Apophis Encounter 2029: Differential Algebra and Taylor Model Approaches

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INTRODUCTION

Orbit uncertainty propagation usually requires linearized propagation models [1–3] or full nonlinear Monte Carlo simulations [4]. The linear assumption simplifies the problem, but fails to characterize trajectory statistics when the system is highly nonlinear or when mapped over a long time period. On the other hand, Monte Carlo simulations provide true trajectory statistics, but are computationally intensive. The tools currently used for the robust detection and prediction of planetary encounters and potential impacts with Near Earth Objects (NEO) are based on these two techinques [5–7], and thus suffer the same limitations. A different approach to orbit uncertainty propagation has been discussed by Junkins et al. [8,9], in which the effect of the coordinate system on the propagated statistics is thoroughly analyzed; however, the propagation method was based on the linear assumption and the system nonlinearity was not incorporated in the mapping. An alternate way to analyze trajectory statistics by incorporating higher-order Taylor series terms that describe localized nonlinear motion was proposed by Park and Scheeres [10]. Their appoach is based on proving the integral invariance of the probability density function via solutions of the Fokker–Planck equations for diffusionless systems, and by combining this result with the nonlinear state propagation to derive an analytic representation of the nonlinear uncertainty propagation. This method is limited to systems derived from a single potential.

Differential algebraic (DA) techniques are proposed as a valuable tool to develop alternative approaches to tackle the previous tasks. Differential algebra provides the tools to compute the derivatives of functions within a computer environment [11–13]. More specifically, by substituting the classical implementation of real algebra with the implementation of a new algebra of Taylor polynomials, any function f of v variables is expanded into its Taylor series up to an arbitrary order n. This has an important consequence when the numerical integration of an ordinary differential equation (ODE) is performed by means of an arbitrary integration scheme. Any explicit integration scheme is based on algebraic operations, involving the evaluations of the ODE right hand side at several integration points. Therefore, carrying out all the evaluation in the DA framework allows differential algebra to compute the arbitrary order expansion of the flow of a general ODE initial value problem. The availability of such high order expansions is exploited to improve the Monte Carlo simulation approach by replacing thousands of integrations with evaluations of the high order expansion of the flow, reducing the computational time significantly. This algorithm is applied to the prediction of Apophis planetary encounter and potential impact taking into account its measurement uncertainties. The availability of high order maps in space and time and intrinsic tools for their inversion are then exploited in an algorithm that reduces the computation of the minimum distance from the Earth of all the asteroids belonging to the initial uncertainties cloud to the simple evaluation of polynomials.

The second part of the paper deals with the rigorous study of Apophis close encounter by means of Taylor models (TM). The first methods introduced to perform validated integrations of dynamical systems [14] were based on the use of interval analysis, originally formalized by Moore in 1966 [15]. The main idea beneath this theory is the substitution of real numbers with intervals of real numbers; consequently, interval arithmetic and analysis are developed in order to operate on the set of interval numbers in place of the real numbers. This turned out to be an effective tool for error and uncertainty propagation, as both the numerical errors and the uncertainties can be bounded by intervals and rigorously propagated using interval analysis. Several codes, which implement a variety of features to improve the performances of naive interval algebra based on interval analysis, have been implemented for the rigorous integration of ODE [16–19]. Unfortunately, all of them produce an unacceptable overestimation of the solution when applied to Solar system dynamics [20]. The reasons for such an overestimation are the so-called dependency problem and wrapping effect [21].

Taylor model integrators have shown to be a powerful tool for the validated integration of ordinary differential equations as they successflully address both these problems [22,23]. The Taylor Model approach combines high-order multivariate polynomial techniques and the interval technique for verification. In particular, it represents a multivariate functional dependence *f* by a high order multivariate Taylor polynomial *P* and the remainder bound interval *I*. The *n*-th order Taylor polynomial *P* captures the bulk of functional dependency. Because the manipulation of those polynomials can be performed by operations on the coefficients where the minor errors due to their floating point nature are moved into the remainder bound, the major source of interval overestimation is removed. Thus, the overestimation only occurs in the remainder bound, the size of which scales with order *n* of the width of the domain. When applied to the verified integration of ODE, the relationships between the state vector at a generic time *t* and the initial conditions are expressed in terms of a Taylor model (*P*, *I*) and a tight enclosure for the action of the differential equations on an extended region is then provided. The TM-based integrator implemented in COSY VI [22] has been already succesfully exploited for the long-term rigourous integration of asteroids motion [24]. In this paper, an improved version of the integrator that exploits dynamic domain decomposition, automatic step size control, and a flow operator based on Lie derivatives [25], is applied to the more challenging task of Apophis close approach rigorous integration.

