High Order Optimal Feedback Control of Lunar Landing and Rendezvous Trajectories

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Abstract

Optimal feedback control is classically based on linear approximations, whose accuracy drops off rapidly in highly nonlinear dynamics. A high order optimal control strategy is proposed in this work, based on the use of differential algebraic techniques. In the frame of orbital mechanics, differential algebra allows the dependency of the spacecraft state on initial conditions and environmental parameters to be represented by high order Taylor polynomials. The resulting polynomials can be manipulated to obtain the high order expansion of the solution of two-point boundary value problems. Based on the reduction of the optimal control problem to an equivalent two-point boundary value problem, differential algebra is used in this work to compute the high order expansion of the solution of the optimal control problem about a reference trajectory. New optimal control laws for displaced initial states are then obtained by the mere evaluation of polynomials.

1 Introduction

Nominal space trajectories are usually designed by solving optimal control problems that maximize the payload launch-mass ratio while achieving the primary mission goals. However, uncertainties and disturbances affect the spacecraft dynamics in real scenarios. Moreover, state identification is influenced by navigation errors; consequently, the spacecraft state is only known with a given accuracy. Thus, after the nominal solution is computed, an optimal control strategy that assures the satisfaction of mission goals in the real scenario must be implemented. More specifically, given an initial deviation of the spacecraft state from its nominal value, the optimal control strategy aims at canceling the effects of such a deviation on the satisfaction of the mission requirements by correcting the nominal control law, while minimizing propellant consumption.

Classical optimal feedback control strategies are based on linear approximations, whose main advantage is the simplification of the problem. However, their accuracy drops off rapidly with increasing errors and decreasing control frequencies in highly nonlinear dynamics. Thus, nonlinear optimal feedback control has gained particular interest in recent years, and several strategies have appeared to tackle nonlinearities. One of the highly promising and rapidly emerging methodologies for designing nonlinear controllers is the state-dependent Riccati equation (SDRE) approach, which was originally proposed by Pearson and Burghart and then described in details by Cloutier, Hammett and

Beeler [1]. This approach involves manipulating the governing dynamic equations into a pseudo-linear non-unique form in which system matrices are given as a function of the current state and minimizing a quadratic-like performance index. An algebraic Riccati equation using the system matrices is then solved repetitivily online to give the optimal control law. Thus, the SDRE approach might turn out to be computationally expensive when the solution of the Riccati equation is not properly managed. This can prevent its use for real-time optimal control. A sub-optimal solution can be obtained using the approximating sequence of Riccati equations (ASRE) method [6]. Based on an approximate solution to the Hamilton-Jacobi-Bellman equation, the ASRE method avoids solving the Riccati equation repetitively at every instant and provides a closed-form feedback controller.

An alternative approach was recently introduced by Park and Scheeres, which relies on the theory of canonical transformations and their generating functions for Hamiltonian systems [8]. More specifically, canonical transformations are able to solve boundary value problems between Hamiltonian coordinates and momenta for a single flow field. Thus, based on the reduction of the optimal control problem to an equivalent boundary value problem, they can be effectively used to solve the optimal control problem analytically as a function of the boundary conditions, which is instrumental to optimal feedback control. The main difficulty of this approach is finding the generating functions via the solution of the Hamilton-Jacobi equation. This problem was solved by Park and Scheeres by expanding the generating function in power series of its arguments.

Differential algebraic (DA) techniques [3] are used in this work to develop an alternative approach to the generating function method. Differential algebra serves the purpose of computing the derivatives of functions in a computer environment. More specifically, by substituting the classical implementation of real algebra with the implementation of a new algebra of Taylor polynomials, it expands any function f of v variables into its Taylor series up to an arbitrary order n. DA techniques are used in this work to represent the dependency of the spacecraft state on the initial conditions by means of high order Taylor polynomials. Then, the resulting Taylor polynomials are manipulated to impose the boundary and optimality conditions of the optimal control problem. This enables the expansion of the solution of the optimal control problem with respect to the initial conditions about an available reference trajectory. The resulting Taylor polynomials can be evaluated for new solutions of the original optimal control problem, so avoiding repetitive runs of classical iterative procedures.

The paper is organized as follows. A brief introduction to differential algebra is given in Sect. 2. Being at the basis of the proposed methods, the possibility of expanding the flow of ODEs is presented in Sect. 3. The optimal control problem and the algorithm for the high order expansion of its solution are illustrated in Sect. 4 and 5, respectively. The application of the algorithm to lunar landing and rendezvous maneuvers is addressed in Sect. 6 and 7, respectively.

