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Verified Integration of Dynamics in the Solar System

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Abstract

The motion of objects in the solar system is studied with the use of verified integration methods. Providing rigorous bounds for the possible coordinates of objects whose initial coordinates are known to lie in a certain region, the methods are applied to the study of near-earth asteroids within an advanced relativistic NASA model of the solar system with the ultimate goal of assessing the possibility of collision with earth.

Because of the relatively large set of initial conditions compatible with measured orbit data, great care has to be taken to limit overestimation of the possible range of final coordinates. This is achieved using the approach of Taylor models. Within this framework, it is possible to control the so-called dependency problem as well as the wrapping effect commonly observed in verified integration. This approach yields accuracies that are sufficient to guarantee absence of collisions.

Examples of orbit integrations are given, showing that even relatively large domain boxes can be transported over extended time periods with a relative overestimation in the range of only around 10^{-5}.

Key words: Taylor model, high order multivariate polynomial, interval method, verified integration of ODEs, dynamics in the solar system
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1 Introduction

Orbit calculations of near-earth asteroids pose several difficult challenges. First of all, due to the fact that the impact of such an object is potentially serious, the problem requires as precise an answer as possible, for a relatively large range of initial conditions determined by measurement inaccuracies. Secondly, while the primary force to govern the equation of motion is the gravity of the sun, there are many other factors that influence the details of asteroid motion, making the resulting ODE quite complicated.

In the following, we outline an approach that allows the verification of the absence of the possibility of collisions for a given range of initial conditions of an asteroid based on verified ODE integration using the Taylor Model approach [1,2].

While interval-based methods can provide verification, it is commonly known that the overestimation due to the cancellation problem sometimes limits their application for practical problems. To solve a system of multidimensional ordinary differential equations

$$\frac{d}{dt}\vec{r}(t) = \vec{F}(\vec{r}(t), t) \tag{1}$$

over long time with verification, the so-called wrapping effect adds a further difficulty. This effect is caused by the inflation of the size of the geometric set at each time step containing the validated solution set. Details about interval methods can be found in [3–7] as well as many other sources.

We have been proposing a new method, the Taylor model approach [1,2], which combines high-order multivariate polynomial techniques and the interval technique for verification. Any $(n+1)$ times continuously partially differentiable function f in a domain D can be expressed by its nth order Taylor polynomial $P_{n,f}$ at the expansion point $\vec{x}_0 \in D$, and a remainder bounded by an interval $I_{n,f}$ via

$$\forall \vec{x} \in D, \quad f(\vec{x}) \in P_{n,f}(\vec{x} - \vec{x}_0) + I_{n,f}. \tag{2}$$

From Taylor's theorem, the width of the remainder interval $I_{n,f}$ can be chosen to scale with the domain size proportional to $(\vec{x} - \vec{x}_0)^{n+1}$. By choosing the size $|\vec{x} - \vec{x}_0|$ small and the order n sufficiently high, the size of the remainder interval $I_{n,f}$ can be kept very small in practice. The bulk of the functional dependency is kept in the polynomial part $P_{n,f}$ with point coefficients, and there is no interval arithmetic associated inflation that happens in the polynomial part. Thus, the interval related overestimation is rather optimally suppressed

with the Taylor model method [8]. The implementation of the method in the code COSY Infinity [2,9] supports binary operations and standard intrinsic functions, as well as the antiderivative operation which widens the applications of the method.

The method can be applied to verified global optimizations straightforwardly; refer to [8] for example. The Taylor model based algorithm for a verified ODE integrator carries the functional dependency of the solutions on the initial conditions in the Taylor polynomial part. Thus it can optimally eliminate the wrapping effect, making possible not only to integrate over long time but also to deal with much larger domains of initial conditions. The algorithm can be naturally extended to be a verified solver of differential algebraic equations (DAEs), when combined with methods for verified solutions of constraint conditions over extended domains [10]. The high order differential algebraic method for multidimensional systems, which is the backbone of the Taylor model method, presents an algorithm for an efficient prescribed path control.

The following two sections provide a brief summary of the aspects of the Taylor model methods as well as their use in the setting of verified integration of ODEs necessary for the further developments. Subsequently, we outline the tools necessary for verified integration in the solar system, and study the behavior of the verified integrator and in particular the overestimation of the range of initial conditions for a typical asteroid problem.