The paper is organized as follows. The models developed to describe Apophis dynamics and to evaluate the planetary ephemerides are illustrated first. The improved version of the Monte Carlo simulation together with the minimum close encounter distance algorithm are then presented. The last part of the paper is devoted to the presentation of the rigorous integration of Apophis flyby. Conclusions end the paper.

MODELS

The set of ODE that describes Apophis motion and the planetary ephemeris functions implemented to study the close encounter are presented. In particular, two different dynamical models are used to deal with the heliocentric or the geocentric phase of the trajectory. Furthermore, two ephemeris models are implemented to deal with DA and TM evaluations.

Dynamics

The dynamical model that describes the heliocentric phase of Apophis trajectory is written in the J2000.0 coordinates, in which the *x*-axis is aligned with the mean equinox at the given reference epoch and the *z*-axis is orthogonal to the ecliptic plane. As an accurate modeling of the dynamics is required to perform NEO impact analysis, various relativistic corrections to the well-known Newtonian forces are implemented. Specifically, the full equation of motion in the Solar system including the relevant relativistic effects is given by

$$\begin{aligned}\ddot{\boldsymbol{r}} = G\sum_i \frac{m_i(\boldsymbol{r}_i - \boldsymbol{r})}{r_i^3}\Bigg\{1 - \frac{2(\beta+\gamma)}{c^2}G\sum_j \frac{m_j}{r_j} - \frac{2\beta+1}{c^2}G\sum_{j\neq i}\frac{m_j}{r_{ij}} + \gamma(v/c)^2 \\ +(1+\gamma)(v/c)^2 - \frac{2(1+\gamma)}{c^2}\cdot\boldsymbol{v}_i - \frac{3}{2c^2}\left[\frac{(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\boldsymbol{r}_i}{r_i}\right]^2 + \frac{1}{2c^2}(\boldsymbol{r}-\boldsymbol{r}_i)\cdot\ddot{\boldsymbol{r}}_i\Bigg\} \\ +G\sum_i \frac{m}{c^2 r_i}\left\{\frac{3+4\gamma}{2}\ddot{\boldsymbol{r}} + \frac{[\boldsymbol{r}-\boldsymbol{r}_i]\cdot[(2+2\gamma)\dot{\boldsymbol{r}} - (1+2\gamma)\dot{\boldsymbol{r}}_i](\boldsymbol{r}-\boldsymbol{r}_i)}{r_i^2}\right\}\end{aligned} \tag{1}$$

where **r** is the point of interest and v its speed, G = 6.67529x10^{-11} m^3/(kg s^2) is the gravitational constant; m_i and r_i are the mass and the solar-system barycentric position of body or system i; $\mathbf{r}_i = |\mathbf{r}_i - \mathbf{r}|$; β and γ are the parameterized post-Newtonian parameters measuring the nonlinearity in superposition of gravity and the space curvature produced by unit rest mass [26]. In (1) it is supposed that the object we are integrating is affected by the gravitational attraction of n bodies, but has no gravitational effect on them; i.e., we are adopting the restricted (n+1)-body problem approximation. In our particular case n includes the Sun, planets, the Moon, Ceres, Pallas, and Vesta. Note that for planets given with moons, with the exception of the Earth, the system center of mass is considered. To improve the integration accuracy the dynamics are scaled by Earth semi-major axis and Sun gravitational parameter (i.e., $a_E = 1$ and $\mu_S = Gm_S = 1$).

When the asteroid approaches the Earth, a different set of ODE is integrated to improve the integration accuracy. These ODE are written in a non-inertial reference frame centered on the Earth center of mass, with the x-axis in the mean equinox direction and the z-axis either aligned with Earth spin axis or normal to the ecliptic plane at the reference epoch. Relativistic effects are neglected as their effects during a fast close encounter are negligible, whereas Earth oblateness is taken into account through J_2 harmonic. To improve the integration accuracy of this phase, the dynamics is scaled by the radius of the Earth and by Earth gravitational parameter (i.e., $R_E = 1$ and $\mu_E = Gm_E = 1$).