2 Differential Algebra

DA techniques find their origin in the attempt to solve analytical problems by an algebraic approach [3]. Historically, the treatment of functions in numerics has been based on the treatment of numbers, and the classical numerical algorithms are based on the mere evaluation of functions at specific points. DA techniques are based on the observation that it is possible to extract more information on a function rather than its mere values. The basic idea is to bring the treatment of functions and the operations on them to the computer environment in a similar way as the treatment of real numbers. Referring to Fig. 1, consider two real numbers a and b. Their transformation into the floating point representation, $\overline{a}$ and $\overline{b}$ respectively, is performed to operate on them in a computer environment. Then, given any operation $\times$ in the set of real numbers, an adjoint operation $\otimes$ is defined in the set of FP numbers such that the diagram in figure commutes. (The diagram commutes approximately in practice, due to truncation errors.) Consequently, transforming the real num-

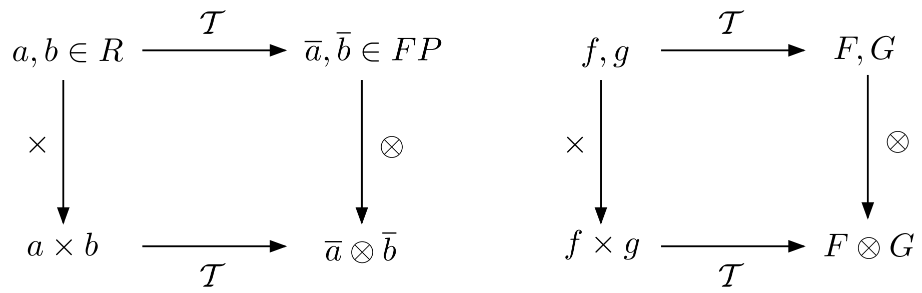

Figure 1: Analogy between the floating point representation of real numbers in a computer environment (left figure) and the introduction of the algebra of Taylor polynomials in the differential algebraic framework (right figure).

bers a and b in their FP representation and operating on them in the set of FP numbers returns the same result as carrying out the operation in the set of real numbers and then transforming the achieved result in its FP representation. In a similar way, suppose two sufficiently regular functions f and g are given. In the framework of differential algebra, the computer operates on them using their Taylor series expansions, F and G respectively. Therefore, the transformation of real numbers in their FP representation is now substituted by the extraction of the Taylor expansions of f and g. For each operation in the function space, an adjoint operation in the space of Taylor polynomials is defined such that the corresponding diagram commutes; i.e., extracting the Taylor expansions of f and g and operating on them in the function space returns the same result as operating on f and g in the original space and then computing the Taylor expansion of the resulting function. The straightforward implementation of differential algebra in a computer allows to compute the Taylor coefficients of a function up to a specified order n, along with the function evaluation, with a fixed amount of effort. The Taylor coefficients of order n for sums and product of functions, as well as scalar products with reals, can be computed from those of summands and factors; therefore, the set of equivalence classes of functions can be endowed with well-defined operations, leading to the so-called truncated power series algebra [2].

Similarly to the algorithms for floating point arithmetic, the algorithm for functions followed, including methods to perform composition of functions, to invert them, to solve nonlinear systems explicitly, and to treat common elementary functions [3]. In addition to these algebraic operations, also the analytic operations of differentiation and integration are introduced, so finalizing the definition of the DA structure. The differential algebra sketched in this section was implemented by Berz and Makino in the software COSY-Infinity [4].

3 High Order Expansion of ODE Flow

The differential algebra introduced in the previous section allows to compute the derivatives of any function f of v variables up to an arbitrary order n, along with the function evaluation. This has an important consequence when the numerical integration of an ODE is performed by means of an arbitrary integration scheme. Any explicit integration scheme is based on algebraic operations, involving the evaluations of the ODE right hand side at several integration points. Therefore, carrying out all the evaluations in the DA framework allows differential algebra to compute the arbitrary order expansion of the flow of a general ODE initial value problem.

