covers the cases of gravitational as well as electromagnetic interaction. Let us assume that in the Cartesian

system, the fields $\vec{E}$ and $\vec{B}$ are expressed in terms of a scalar potential Φ and a vector potential $\vec{A}$ as

$$\vec{E}^{ct} = -\text{grad}^{ct}\Phi - \frac{\partial \vec{A}^{ct}}{\partial t} = -\vec{\nabla}^{ct}\Phi - \frac{\partial \vec{A}^{ct}}{\partial t}$$

$$\vec{B}^{ct} = \text{curl}^{ct}\vec{A} = \vec{\nabla}^{ct} \times \vec{A}^{ct}.$$

Choosing Φ and $\vec{A}$ as the electric and magnetic potentials, this covers the case of dynamics in electrodynamics. Choosing Φ as the gravitational potential and $\vec{A} = 0$, covers the gravitational case. Our goal here is to express these fields and the resulting force laws in terms of the curvilinear coordinates. We need to find $\vec{E}^C$ and $\vec{B}^C$ and their relationships to the potentials such that

$$\vec{E}^{ct} = \hat{O} \cdot \vec{E}^C, \text{ and } \vec{B}^{ct} = \hat{O} \cdot \vec{B}^C,$$

where

$$\vec{E}^{ct} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \ \vec{E}^C = \begin{pmatrix} E_s \\ E_x \\ E_y \end{pmatrix}, \text{ and } \vec{B}^{ct} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \ \vec{B}^C = \begin{pmatrix} B_s \\ B_x \\ B_y \end{pmatrix}.$$

Using the differential operators from the last section, we have

$$\vec{E}^C = \hat{O}^t \vec{E}^{ct} = \hat{O}^t \left(-\vec{\nabla}^{ct}\Phi - \frac{\partial \vec{A}^{ct}}{\partial t} \right) = -\hat{O}^t \vec{\nabla}^{ct}\Phi - \hat{O}^t \frac{\partial \vec{A}^{ct}}{\partial t}$$

$$= -\vec{\nabla}^C \Phi - \frac{\partial}{\partial t}(\hat{O}^t \vec{A}^{ct}) = -\vec{\nabla}^C \Phi - \frac{\partial \vec{A}^C}{\partial t} = - \begin{pmatrix} \nabla_s \\ \nabla_x \\ \nabla_y \end{pmatrix} \Phi - \frac{\partial}{\partial t} \begin{pmatrix} A_s \\ A_x \\ A_y \end{pmatrix},$$

or explicitly expressed in terms of partials of curvilinear coordinates:

$$\vec{E}^C = \begin{pmatrix} -\dfrac{1}{\alpha}\left(\dfrac{\partial}{\partial s} + \tau_1 y \dfrac{\partial}{\partial x} - \tau_1 x \dfrac{\partial}{\partial y} \right)\Phi - \dfrac{\partial A_s}{\partial t} \\ -\dfrac{\partial \Phi}{\partial x} - \dfrac{\partial A_x}{\partial t} \\ -\dfrac{\partial \Phi}{\partial y} - \dfrac{\partial A_y}{\partial t} \end{pmatrix}. \tag{30}$$

The field $\vec{B}^C$ can be determined in a straightforward way from the transformation rule for the curl (25), and we have

$$\vec{B}^C = \text{curl}^C \vec{A} = \begin{pmatrix} \nabla_x A_y - \nabla_y A_x \\ \nabla_y A_s - \nabla_s A_y + \dfrac{1}{\alpha}(\tau_2 A_s - \tau_1 A_x) \\ \nabla_s A_x - \nabla_x A_s + \dfrac{1}{\alpha}(\tau_3 A_s - \tau_1 A_y) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} \\ \dfrac{\partial A_s}{\partial y} - \dfrac{1}{\alpha}\left(\dfrac{\partial}{\partial s} + \tau_1 y \dfrac{\partial}{\partial x} - \tau_1 x \dfrac{\partial}{\partial y}\right) A_y + \dfrac{1}{\alpha}(\tau_2 A_s - \tau_1 A_x) \\ \dfrac{1}{\alpha}\left(\dfrac{\partial}{\partial s} + \tau_1 y \dfrac{\partial}{\partial x} - \tau_1 x \dfrac{\partial}{\partial y}\right) A_x - \dfrac{\partial A_s}{\partial x} + \dfrac{1}{\alpha}(\tau_3 A_s - \tau_1 A_y) \end{pmatrix}. \tag{31}$$

5. Relativistic Motion in Gravitation or Electromagnetic Fields

As an important application of the methods, we study the transformation of the relativistic equations of motion describing the dynamics in gravitational fields, which are derivable from a scalar potential, as well as electromagnetic fields derivable from scalar and vector potentials. We begin with the transformation for the Cartesian quantity

$$\vec{f}^{ct} = \vec{E}^{ct} + \vec{v}^{ct} \times \vec{B}^{ct},$$

which in the electromagnetic case corresponds to the Lorentz force per unit charge, and in the gravitational case, with $\vec{B}^{ct} = 0$, to the gravitational force per unit mass. We want to find $\vec{f}^{C}$ such that $\vec{f}^{ct} = \hat{O} \cdot \vec{f}^{C}$, where we write the components as

$$\vec{f}^{ct} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad \vec{f}^{C} = \begin{pmatrix} f_s \\ f_x \\ f_y \end{pmatrix}.$$