Ephemeris Functions

When a restricted (n+1)-body problem is considered, the positions, velocities, and accelerations of n bodies are evaluated by an ephemeris function. As both in DA and TM framework the planetary ephemerides cannot be computed by external codes, interpolations in time of either planets states or orbital parameters, obtained through JPL DE405, are carried out.

A first model of ephemerides is based on cubic spline interpolation of cartesian position and velocity of planets. More specifically, two different interpolations are carried out to deal with the heliocentric and the close encounter phase of Apophis trajectory, respectively. In the first one the Solar System barycentric coordinates of the planets, the Moon, Ceres, Pallas, and Vesta are interpolated. The ephemerides of the Sun are computed using the Solar System barycenter definition. In order to assure homogeneous interpolation accuracies, a planet dependent grid is adopted, ranging from 1 day for the Moon up to 90 for Pluto system. For the close encounter phase, the ephemerides of the Sun, planets, and the Moon are interpolated in the geocentric reference frame. Note that a grid of 0.1 day is adopted to accurately describe the Moon's motion.

A simple osculating ellipses ephemeris model is developed for the validated integration of Apophis flyby. In this model the gravitational bodies are supposed to move on conic arcs; i.e., their motion is affected only by the gravitational force of the Sun. Within this approximation, the orbit semi-major axis, the eccentricity, the inclination, the right ascension of the ascending node, and the true anomaly, (a, e, i, Ω, ω), are constant for each planet. The mean anomaly M varies as

$$M = M_0 + n(t - t_0), \tag{2}$$

in which M_0 is the mean anomaly at the reference epoch, n is the mean motion, and (t-t_0) the time elapsed from the reference epoch. Note that the result of evaluating the ephemeris functions in the DA and TM frames is an arbitrary order Taylor expansion or Taylor model representation of the position, velocity, and acceleration of the planet with respect to the epoch.

Initial Data

The nominal initial state of Apophis, expressed in cartesian coordinates, is taken from the JPL Horizons system (http://ssd.jpl.nasa.gov/?horizons). The measurement errors reported in [27] are assumed as 3σ values for Apophis' state knowledge. The considered data are summarized in Table 1. For all the simulations, the starting epoch is fixed to 2656 MJD2000 (April 10, 2007). It has to be stressed that the goal of the paper is more to show the potential use of DA and TM for the study of NEO close encounters rather than to specifically evaluate the probability of Apophis impact with the Earth. It was decided to use the aforementioned uncertainty values as they produce a final solution set width larger than that associated to uncertainties on orbital parameters (as given in JPL Near Earth Object Program website at the link http://neo.jpl.nasa.gov/neo/); thus, the robustness of the algorithms is proven.

Table 1. Apophis' coordinates and velocity components at 2656 MJD2000 (April 10, 2007) and associated 3σ values

	Initial state	3σ
x [AU]	$-1.691570577200279\text{x}10^{-1}$	$3.580100000000000\ \text{x}10^{-7}$
y [AU]	$-8.174631401511659\text{x}10^{-1}$	$2.836528213999180\ \text{x}10^{-8}$
z [AU]	$3.933161414674091\text{x}10^{-2}$	$7.427871371978804\ \text{x}10^{-9}$
v_x [AU/day]	$1.955587532182638\text{x}10^{-2}$	$2.087600000000000\ \text{x}10^{-9}$
v_y [AU/day]	$-6.405009915627138\text{x}10^{-4}$	$7.589683940414699\ \text{x}10^{-9}$
v_z [AU/day]	$5.056342169384057\text{x}10^{-4}$	$-1.812601020942239\ \text{x}10^{-10}$