Without loss of generality, consider the scalar initial value problem

$$\begin{cases} \dot{x} = f(x) \\ x(t_i) = x_i. \end{cases} \tag{1}$$

Replace the point initial condition x_i with the DA representative of its identity function, $[x_i] = x_i^0 + \delta x_i$, where x_i^0 is the reference point for the expansion. If all the operations of the numerical integration scheme are carried out in the framework of differential algebra, the Taylor expansion of the solution with respect to the initial condition is obtained at each step. As an example, consider the forward Euler scheme

$$x_k = x_{k-1} + \Delta t \cdot f(x_{k-1}) \tag{2}$$

and analyze the first integration step; i.e.,

$$x_1 = x_0 + \Delta t \cdot f(x_0), \tag{3}$$

where $x_0 = x_i$. Substitute the initial value with $[x_0] = [x_i] = x_i^0 + \delta x_i$ in Eq. 3 for

$$[x_1] = [x_0] + \Delta t \cdot f([x_0]). \tag{4}$$

If the function f is evaluated in the DA framework, the output of the first step, $[x_1]$, is the Taylor expansion of the solution x_1 at t_1 with respect to the initial condition about the reference point x_i^0. The previous procedure can be inferred through the subsequent steps until the last integration step is reached. The result at the final step is the n-th order Taylor expansion of the flow of the initial value problem of Eq. 1 at the final time t_f. Thus, the expansion of the flow of a dynamical system can be computed up to order n with a fixed amount of effort.

4 Optimal Control Problem

Suppose the spacecraft moves under the general dynamics

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t), \tag{5}$$

where $\boldsymbol{x} = \{x_1, \ldots, x_v\}$ is the state vector and $\boldsymbol{u} = \{x_1, \ldots, x_m\}$ is the control vector ($m \leq v$). The optimal control problem aims at finding the m control functions $\boldsymbol{u}(t)$ that minimize the performance index

$$J = \varphi(\boldsymbol{x}_f, t_f) + \int_{t_i}^{t_f} L(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\, dt. \tag{6}$$

The initial state vector, $\boldsymbol{x}_i$, and the final state vector, $\boldsymbol{x}_f$, are not necessarily fixed, as well as the final time t_f. In addition to the previous statements, boundary constraints

$$\boldsymbol{\psi}(\boldsymbol{x}_f, t_f) = 0, \tag{7}$$

where $\boldsymbol{\psi} = \{\psi_1, \ldots, \psi_p\}$, and path constraints

$$\boldsymbol{C}(\boldsymbol{u}(t), t) \leq 0, \tag{8}$$

where $\boldsymbol{C} = \{C_1, \ldots, C_q\}$, can be imposed.

The above problem can be solved by reducing it to a boundary value problem on a set of differential algebraic equations (DAEs) [5]. To this aim, the dynamics and constraints are added to the performance index J to form the so-called augmented performance index

$$\begin{aligned} \bar{J} \;=\; & \varphi(\boldsymbol{x}_f, t_f) + \boldsymbol{\nu}^T \boldsymbol{\psi}(\boldsymbol{x}_f, t_f) + \\ & + \textstyle\int_{t_i}^{t_f} [L(\boldsymbol{x}, \boldsymbol{u}, t) + \boldsymbol{\lambda}^T (\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) + \\ & - \dot{\boldsymbol{x}}) + \boldsymbol{\mu}^T \boldsymbol{C}(\boldsymbol{u}, t)] dt, \end{aligned} \tag{9}$$

where two kind of Lagrange multipliers are introduced:

- a p-dimensional vector of constants, $\boldsymbol{\nu}$, for the final constraints of Eq. (7);
- an n-dimensional and a q-dimensional vector of functions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ for the dynamics of Eq. (5) and the path constraints of Eq. (8), which are usually referred to as adjoint or costate variables.

The optimal control problem is then reduced to identifying a stationary point of the augmented performance index $\bar{J}$. This is achieved by imposing the gradient of $\bar{J}$ to be zero with respect to all optimization variables; specifically, the state vector $\boldsymbol{x}$ and the control vector $\boldsymbol{u}$, the Lagrange multipliers $\boldsymbol{\nu}$ and the costate variables $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, the unknown components of the initial state $\boldsymbol{x}_i$ and the final state $\boldsymbol{x}_f$, and the final time t_f. In particular, the optimality with respect to $\boldsymbol{\lambda}$ and $\boldsymbol{x}$ leads to the following relations:

$$\begin{aligned} & \frac{\partial \bar{J}}{\partial \boldsymbol{\lambda}} = 0 \;\Rightarrow \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \\ & \frac{\partial \bar{J}}{\partial \boldsymbol{x}} = 0 \;\Rightarrow \dot{\boldsymbol{\lambda}} = -\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{\lambda} - \left(\frac{\partial L}{\partial \boldsymbol{x}}\right)^T, \end{aligned} \tag{10}$$

whereas $\partial \bar{J}/\partial \boldsymbol{u} = 0$ yields

$$\left(\frac{\partial L}{\partial \boldsymbol{u}}\right)^T + \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right)^T \boldsymbol{\lambda} + \left(\frac{\partial \boldsymbol{C}}{\partial \boldsymbol{u}}\right)^T \boldsymbol{\mu} = \boldsymbol{0}. \tag{11}$$