Because of the orthonormality of $\hat{O}$, we first observe

$$\vec{f}^{C} = \hat{O}^t \vec{f}^{ct} = \hat{O}^t(\vec{E}^{ct} + \vec{v}^{ct} \times \vec{B}^{ct}) = \hat{O}^t(\hat{O}\vec{E}^{C} + (\hat{O}\vec{v}^{C}) \times (\hat{O}\vec{B}^{C}))$$

$$= \vec{E}^{C} + \vec{v}^{C} \times \vec{B}^{C} = \begin{pmatrix} E_s + v_x B_y - v_y B_x \\ E_x + v_y B_s - v_s B_y \\ E_y + v_s B_x - v_x B_s \end{pmatrix},$$

where the invariance of the cross product under SO(3) transformations (6) was used. Expressed in terms of curvilinear coordinates, we have explicitly for the components of $\vec{f}^{C}$ the forms

$$f_s = E_s + v_x B_y - v_y B_x$$

$$= -\nabla_s \Phi - \frac{\partial A_s}{\partial t} + v_x \left\{ \nabla_s A_x - \nabla_x A_s + \frac{1}{\alpha}(\tau_3 A_s - \tau_1 A_y) \right\}$$

$$- v_y \left\{ \nabla_y A_s - \nabla_s A_y + \frac{1}{\alpha}(\tau_2 A_s - \tau_1 A_x) \right\}$$

$$= -\nabla_s \Phi - \frac{dA_s}{dt} + \dot{s}\frac{\partial A_s}{\partial s} + \dot{x}\frac{\partial A_s}{\partial x} + \dot{y}\frac{\partial A_s}{\partial y} + v_x \nabla_s A_x - v_x \frac{\partial A_s}{\partial x}$$

$$+ \frac{v_x}{\alpha}(\tau_3 A_s - \tau_1 A_y) - v_y \frac{\partial A_s}{\partial y} + v_y \nabla_s A_y - \frac{v_y}{\alpha}(\tau_2 A_s - \tau_1 A_x)$$

$$= -\frac{dA_s}{dt} - \nabla_s(\Phi - v_s A_s - v_x A_x - v_y A_y) - A_s \nabla_s v_s - A_x \nabla_s v_x$$

$$- A_y \nabla_s v_y - v_s \nabla_s A_s + \frac{v_s}{\alpha}\frac{\partial A_s}{\partial s} + (v_x + \dot{s}\tau_1 y)\frac{\partial A_s}{\partial x} + (v_y - \dot{s}\tau_1 x)\frac{\partial A_s}{\partial y}$$

$$- v_x \frac{\partial A_s}{\partial x} + \frac{v_x}{\alpha}(\tau_3 A_s - \tau_1 A_y) - v_y \frac{\partial A_s}{\partial y} - \frac{v_y}{\alpha}(\tau_2 A_s - \tau_1 A_x),$$

where (27) is used in the last step. Utilizing the relationships

$$\nabla_s v_s = \frac{1}{\alpha}\left(-\dot{s}\frac{d\tau_3}{ds}x + \dot{s}\frac{d\tau_2}{ds}y - \dot{s}\tau_1\tau_3 y - \dot{s}\tau_1\tau_2 x\right)$$

$$\nabla_s v_x = \frac{1}{\alpha}\left(-\dot{s}\frac{d\tau_1}{ds}y + \dot{s}\tau_1^2 x\right)$$

$$\nabla_s v_y = \frac{1}{\alpha}\left(\dot{s}\frac{d\tau_1}{ds}x + \dot{s}\tau_1^2 y\right)$$

$$\nabla_s A_s = \frac{1}{\alpha}\left(\frac{\partial A_s}{\partial s} + \tau_1 y\frac{\partial A_s}{\partial x} - \tau_1 x\frac{\partial A_s}{\partial y}\right),$$

we have

$$f_s = -\frac{dA_s}{dt} - \nabla_s(\Phi - \vec{v}^C \cdot \vec{A}^C)$$

$$+ \frac{A_s}{\alpha}\left(\dot{s}\frac{d\tau_3}{ds}x - \dot{s}\frac{d\tau_2}{ds}y + \dot{s}\tau_1\tau_3 y + \dot{s}\tau_1\tau_2 x + v_x\tau_3 - v_y\tau_2\right)$$

$$+ \frac{A_x}{\alpha}\left(\dot{s}\frac{d\tau_1}{ds}y - \dot{s}\tau_1^2 x + v_y\tau_1\right) + \frac{A_y}{\alpha}\left(-\dot{s}\frac{d\tau_1}{ds}x - \dot{s}\tau_1^2 y - v_x\tau_1\right)$$