2 Taylor models

Following the reasoning of the last section, a pair $(P_{n,f}, I_{n,f})$ satisfying (2) is called a Taylor model of f and denoted by

$$T_{n,f} = (P_{n,f}, I_{n,f}). \tag{3}$$

Taylor models of complicated functions f can be determined by carrying Taylor model arithmetic through binary operations and intrinsic functions which compose the function f sequentially. Suppose we have Taylor models for g and h as $T_{n,g} = (P_{n,g}, I_{n,g})$ and $T_{n,h} = (P_{n,h}, I_{n,h})$. Then Taylor models of the sum and difference of g and h can be obtained as

$$T_{n,g} \pm T_{n,h} = (P_{n,g} \pm P_{n,h}, I_{n,g} \pm I_{n,h}). \tag{4}$$

The Taylor model for the product of g and h can be obtained as

$$T_{n,g} \cdot T_{n,h} = (P_{n,g \cdot h}, I_{n,g \cdot h}), \tag{5}$$

where $P_{n,g} \cdot P_{n,h} = P_{n,g\cdot h} + P_e$ with $P_{n,g\cdot h}$ being the nth order polynomial of the result of the left hand side, and P_e the part of the product polynomial with order from $n+1$ to $2n$, and

$$I_{n,g\cdot h} = B(P_e) + B(P_{n,g}) \cdot I_{n,h} + B(P_{n,h}) \cdot I_{n,g} + I_{n,g} \cdot I_{n,h}, \tag{6}$$

where B denotes the bounds of the argument over the domain D. Refer to [1,2] for the details on intrinsic functions including a multiplicative inverse of Taylor models.

The other important operation is the antiderivation operation, which is naturally available as an intrinsic function on the space of Taylor models. It has the form

$$\partial_i^{-1}(P_{n,f}, I_{n,f}) = \left(\int P_{n-1,f} dx_i \,,\; I_{n,\partial_i^{-1} f} \right), \tag{7}$$

where $I_{n,\partial_i^{-1} f} = (B(P_{n,f} - P_{n-1,f}) + I_{n,f}) \cdot B(x_i)$.

3 Verified integration based on Taylor model methods

In this section we review the key elements of the verified integration of ODEs through Taylor models; for details refer to [2,11]. Compared to other verified ODE integrators, the method has the following characteristics:

- Because the solution set is described as a Taylor model describing the dependence on initial conditions, in subsequent operations the dependency problem based on the repeated use of the solution set, which is the source of the wrapping effect, can be avoided.
- For nonlinear ODEs, the sharpness of the inclusion of the true solution set for each time step scales with order $(n+1)$ in the original domain width, and not order 2 as in the case of inclusion in interval boxes, polygons, or zonotopes used by other approaches. This allows larger domains to be transported without significant overestimation.
- The method naturally combines both first inclusion, commonly known as Algorithm I, as well as verification inclusion, commonly known as Algorithm II, into one high-order step.
- Because of the use of the intrinsic antiderivation operation, there is no need for explicit use of any error bounding formulas based on higher derivatives; rather, the error verification happens automatically as part of the integration step.

More specifically, the algorithm consists of the following basic steps. First, as is commonly done, we re-write the ODE (1) in the form of an integral equation

$$\vec{r}(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{F}(\vec{r}(t'), t')\, dt', \tag{8}$$

and with the introduction of the operator on the space of continuous functions from $[t_0, t_1]$ to R^v given by $A(\vec{f})(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{F}(\vec{f}(t'), t')\, dt'$, we arrive at a fixed point problem $\vec{r} = A(\vec{r})$. We apply Schauder's fixed point theorem to obtain a Taylor model for the flow $\mathcal{M}(\vec{r}_0, t)$ of the ODE (1). An important aspect is that the quantity $\vec{r}_0$ in the fixed point problem can be either a point expressed by real numbers, or an interval box, or Taylor models depending on the initial conditions, or a combination of the above.

To apply Schauder's theorem, we follow the following steps.