Equations (10) and (11) together are usually referred to as Euler-Lagrange equations. It is worth observing that the Euler-Lagrange equations form a system of DAEs: the differential part is represented by Eq. (10), which defines the dynamics for the state variables $\boldsymbol{x}$ and the costate variables $\boldsymbol{\lambda}$; the role of the algebraic constraint is played by Eq. (11). The previous system must be coupled with the boundary conditions ensuing from the optimality conditions with respect to the remaining optimization variables (see [5] for further details). The optimal control problem is eventually reduced to a boundary value problem on a system of DAEs.

A particular optimal control problem is addressed in this work. The dynamics is supposed to be affine in the control vector $\boldsymbol{u}$; i.e.

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) = \tilde{\boldsymbol{f}}(\boldsymbol{x}, t) + \mathrm{B}(\boldsymbol{x})\, \boldsymbol{u}, \tag{12}$$

where $\mathrm{B}(\boldsymbol{x})$ is a $v \times m$ matrix, whose elements do not depend on the controls. Moreover the control functions are sought to minimize the performance index

$$J = \frac{1}{2} \int_{t_i}^{t_f} \boldsymbol{u}^T \boldsymbol{u}\, dt \tag{13}$$

and no path constraints are imposed. Based on the previous hypotheses, Eq. (11) assumes the simpler form

$$\boldsymbol{u} + \mathrm{B}^T(\boldsymbol{x})\, \boldsymbol{\lambda} = 0. \tag{14}$$

Equation (14) supplies an explicit relation between the control functions $\boldsymbol{u}$ and the costate variables $\boldsymbol{\lambda}$, which can be substituted in Eq. (10). The original system of DAEs of the Euler-Lagrange equations translates into the system of ODEs

$$\begin{aligned} \dot{\boldsymbol{x}} &= \tilde{\boldsymbol{f}}(\boldsymbol{x}, t) - \mathrm{B}(\boldsymbol{x})\, \mathrm{B}^T(\boldsymbol{x}) \boldsymbol{\lambda} \\ \dot{\boldsymbol{\lambda}} &= - \left(\frac{\partial \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\lambda}, t)}{\partial \boldsymbol{x}} \right)^T \boldsymbol{\lambda}. \end{aligned} \tag{15}$$

Therefore, the original optimal control problem reduces to a two-point boundary value problem (TPBVP) on the set of ODEs in Eq. (15), where boundary conditions are imposed on the initial and final values of the state and costate variables, depending on the optimal control problem at hand.

5 High Order Optimal Feedback

Suppose the problem of transferring a spacecraft from a fixed initial state to a fixed final state with fixed t_i and t_f is of interest; i.e., boundary conditions assume the simpler form

$$\begin{cases} \boldsymbol{x}_i &= \overline{\boldsymbol{x}}_i \\ \boldsymbol{x}_f &= \overline{\boldsymbol{x}}_f. \end{cases} \tag{16}$$

The optimal control problem is then reduced to the problem of solving Eq. (15) subject to the boundary conditions in Eq. (16).

Several techniques are available in the literature to solve the previous problem for assigned $\overline{\boldsymbol{x}}_i$ and $\overline{\boldsymbol{x}}_f$, like the simple and multiple shooting schemes or difference methods [9]. This means that, given $\overline{\boldsymbol{x}}_i$ and $\overline{\boldsymbol{x}}_f$, the previous techniques are applied to compute the initial values of the costate variables that solve the TPBVP, which will be indicated as $\boldsymbol{\lambda}_i^0$. The solution is then uniquely identified by the initial state and costate vectors, $\overline{\boldsymbol{x}}_i$ and $\boldsymbol{\lambda}_i^0$ respectively.

Assume now a reference solution $\boldsymbol{\lambda}_i^0$ is available and suppose the Taylor expansion of the solution of the optimal control problem with respect to the initial state $\boldsymbol{x}_i$ is of interest. Differential algebra can effectively serve this purpose. To this aim, initialize both the initial state $\boldsymbol{x}_i$ and the initial costate $\boldsymbol{\lambda}_i$ as DA variables. This means the variations

$$\begin{aligned} [\boldsymbol{x}_i] &= \overline{\boldsymbol{x}}_i + \delta \boldsymbol{x}_i \\ [\boldsymbol{\lambda}_i] &= \boldsymbol{\lambda}_i^0 + \delta \boldsymbol{\lambda}_i \end{aligned} \tag{17}$$

to the fixed initial state $\overline{\boldsymbol{x}}_i$ and the reference solution $\boldsymbol{\lambda}_i^0$ are considered.