MINIMUM CLOSE ENCOUNTER DISTANCE ALGORITHM

The algorithm that computes for all the asteroids belonging to the initial uncertainty cloud both the minimum close encounter distance form the Earth and the associated epoch requires many steps. The first step is to perform an accurate integration of the motion and to compute an accurate Taylor expansion of the flow. These goals are obtained by applying the DA version of the 8-th order Runge–Kutta–Fehlberg scheme implemented in COSY Infinity [13] with absolute and relative tolerance of 10^{-14}. The accuracy of the result is checked by comparing the computed nominal solution with that downloadable form JPL Horizon system, which is taken as reference. Errors on position of 10^{-8} and 10^{-7} AU are obtained right before entering the Earth's sphere of influence and at t_f = 10695.907098 MJD2000 (the nominal close encounter epoch), respectively. These errors are sufficiently small to the performance of an accurate analysis of the close encounter. The result of the DA integration is the n-th order expansion of the flow with respect to initial condition and final time; i.e,

$$x_f = M_{x_f}(\delta x_0, \delta t_f). \quad (3)$$

The evaluation of the Taylor map (3) delivers the final state as a function of changes in both the initial asteroid state and the integration final time. The accuracy of this map is assessed by comparing the result of its evaluation with point-wise integrations of the dynamics. Numerical tests show that a third order expansion guarantees a maximum error of 10 km AU for the range of initial conditions of interest in a time window of 0.1 days around the final epoch. This accuracy can be considered satisfactory for the purpose of this work. Map (3) and the DA evaluation of the Earth ephemerides are then used to compute the Taylor expansion of the asteroid's distance from our planet,

$$d = M_d(\delta x_0, \delta t_f). \quad (4)$$

Furthermore, in the DA framework, the Taylor expansion of the derivative of the Earth's distance with respect to the final time is straightforwardly computed:

$$d' = M_{d'}(\delta x_0, \delta t_f). \quad (5)$$

This map is then augmented with an identity map in the initial state to obtain a square map and inverted to achieve the Taylor expansion of variation of the final epoch with respect to variation in both the initial condition and in the derivative of the Earth's distance

$$\delta t_f = M_{\delta t_f}(\delta x_0, \delta d'). \quad (6)$$

As the goal is to compute the epoch of the close encounter of all the asteroids belonging to the uncertainty cloud, the map (6) needs to be evaluated only in $\delta d' = 0$, thus obtaining the polynomial relation between the variation of the close approach epoch and the variation of initial conditions:

$$\delta t^* = M_{\delta t^*}(\delta x_0). \quad (7)$$

The Taylor expansion (7) can be then composed with (4) to obtain the Taylor expansion of the close encounter distance as function of initial conditions

$$d^* = M_{d^*}(\delta x_0). \quad (8)$$

The evaluation of map (8) can be used to perform a DA-based Monte Carlo simulation that computes the close encounter distance statistics of Apophis. A simulation of 10000 normally random distributed initial conditions, usually referred to as virtual asteroids, is performed producing the result of Fig. 1. The computed mean close encounter distance from the Earth is 38175.9 km with a standard deviation of 2298.1 km. Note that this result is obtained by

- a single third order DA integration,
- DA map manipulations (4-8),
- and generation of 10000 samples and polynomial evaluations.

On the other hand, a classical Monte Carlo simulation requires 10000 point-wise accurate integration. Fig. 2 shows the ratio between the computational time required by a DA-based Monte Carlo simulation and a classical Monte Carlo run, as a function of expansion order for 10000 samples. As the entire computational time of the DA run is almost taken by the single DA integration, the computational time ratio increases along with the expansion order. A third order DA-based Monte Carlo run takes only 0.02 % of its point-wise counterpart; thus solving the major drawback of Monte Carlo simulations of being computationally intensive.

Once that map (8) is available, the range of the close encounter distances can be computed by using proper validated polynomial bounder, like the linear-dominated bounder (LDB) implemented in COSY-GO [28]. Furthermore, by applying the verified global optimum solver available in COSY-GO we can compute the minimum close encounter distance for initial conditions compatible with the considered measurement uncertainty.