$$= -\frac{dA_s}{dt} - \nabla_s(\Phi - \vec{v}^C \cdot \vec{A}^C)$$

$$+ \frac{1}{\alpha}\left\{A_s\frac{d}{dt}(\tau_3 x - \tau_2 y) + A_x\frac{d}{dt}(\tau_1 y) + A_y\frac{d}{dt}(-\tau_1 x)\right\}.$$

f_x and f_y are computed in the similar but more straightforward way to yield

$$f_x = -\frac{dA_x}{dt} - \nabla_x(\Phi - \vec{v}^C \cdot \vec{A}^C)$$

$$f_y = -\frac{dA_y}{dt} - \nabla_y(\Phi - \vec{v}^C \cdot \vec{A}^C).$$

Thus, the Lorentz force expressed in curvilinear coordinates is

$$\vec{f}^C = \vec{E}^C + \vec{v}^C \times \vec{B}^C = -\frac{d\vec{A}^C}{dt} - \vec{\nabla}^C(\Phi - \vec{v}^C \cdot \vec{A}^C)$$

$$+ \frac{1}{\alpha}\left\{A_s\frac{d}{dt}(\tau_3 x - \tau_2 y) + A_x\frac{d}{dt}(\tau_1 y) + A_y\frac{d}{dt}(-\tau_1 x)\right\} \cdot \vec{e}_s, \qquad (32)$$

knowing the Lorentz force is also the first step towards determining Newton's equations of a charged particle in the curvilinear system.

The momentum of the particle $\vec{p}$ is expressed as

$$\vec{p} = p_1\vec{e}_1 + p_2\vec{e}_2 + p_3\vec{e}_3 = p_s\vec{e}_s + p_x\vec{e}_x + p_y\vec{e}_y,$$

and similarly before, we define

$$\vec{p}^{ct} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad \vec{p}^C = \begin{pmatrix} p_s \\ p_x \\ p_y \end{pmatrix},$$

and have as before that $\vec{p}^{ct} = \hat{O} \cdot \vec{p}^C$. In the Cartesian system, a particle with the charge e obeys Newton's equation

$$\frac{d\vec{p}^{ct}}{dt} = e\vec{f}^{ct}. \tag{33}$$

Now we merely have to express the momentum derivatives in terms of curvilinear coordinates:

$$\frac{d\vec{p}^{ct}}{dt} = \frac{d}{dt}(\hat{O}\vec{p}^C) = \hat{O} \cdot \frac{d\vec{p}^C}{dt} + \frac{d\hat{O}}{dt} \cdot \vec{p}^C = \hat{O} \cdot \left(\frac{d\vec{p}^C}{dt} + \dot{s}\hat{O}^t\frac{d\hat{O}}{ds}\vec{p}^C \right)$$

$$= \hat{O} \cdot \left(\frac{d\vec{p}^C}{dt} + \dot{s}\hat{T}\vec{p}^C \right) = \hat{O} \cdot \left(\frac{d\vec{p}^C}{dt} + \dot{s}\vec{\tau} \times \vec{p}^C \right),$$

and then (33) can be written as $\hat{O} \cdot (d\vec{p}^C/dt + \dot{s}\vec{\tau} \times \vec{p}^C) = e\,\hat{O} \cdot \vec{f}^C$, or directly

$$\frac{d\vec{p}^C}{dt} + \dot{s}\vec{\tau} \times \vec{p}^C = e\,\vec{f}^C. \tag{34}$$

Explicitly we then have

$$\frac{d}{dt}\begin{pmatrix} p_s \\ p_x \\ p_y \end{pmatrix} + \dot{s} \cdot \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \times \begin{pmatrix} p_s \\ p_x \\ p_y \end{pmatrix} = e \cdot \left[\begin{pmatrix} E_s \\ E_x \\ E_y \end{pmatrix} + \begin{pmatrix} v_s \\ v_x \\ v_y \end{pmatrix} \times \begin{pmatrix} B_s \\ B_x \\ B_y \end{pmatrix} \right]$$

$$= e \cdot \left[-\frac{d}{dt}\begin{pmatrix} A_s \\ A_x \\ A_y \end{pmatrix} - \begin{pmatrix} \dfrac{1}{\alpha}\left(\dfrac{\partial}{\partial s} + \tau_1 y\dfrac{\partial}{\partial x} - \tau_1 x\dfrac{\partial}{\partial y} \right) \\ \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} \cdot (\Phi - \vec{v}^C \cdot \vec{A}^C) \right.$$

$$\left. + \frac{1}{\alpha}\left\{ A_s\frac{d}{dt}(\tau_3 x - \tau_2 y) + A_x\frac{d}{dt}(\tau_1 y) + A_y\frac{d}{dt}(-\tau_1 x) \right\} \cdot \vec{e}_s \right]. \tag{35}$$

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