Using the techniques introduced in Sect. 3, the solution of Eq. (15) is expanded with respect to the initial state and costate vectors. More specifically, the dependence of the final state and costate vectors on their initial values are obtained in terms of the high order polynomial map

$$\begin{aligned} \begin{pmatrix} [\boldsymbol{x}_f] \\ [\boldsymbol{\lambda}_f] \end{pmatrix} &= \begin{pmatrix} \overline{\boldsymbol{x}}_f + \delta \boldsymbol{x}_f \\ \boldsymbol{\lambda}_f^0 + \delta \boldsymbol{\lambda}_f \end{pmatrix} \\ &= \begin{pmatrix} \overline{\boldsymbol{x}}_f \\ \boldsymbol{\lambda}_f^0 \end{pmatrix} + \begin{pmatrix} \mathcal{M}_{\boldsymbol{x}_f} \\ \mathcal{M}_{\boldsymbol{\lambda}_f} \end{pmatrix} \begin{pmatrix} \delta \boldsymbol{x}_i \\ \delta \boldsymbol{\lambda}_i \end{pmatrix}, \end{aligned} \tag{18}$$

where $\overline{\boldsymbol{x}}_f$ and $\boldsymbol{\lambda}_f^0$ are the constant part of the map (i.e., the reference solution flowing from $\overline{\boldsymbol{x}}_i$

and $\boldsymbol{\lambda}_i^0$ under the ODEs in Eq. (15)), whereas all higher order terms are included in $\mathcal{M}_{\boldsymbol{x}_f}$ and $\mathcal{M}_{\boldsymbol{\lambda}_f}$.

Subtract now the constant part from Eq. (18) for

$$\begin{pmatrix} \delta \boldsymbol{x}_f \\ \delta \boldsymbol{\lambda}_f \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{\boldsymbol{x}_f} \\ \mathcal{M}_{\boldsymbol{\lambda}_f} \end{pmatrix} \begin{pmatrix} \delta \boldsymbol{x}_i \\ \delta \boldsymbol{\lambda}_i \end{pmatrix}. \quad (19)$$

Then, extract $\mathcal{M}_{\boldsymbol{x}_f}$ from Eq. (19) and consider the map

$$\begin{pmatrix} \delta \boldsymbol{x}_f \\ \delta \boldsymbol{x}_i \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{\boldsymbol{x}_f} \\ \mathcal{I}_{\boldsymbol{x}_i} \end{pmatrix} \begin{pmatrix} \delta \boldsymbol{x}_i \\ \delta \boldsymbol{\lambda}_i \end{pmatrix}, \quad (20)$$

which is built by concatenating $\mathcal{M}_{\boldsymbol{x}_f}$ with the identity map for $\delta \boldsymbol{x}_i$, $\mathcal{I}_{\boldsymbol{x}_i}$.

Using suitable inversion techniques for high order polynomials [3], the map in Eq. (20) can be inverted to obtain

$$\begin{pmatrix} \delta \boldsymbol{x}_i \\ \delta \boldsymbol{\lambda}_i \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{\boldsymbol{x}_f} \\ \mathcal{I}_{\boldsymbol{x}_i} \end{pmatrix}^{-1} \begin{pmatrix} \delta \boldsymbol{x}_f \\ \delta \boldsymbol{x}_i \end{pmatrix}. \quad (21)$$

The high order polynomial map in Eq. (21) relates the displacements of the initial state and costate vectors from their reference values $\overline{\boldsymbol{x}}_i$ and $\boldsymbol{\lambda}_i^0$, $\delta \boldsymbol{x}_i$ and $\delta \boldsymbol{\lambda}_i$ respectively, to the displacement of the final state vector from its fixed value $\overline{\boldsymbol{x}}_f$, $\delta \boldsymbol{x}_f$, and again $\delta \boldsymbol{x}_i$.