- Determine a family Y of subsets of X, the Schauder Candidate Sets. Each set in Y should be compact and convex, it should be contained in a suitable Taylor model, and its image under A should be in Y. The mathematical details of the choice of the candidate sets are described in [11].
- Using differential algebraic methods on Taylor models, we determine an initial set $M_0 \in Y$ satisfying the inclusion property $A(M_0) \subset M_0$. Then all requirements of Schauder's theorem are satisfied, and M_0 contains a solution. Note that different from other verified integrators, in this procedure we simultaneously determine the so-called a priori (initial) inclusion as well as the high order inclusion necessary to execute the step.
- Iteratively generate the sequence $M_i = A(M_{i-1})$ for $i = 1, 2, \ldots$ Each M_i also satisfies $A(M_i) \subset M_i$, and we have $M_1 \supset M_2 \supset \ldots$. We continue the iteration until the size stabilizes sufficiently.

For computational purposes, the only requirement for Schauder's theorem is to find a Taylor model $\vec{P} + \vec{I}$ such that

$$A(\vec{P} + \vec{I}) \subset \vec{P} + \vec{I}, \tag{9}$$

which can be checked computationally. The task depends on finding a suitable choice for $\vec{P}$ and $\vec{I}$, and furthermore it is desirable to have $\vec{I}$ as tight as possible. By choosing a polynomial $\vec{P}$ that is already close to the true solution of the ODE in the following way, we can get the desired answer.

The nth order expansion $\mathcal{M}_n(\vec{r}_0, t)$ of the flow including time dependence can be obtained in conventional Differential Algebraic scheme [12]. Choose an

initial function $\mathcal{M}_n^{(0)}$ to be the identity function $\mathcal{I}$, then iteratively determine

$$\mathcal{M}_n^{(k+1)} =_n A(\mathcal{M}_n^{(k)}). \tag{10}$$

This process converges to the exact result $\mathcal{M}_n$ in $n+1$ steps.

Now try to find $\vec{I}^*$ such that

$$\mathcal{M}_n + \vec{I}^* \subset A(\mathcal{M}_n + \vec{I}^*), \tag{11}$$

the Schauder inclusion requirement. The suitable choice for $\vec{I}^*$ requires some trial and error. $\vec{I}^{(0)}$ obtained as $A(\mathcal{M}_n + [\vec{0}, \vec{0}]) = \mathcal{M}_n + \vec{I}^{(0)}$ can serve as a good estimate for a lower bound for $\vec{I}^*$ because $\vec{I}^* \supset \vec{I}^{(0)}$.Now iteratively try $\vec{I}^{(k)} = q^k \cdot \vec{I}^{(0)}$ with $q > 1$, until a computational inclusion is found, i.e. $A(\mathcal{M}_n + \vec{I}^{(k)}) \subset \mathcal{M}_n + \vec{I}^{(k)}$. In practice, a computational inclusion can be found in a few iterations with q between 1 and 2. Should this however not be the case, it can almost always be forced by slightly reducing the integration step size, which reduces the main contribution of the remainder bound of the mapped set.

Once a computational inclusion has been determined, the solution of the ODE is known to be contained in the Taylor model $\mathcal{M}_n + \vec{I}^{(k)}$. Set $\vec{I}_{(1)} = \vec{I}^{(k)}$. Since the solution is a fixed point of A, it is even contained in $A^k(\mathcal{M}_n + \vec{I}_{(1)})\ \forall k$. Furthermore, the iterates of A are shrinking in size, i.e. $A^k(\mathcal{M}_n + \vec{I}_{(1)}) \subset A^{k-1}(\mathcal{M}_n + \vec{I}_{(1)})\ \forall k$. So the width of the remainder bound of the flow can be decreased by iteratively determining $\mathcal{M}_n + \vec{I}_{(k)} = A(\mathcal{M}_n + \vec{I}_{(k-1)})$, until no further significant decrease in size is achieved. As a result, $\mathcal{M}_n + \vec{I}_{(k)}$ is the desired sharp inclusion of the flow of the ODE.

To integrate over time, we apply the procedure at each time step. Automatic step size controllers are utilized that assure that if the solution at a time step is not favorable, the step size is decreased, on the other hand, if a time step can proceed without much growth of errors, the next step size is increased.