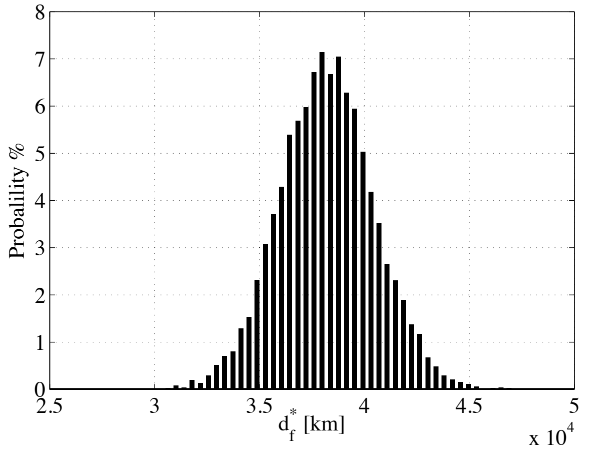

Fig.1. DA-based Monte Carlo simulation of close encounter distances for 10000 virtual asteroids

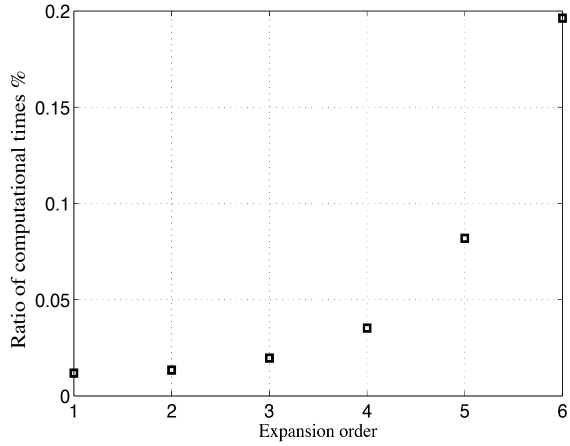

Fig. 2. Computational time comparison: percentage of computational time required by the Monte Carlo versus the classical Monte Carlo simulation for 10000 virtual asteroids

Being based on a branch and bound scheme, if more than one global minimum exists, COSY-GO is able to keep track of all of them, delivering validated enclosures of all the global minima. Fig 3 shows the result of both the range and the minimum close approach distance determination. LDB bound seems to largely overestimate the cloud of the Monte Carlo evaluation. However, it must be noticed that the LDB algorithm computes the bounds of the polynomial over an interval box defined on its variables. Consequently, the bounds take account of the edges of the interval box, which can significantly contribute to the identification of the actual range of the polynomial. These regions are unlikely sampled by the statistical distributions used in the Monte Carlo simulation, due to the different uncertainty modeling and treatment.

On the other hand, the minimum close encounter distance computed by the verified optimizer lies on the lower bound of the range estimated by the LDB; thus the LDB effectiveness is proven. Furthermore, the computation of a minimum close encounter distance of 26833.1 km allows us to rule out the possibility of an Earth's impact. By showing the Earth's distance profile of the most dangerous virtual asteroid, Fig. 4 concludes the DA-based analysis.

CLOSE ENCOUNTER VALIDATED INTEGRATION

The previous section showed a detailed analysis of Apophis close encounter based on DA. As the achieved results are affected by modeling errors, numerical errors, and expansion errors, they can be considered valid only if the estimate of such errors is reliable. On the other hand, the property of TM-based computations of being rigorous can then be exploited to avoid this problem.

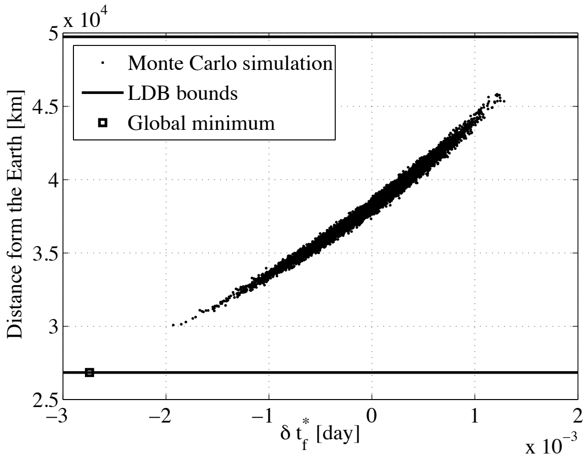

Fig. 3. Close encounter distances of 10000 virtual asteroids: Monte Carlo simulation, range, and global minimum of close encounter distances