Suppose now the problem of reaching the fixed final state $\overline{\boldsymbol{x}}_f$ regardless of any error on the initial state $\delta \boldsymbol{x}_i$ is of interest. This means the new solution $\boldsymbol{\lambda}_i$ of the optimal control problem corresponding to the new initial condition $\boldsymbol{x}_i = \overline{\boldsymbol{x}}_i + \delta \boldsymbol{x}_i$ is to be computed. The map in Eq. (21) can effectively serve this purpose. More specifically, the boundary condition

$$\boldsymbol{x}_f = \overline{\boldsymbol{x}}_f \quad (22)$$

must be imposed. To this aim, note that $\boldsymbol{x}_f = \overline{\boldsymbol{x}}_f + \delta \boldsymbol{x}_f$. Consequently, Eq. (22) reduces to

$$\delta \boldsymbol{x}_f = 0. \quad (23)$$

Substituting Eq. (23) into the high order polynomial map of Eq. (21) yields

$$\begin{pmatrix} \delta \boldsymbol{x}_i \\ \delta \boldsymbol{\lambda}_i \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{\boldsymbol{x}_f} \\ \mathcal{I}_{\boldsymbol{x}_i} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ \delta \boldsymbol{x}_i \end{pmatrix}. \quad (24)$$

Extract the last v components of the map in Eq. (24), which will be indicated as

$$\delta \boldsymbol{\lambda}_i = \mathcal{M}_{\boldsymbol{\lambda}_i}(\delta \boldsymbol{x}_i). \quad (25)$$

The polynomial map in Eq. (25) is the high order Taylor expansion of the solution of the optimal control problem with respect to the initial state $\boldsymbol{x}_i$: given any displacement $\delta \boldsymbol{x}_i$ of $\boldsymbol{x}_i$ from the reference value $\overline{\boldsymbol{x}}_i$, the mere evaluation of the polynomials in Eq. (25) delivers the high order correction $\delta \boldsymbol{\lambda}_i$ to $\boldsymbol{\lambda}_i^0$ to obtain the corresponding solution $\boldsymbol{\lambda}_i$ of the optimal control problem.

It is worth observing that a possible alternative approach to solve the previous problem could have consisted in solving the TPBVP for the new solution $\boldsymbol{\lambda}_i$ using classical techniques. However, the main disadvantage of this approach is that a new TPBVP must be solved for each displaced initial condition. This involves running through the iterative procedures of the classical TPBVP solvers. Each iterative procedure is able to deliver one solution, whose validity is limited to the corresponding $\delta \boldsymbol{x}_i$. Consequently, the classical TPBVP solvers should be applied for each new $\delta \boldsymbol{x}_i$. The Taylor expansion of the optimal control problem supplies an effective alternative method to overcome this issue. First of all, analytical information are gained, which can supply a valuable insight on the underlying dynamics. Moreover, for any error $\delta \boldsymbol{x}_i$, the mere evaluation of polynomials suffices to obtain the new optimal control law to reach $\overline{\boldsymbol{x}}_f$, so avoiding the use of iterative algorithms. Nevertheless, the polynomial relation between $\delta \boldsymbol{\lambda}_i$ and $\delta \boldsymbol{x}_i$ given by Eq. (25) is accurate up to the order of the DA-based computation.

6 Lunar Landing

The technique introduced in Sect. 5 is used here for the optimal feedback control of a probe landing on Moon's south pole. The control profile is designed in the frame of the controlled two-body problem. Referring to Figure 2, the lander is supposed to originally move on an elliptical polar descent orbit, taking it from an altitude of 100 km (apocenter) to an altitude of 20 km (pericenter). The landing phase is supposed to start at the pericenter of the descent orbit. Final conditions are imposed to position the lander over Moon's south pole at an altitude of 2 m, with a downward velocity of 3 m/s, from which the final phase of the landing maneu-

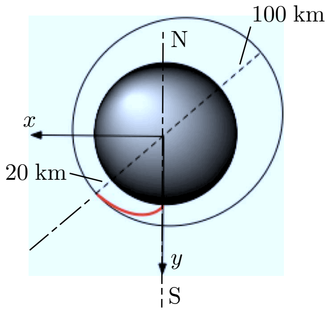

Figure 2: Lunar landing problem.

ver is supposed to start. A cartesian reference frame is selected to describe the dynamics: the y-axis is aligned with Moon's south pole; the x-axis lies on Moon's equatorial plane, pointing towards the orbital descending node; the z-axis is selected to form a right-handed reference system. The landing dynamics is described by the set of ODEs:

$$\begin{aligned} \dot{\boldsymbol{r}} &= \boldsymbol{v} \\ \dot{\boldsymbol{v}} &= -\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{u}, \end{aligned} \tag{26}$$

in which $\boldsymbol{r}$ and $\boldsymbol{v}$ are the probe position and velocity, respectively; $r = ||\boldsymbol{r}||$; μ is Moon's gravitational parameter; and $\boldsymbol{u}$ is the control vector. As from Eq. (26), the dynamics is affine in the control vector $\boldsymbol{u}$. Thus, Eq. (15) holds for the problem at hand and the optimal control problem is then reduced to a TPBVP with fixed initial and final states for the landing probe.