4 Dynamics in the solar system

In order to study the motion in the solar system with an accuracy that is sufficient to predict collisions, it is necessary to include various relativistic corrections to the well-known Newtonian forces based on the Kepler force law. Specifically, the full equation of motion in the solar system including the relevant relativistic effects is given by [13]

$$\ddot{\mathbf{r}} = G\sum_i \frac{m_i(\mathbf{r}_i - \mathbf{r})}{r_i^3}\left\{1 - \frac{2(\beta+\gamma)}{c^2}G\sum_j \frac{m_j}{r_j} - \frac{2\beta-1}{c^2}G\sum_{j\neq i}\frac{m_j}{r_{ij}} + \frac{\gamma|\dot{\mathbf{r}}|^2}{c^2} + \frac{(1+\gamma)|\dot{\mathbf{r}}_i|^2}{c^2} - \frac{2(1+\gamma)}{c^2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}_i - \frac{3}{2c^2}\left[\frac{(\mathbf{r}-\mathbf{r}_i)\cdot\dot{\mathbf{r}}_i}{r_i}\right]^2 + \frac{1}{2c^2}(\mathbf{r}_i - \mathbf{r})\cdot\ddot{\mathbf{r}}_i\right\} + G\sum_i \frac{m_i}{c^2 r_i}\left\{\frac{3+4\gamma}{2}\ddot{\mathbf{r}}_i + \frac{\{[\mathbf{r}-\mathbf{r}_i]\cdot[(2+2\gamma)\dot{\mathbf{r}} - (1+2\gamma)\dot{\mathbf{r}}_i]\}(\dot{\mathbf{r}}-\dot{\mathbf{r}}_i)}{r_i^2}\right\}, \tag{12}$$

where $\mathbf{r}$ is the point of interest, G is the gravitational constant; m_i and $\mathbf{r}_i$ are the mass and the solar-system barycentric position of body i, including the sun, the planets, the moon and the five major asteroids, respectively; $r_i = |\mathbf{r}_i - \mathbf{r}|$; and β and γ are the parametrized post-Newtonian parameters measuring the nonlinearity in superposition of gravity, and space curvature produced by unit rest mass [13].

Based on the rather accurately known positions $\mathbf{r}_i$ of the celestial bodies, which are the result of measurements over long periods of time and fitting of orbits through large numbers of data points and are thus orders of magnitude more precisely known as those of asteroids, the equation of motion as written is believed to allow prediction of orbit coordinates with an accuracy in the range of a few kilometers for integration over time spans of around one century [13]. Based on this information, an interval bound for the error of the right hand side can be estimated and folded into the computation as an additional term to be added to the remainder bound of the respective Taylor model of the right hand side. For the practical calculations below, it turns out that this additional term proves rather inconsequential. However, it needs to be stressed that within the framework of conventional reasoning of the community in celestial mechanics, it is sometimes difficult to find error bounds at a level of rigor needed for interval-based arguments.

To study potentially hazardous near-earth asteroids [14], we apply the verified Taylor model based ODE integrator to the equation of motion (12). Due to the limitation of the means of measurements, i.e. the observation of an asteroid at the observatories situated on the earth or near earth, the inaccuracies of the position and velocity of the asteroid are large, especially in the direction from the earth toward the asteroid. This means that the ranges of initial conditions for the ODE may not only be quite large (in the range of 10^{-6} relative), but also have different magnitudes for different variables. These errors have to be accounted for, as they represent the main source for the large ranges of possible later positions that are typical for predictions of asteroid dynamics. Overall, it is these large ranges that make the system very susceptible to wrapping effect problems as well as the common problem of overestimation.

As an example to assess the performance of our verified integration scheme, we study the dynamics of ranges of initial conditions for the near-earth asteroid 1997 XF11. The asteroid travels from outside of the Mars orbit to inside of the earth orbit very close to the Venus orbit, having a period of about 1.7 earth years. The asteroid 1997 XF11 will make a close approach to the earth on October 26th in 2028. The recent analyses predict an approach distance of 0.00636 AU (951,000 km), which is about 2.5 times farther than the moon, replacing an earlier reported extremely close passage of only a few earth diameters [14]. As is the case for most asteroids, compared to the planets the ranges of initial conditions compatible with measurement data are very large, and the challenge lies in the transport of these large ranges without introducing significant overestimation. In the following section we see that this can actually be accomplished, and hence it is possible to perform verified calculations that allow the exclusion of collisions, as was found to be the case for 1997 XF11.