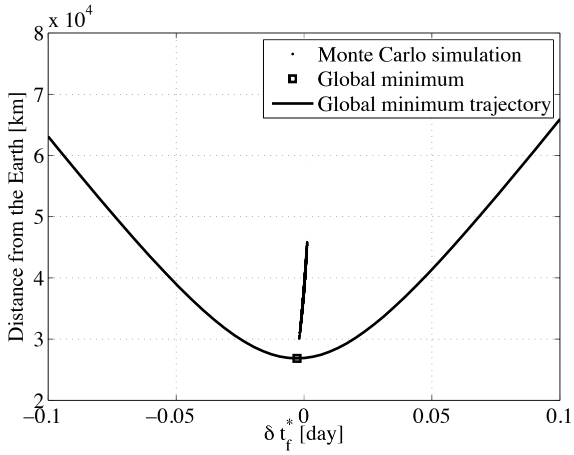

Fig. 4. Actual trajectory corresponding to the virtual asteroid of minimum close encounter distance

A previous work of the authors has already shown the capability of TM-based integrator of addressing long-term rigorous integration of the motion of NEO [24]. This section presents improvements to those results by applying an upgraded version of the validated integrator to the flyby phase only. Such integrator implements new features as dynamic domain splitting, a more efficient operator flow, and an improved step size control algorithm. For details see [23].

The flyby phase is the most challenging one as the high nonlinearity of the dynamics greatly stretches the set of initial conditions. These are obtained by converting into Taylor models the result of a DA-based integration that propagates the set of uncertainties of 2007 up to three days before the closest approach. The rigorous integration is then performed in the geocentric reference frame, considering only the asteroid, the Earth, and the Sun as gravitational bodies; i.e., a restricted three-body problem is integrated.

Fig. 5 and 6 show the result of the validated integration of Apophis flyby using the new validated integration. The integration is performed using 29-th expansion order in time and 9-th in transversal coordinates. It is apparent that the integrator can manage the high nonlinearities that characterize the flyby dynamics. It has to be highlighted that interval enclosures of the TM representation of the flow (black boxes) are used only for visualization aim. The actual volume of the solution is much smaller, as indicated by the blue dots, which are the evaluation of the TM representation of the flow at points belonging to the boundary of the set of initial conditions. Moreover, these dots highlight how the nonlinear dynamics stretch the box of initial conditions. In particular, internal points with low velocity are greatly bent by the Earth's gravitational field, whereas the trajectory of external and fast points is less affected. Fig. 6 shows the effectiveness of automatic domain decomposition algorithm implemented in the integrator. Close to the flyby pericenter, due to the high dynamics nonlinearities, the flow shows a noticeable elongation that would require too many coefficients to be represented accurately. For this reason domain splitting is triggered immediately before the close encounter, reducing the nonlinear terms necessary to describe the flow. In this way, the maximum size of the remainder error is kept suitably small, as shown by Fig. 7. Note that this method increases the number of objects representing the flow, but since the elongation along the orbit is linear in time, the growth of the number of boxes is also only linear. Considering the speed of current Taylor model based integrators, this approach leads to a favourable computational behaviour. When the asteroid trajectory propagation is addressed by means of a rigorous integrator there is no need to develop algorithms for the identification of impact occurrence. This result is immediately obtained by looking for intersections between the validated enclosure of the trajectory of the NEO and the Earth. From Fig. 6 it is clear that, for the considered set on initial conditions and within the implemented dynamical model, the probability of Apophis impact with the Earth in 2029 is zero. Furthermore it has to be stressed that, within the model used, the achieved result is rigorous.

The validated integration of Apophis flyby takes 38136 s on Intel Pentium Dual Core 2.0 GHz, RAM 1 GHz, MacBook laptop. This high value is essentially due to the small step sizes required, starting from the closest encounter (see Fig. 8), to keep the remainder error size small and to the necessity of dealing with five TM objects due to the domain splitting. No box splitting occurs and the integration time lowers to 6544 s when initial uncertainties on orbital parameters consistent with the values available on JPL Near Earth Object Program website. Note that running the integration on parallel machines could considerably lower the computational time.