A reference solution of the optimal control problem is first identified by solving the resulting TPBVP. The initial time is chosen to be zero, whereas the landing duration is set to 31 min. A simple shooting technique is used to solve the TPBVP and to compute the reference trajectory reported in Fig. 3. Figure 4 illustrates the corresponding reference control profile in terms of histories of its components. Due to the symmetry of the problem, the reference trajectory lies completely on the x-y plane.

The initial probe position and velocity, $\boldsymbol{r}_i$ and $\boldsymbol{v}_i$ respectively, are now supposed to be affected by errors. The high order optimal feedback control algorithm introduced in Sect. 5 is applied to optimally correct the reference control law. More specifically, the reference trajectory in Fig. 3 is used as reference solution for the Taylor expansions. The algorithm is then applied to compute the high order polynomial map of Eq. (25) for the problem at hand. Thus, given any displacement $\delta\boldsymbol{x}_i = (\delta\boldsymbol{r}_i, \delta\boldsymbol{v}_i)$ of the initial state vector from its nominal value, the polynomial map is readily evaluated to correct the nominal $\boldsymbol{\lambda}_i^0$ for the new initial value of the costate vector.

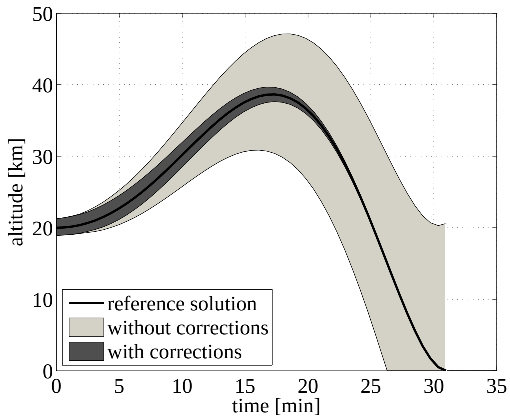

Figure 3: Lunar landing: altitude dispersion.

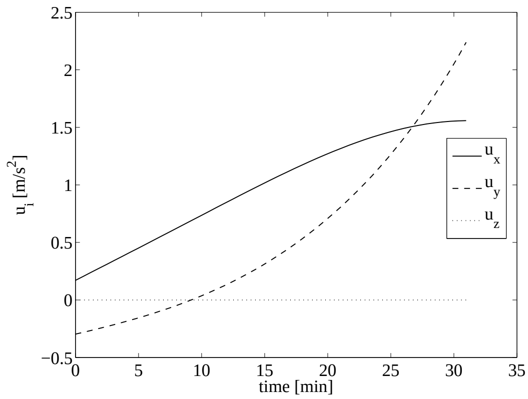

Figure 4: Lunar landing: reference control.

The performances of the procedure are studied in the followings. A maximum position error of 1 km and a maximum velocity error of 5 m/s are supposed to affect each component of the initial lander position and velocity, respectively. The final dispersion at landing is then investigated. First of all, for the sake of a more complete analysis, given any displaced initial conditions, no corrections to the nominal costate variables (and, consequently, to the controls) are supplied. In particular, 100 samples

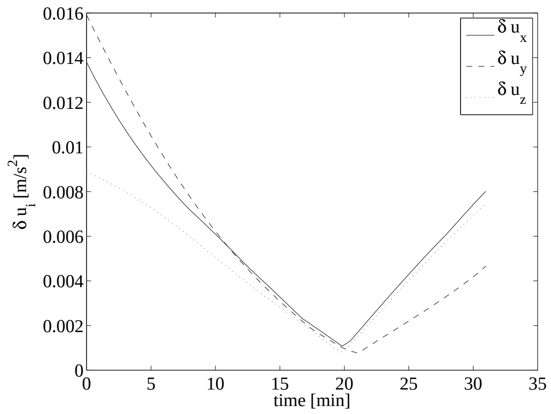

Figure 5: Lunar landing: control corrections.

are randomly generated within the initial uncertainty box with uniform distribution. Each sample is then propagated using the nominal guidance law; once all integrations have been performed, the maximum and minimum lander altitudes at each integration time are computed. Figure 3 shows the resulting altitude dispersion throughout landing. The figure illustrates how initial conditions corresponding to both impacts on Moon's surface (lower area of the strip) and trajectories moving away from the landing site (higher area of the strip) are included in the initial error box.