5 Results of verified calculations

We now present some results of verified calculations based on orbit data of the 1997 XF11 asteroid, and analyze the performance of the verified integrator being used. Denoting by x, y, and z the three Cartesian coordinates of the variables $\vec{r}$ and by $\dot{x}$, $\dot{y}$, and $\dot{z}$ the three Cartesian velocities $\vec{r}$ in eq. (12), we study the dynamics over a domain box with six domain widths of d for the initial conditions. For the sake of better computational performance, we scale the variables around the initial center values so that the relative domain in the new scaled variables is $[-1, 1]$ in each dimension; $\bar{x} = x_c + (x - x_c)/(d/2)$. Thus, the scaled variables as well as the time initially have the following form with the center values as the reference point:

```
VAR  REFERENCE POINT        DOMAIN INTERVAL
 1 -1.772666585569338  [-2.772666585569338 ,-.7726665855693382]
 2  0.237218728844051  [-.7627812711559487 , 1.237218728844051]
 3  0.167950671786751  [-.8320493282132488 , 1.167950671786751]
 4  -.580218647359020  [-1.580218647359021 ,0.4197813526409793]
 5  -.013597900139683  [-1.013597900139683 ,0.9864020998603170]
 6  -.201599807982744  [-1.201599807982745 ,0.7984001920172554]
 7  0.000000000000000  [0.0000000000000000 ,0.0000000000000000]
```

The latter one is initially zero and is only populated in the execution of each integration step through iteration of eq. (10); after each step, the value of the time coordinate is inserted, which again makes the dependence disappear. The first six lines contain the positions (in astronomical units) and velocities (in astronomical units per year times 2π) of the starting point, as well as the width

of the domain interval for the respective quantities. All domain intervals have size ± 1, and the linear parts of the polynomial parts are adjusted to assure that the initial range of possible values for the coordinates is spanned over the domain box.

At the beginning of six dimensional integration to order 10, the Taylor model for the first component of the motion, the Taylor model for the x coordinate of the asteroid, as printed by COSY, has the form

```
 I COEFFICIENT       ORDER EXPONENTS
 1 -1.772666585569       0  000000
 2 0.5000000000000E-06   1  100000
--------------------------------
REMAINDER BOUND INTERVAL [-.4440894318947E-15,0.4440894318947E-15]
```

The first line denotes the center x position in astronomical units. The second line shows the initial linear part of the Taylor model, which is responsible for spanning the initial box of width 10^{-6} astronomical units over the domain box given above, which amounts to the width of possible x values determined by measurement errors. The last line contains the initial remainder bound interval of the Taylor model, which is so far determined only to account for the numerical inaccuracy in the representation of the true numbers by floating point numbers. This Taylor model represents the starting value for the first coordinate of the quantity $\vec{r}_0 = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$ in eq. (8).

It is worth stressing that in a conventional verified integrator, this first component x would be represented merely by an interval of width 10^{-6} centered around the point -1.772666585569338. Such a relatively large width has a tendency to lead to wrapping effect problems very quickly, since in each step a new distorted box with width in the order of 10^{-6} has to be re-packaged by another box. On the other hand, describing the structure of the range of initial conditions and later intermediate values for the dynamical variables by Taylor models avoid the need for re-packaging; the deformed shape is merely described as the range of Taylor polynomials in the initial conditions.

As an additional advantage, the Taylor model approach can avoid most of the overestimation due to the intervals of width 10^{-6} that would occur in the mere interval evaluation of the right hand side of the ODE (12); details of this avoidance of the dependency problem can be found in [8].

After a time of approximately 1.76 years, the Taylor model of the x component as determined by the verified integrator now has the form

```
 I COEFFICIENT       ORDER EXPONENTS                                    
 1 -1.729979938862       0  000000   15 -.1541518999234E-10  2  010100
 2 0.2669995836588E-05   1  100000   16 -.2780687661956E-11  2  001100
```

```
 3 -.1555595145519E-05  1  010000   17 0.2524887623418E-10  2  000200
 4 -.1999596199058E-06  1  001000   18 0.9405928517122E-13  2  100010
 5 0.3983192063746E-05  1  000100   19 -.1304168052733E-12  2  010010
 6 0.1936219598678E-07  1  000010   20 -.6920679984791E-13  2  001010
 7 0.1382206523852E-05  1  000001   21 0.2322553261934E-12  2  000110
 8 0.8813122946243E-11  2  200000   22 0.1032777207550E-10  2  100001
 9 -.8482323930185E-11  2  110000   23 -.5321701018616E-11  2  010001
10 0.3598785862399E-13  2  020000   24 -.9887392833456E-12  2  001001
11 -.1763671873927E-11  2  101000   25 0.1871537734475E-10  2  000101
12 0.7351992613921E-12  2  011000   26 -.2305603850554E-12  2  000020
13 -.2277507757098E-12  2  002000   27 0.1595760887145E-12  2  000011
14 0.2976593389359E-10  2  100100   28 0.1571674237059E-11  2  000002
-------------------------------  -------------------------------
REMAINDER BOUND INTERVAL [-.8670915505993E-11,0.8670738857369E-11]
```