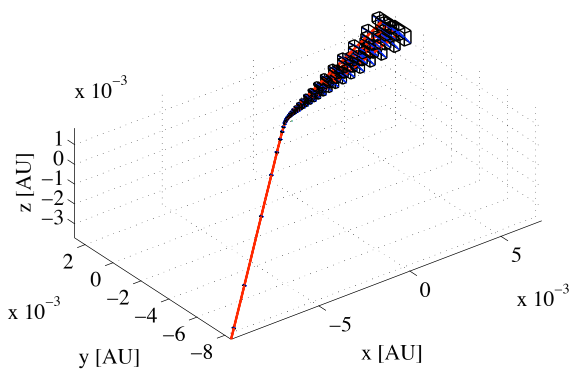

Fig. 5. Apophis flyby validated integration: nominal solution (red line), interval enclosure of the TM (black box), TM evaluations (blue dots)

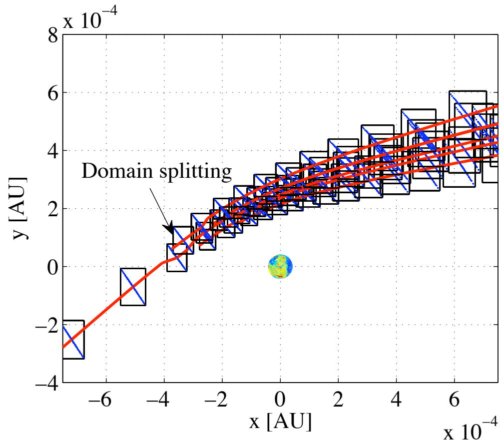

Fig. 6. Details of Apophis flyby validated integration, with dynamic domain splitting triggering

CONCLUSIONS

The paper presented two techniques for the analysis of NEO close encounters with the Earth, using Apophis close encounter in April 2029 as test case. The study has been performed in a dynamical model sufficiently accurate to produce significant results.

The first method exploited the high order Taylor expansion of the flow of ODE to compute the close encounter distance and the close encounter epoch for the entire set of initial conditions. As both these quantities are expressed as Taylor polynomials of the initial state, the computational time of the DA-based Monte Carlo simulation used to compute the statistics of the close encounter is significantly lower than its point-wise counterpart. Furthermore, the polynomial representation can be readily used to obtain bounds for the range of close encounter distance as well as to compute the initial condition of the closest approach. The second part of the work has been devoted to obtain a rigorous integration of Apophis flyby by applying a new version of a TM-based integrator. The property of TM computation of including in the Taylor remainder bound both the numerical and expansion errors is effectively utilized to obtain the validated enclosure of Apophis trajectory. As a result, the possibility of having an impact with the Earth in April 2029, within the adopted dynamical model, is rigorously ruled out.

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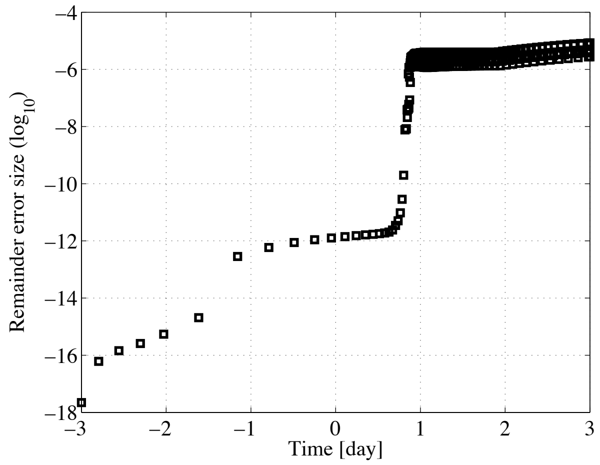

Fig. 7. Maximum remainder error size for Apophis flyby validated integration

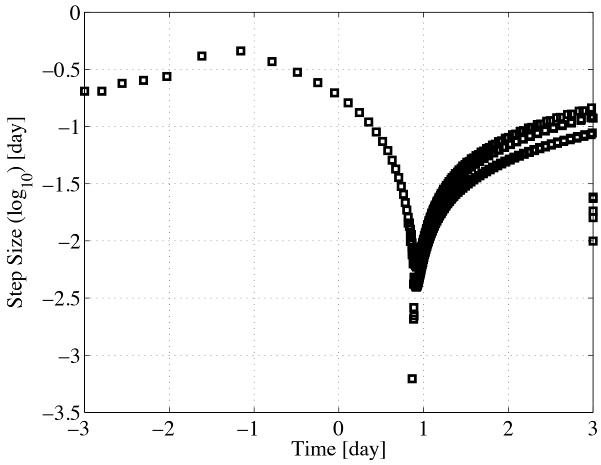

Fig. 8. Step size profile for Apophis flyby validated integration

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