The polynomial now has more terms than before due to the fact that now also the values of the other dynamical coordinates determine the final position of the object. Terms of first and second order appear, the exact meaning of which is described by the exponents of the six dynamical variables and time listed in the end of each line. The zeroth order part of the polynomial of the Taylor model part is now different from before as a result of the integration, indicating that the center of the box containing the range of allowed values has moved. Furthermore, there are now linear dependencies on not only the first variable, but also on the other five variables, indicating that the original box which was aligned with the axes has now been rotated and stretched.

In addition, there are terms of second order appearing, indicating that the actual occupied range is now not a parallelepiped anymore, but rather exhibits some nonlinear deformation. The magnitude of the largest second order terms is in the range of 10^{-10}, above the magnitude of the now larger remainder term. Polynomial terms below a cutoff size of 10^{-17} are lumped into the remainder term to keep the polynomial part simple and of limited size, which is advantageous since the Taylor arithmetic employed here has full sparsity support [15].

It is worth pointing out that had the second order dependence not been kept in the Taylor model calculation, the respective effects would have had to be absorbed into the remainder term, which would lead to an increase of its size by at least one order of magnitude. As a further comparison to other methods, typically the current box is re-packed into a rotated box with right angles, which introduces overestimations of perhaps a few percent of the linear width, and hence at least about four orders of magnitude more than our remainder bound.

As a last example, we show the result of the calculation at a time of about 3.47 years.

I	COEFFICIENT	ORDER	EXPONENTS				
1	-1.754219040644	0	000000	21	0.1021420286277E-11	2	000110
2	0.4360583811370E-05	1	100000	22	0.4278281656362E-10	2	100001
3	-.2882954051752E-05	1	010000	23	-.2749808650290E-10	2	010001
4	-.3551964030358E-06	1	001000	24	-.4016050278429E-11	2	001001
5	0.7093530502070E-05	1	000100	25	0.7839892079064E-10	2	000101
6	0.3458088648861E-07	1	000010	26	-.4036477375995E-12	2	000020
7	0.2461689085201E-05	1	000001	27	0.4821388881932E-12	2	000011
8	0.3470484146983E-10	2	200000	28	0.1057845692876E-10	2	000002
9	-.4321544674158E-10	2	110000	29	0.1513248269030E-14	3	210000
10	0.1042951174372E-10	2	020000	30	-.5550683440348E-14	3	120000
11	-.6646933899734E-11	2	101000	31	0.1379571548696E-13	3	200100
12	0.3930131605285E-11	2	011000	32	-.2376430348886E-13	3	110100
13	-.2239667322416E-12	2	002000	33	-.3725055090868E-13	3	020100
14	0.1232576364442E-09	2	100100	34	-.1581518925090E-14	3	100200
15	-.7941425674937E-10	2	010100	35	-.1339611425873E-14	3	010200
16	-.1141698404321E-10	2	001100	36	0.1405540257892E-13	3	000300
17	0.1098864604671E-09	2	000200	37	0.2269751185586E-14	3	110001
18	0.4810278850719E-12	2	100010	38	0.3157356498990E-14	3	100101
19	-.4745742391081E-12	2	010010	39	0.3828688421870E-14	3	000201
20	-.1471614587873E-12	2	001010				

REMAINDER BOUND INTERVAL [-.2820192825328E-10,0.2820181117737E-10]

Beginning with inspecting the polynomial dependence, the constant part again shows the relative position of the center of the original domain box, and the linear parts provide information about the orientation of the box, which by now has rotated several times. There are now more higher order terms, indicating that the transformation from initial to final coordinates is becoming more and more nonlinear.

The approximate size of the predicted domain region, given largely by the linear terms, is in the range of about 10^{-5}. On the other hand, the remainder bound, which is a measure of the overestimation of the method, is in the range of about 10^{-11}. This means that the relative overestimation of the size of the resulting domain is less of magnitude 10^{-5}, illustrating the far-reaching avoidance of the wrapping effect problem.

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