The DA-based high order algorithm described in Sect. 5 is then applied. In particular, third order corrections are computed using Eq. (25): for the same random samples of Fig. 3, the errors on the initial state are computed and the map is evaluated to correct the reference $\boldsymbol{\lambda}_i^0$. The resulting set of trajectories is reported again in Fig. 3 for the sake of comparison. The corrected optimal control laws take the probe to the final desired conditions and the resulting final dispersion is drastically reduced. The necessary control corrections are analyzed in Fig. 5. More specifically, for each component of the control vector $\boldsymbol{u}$, the maximum control correction is evaluated among the random samples, and the resulting curves are reported in figure. A maximum control correction of about 0.016 m/s^2 is required for the given error box.

7 Rendezvous Maneuver

A rendezvous maneuver is here analyzed as a further test case for the high order optimal feedback technique introduced in Sect. 5. The study of this problem is motivated by the work of Park, Guibout and Scheeres based on the alternative approach of generating functions [7, 8]. The space rendezvous is a maneuver which takes two spacecraft, originally moving on different orbits, to the same final reference orbit, matching their positions and velocities. Referring to Figure 6, this rather general case can be focused on the problem of a spacecraft (referred to as *chaser*) targeting an object (referred to as *target*) on its orbit.

A continuously propelled rendezvous maneuver is considered. The target is supposed to move on a circular orbit of radius R, whereas the chaser is assumed to be subject to a controlled two-body dynamics. In this framework, the rendezvous maneuver is classically designed in a non-inertial reference frame that is centered at the target position, with x-axis constantly aligned with the orbital radius, y-axis directed towards the target orbital velocity, and z-axis chosen to form a right-handed coordinate system with x and y (see Figure 6). Thus, the non-inertial reference frame rotates along the circular target orbit with constant angular velocity ω and the chaser is subject to the relative dynamics

$$\begin{aligned}
\dot{x} &= v_x, \quad \dot{y} = v_y, \quad \dot{z} = v_z \\
\dot{v}_x &= 2\dot{y} - (1+x)(\frac{1}{r^3} - 1) + u_x \\
\dot{v}_y &= -2\dot{x} - y(\frac{1}{r^3} - 1) + u_y \\
\dot{v}_z &= -\frac{1}{r^3}\, z + u_z \,,
\end{aligned} \tag{27}$$

where lengths and time are normalized using R and $1/\omega$ respectively; $\boldsymbol{u} = (u_x, u_y, u_z)$ is the control vector; and $r = \sqrt{(1+x)^2 + y^2 + z^2}$.

The chaser is supposed to have initial offsets from the target in both position and velocity, which are denoted by $\delta\boldsymbol{r}_i$ and $\delta\boldsymbol{v}_i$ respectively. The optimal control problem is solved to design the control functions $\boldsymbol{u}$ that take the chaser from its initial displaced state to the target state (i.e.,

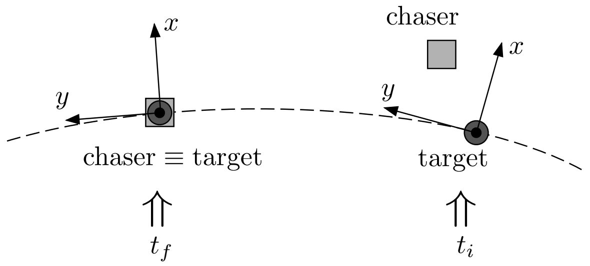

Figure 6: Rendezvous maneuver.

to the origin of the rotating frame with zero velocity) in a given time $t_f - t_i$. The relative dynamics in Eq. (27) is affine in the control vector $\boldsymbol{u}$. Thus, the optimal control problem can be reduced to a TPBVP with fixed initial and final states for the chaser.

Similarly to the previous test case, a nominal solution of the optimal control problem must be identified before applying the high order DA-based technique. To this aim, it is worth observing that the relative dynamics in Eq. (27) satisfies $\boldsymbol{f}(\boldsymbol{x} = 0, \boldsymbol{u} = 0, t) = 0$, with $\boldsymbol{x} = (x, y, z, v_x, v_y, v_z)$. This means that $\boldsymbol{x}(t) = 0$ and $\boldsymbol{u}(t) = 0$ for any t is a trivial solution of the optimal control problem that is used as reference solution for the high order expansion.

The performances of the high order optimal feedback control algorithm are now investigated. The chaser is supposed to have a displaced initial position $\delta \boldsymbol{r}_i = (0.2, 0.2, 0)$ and a displaced initial velocity $\delta \boldsymbol{v}_i = (0.1, 0.1, 0)$. The rendezvous maneuver is designed to take the chaser to the target state in 1 time unit. The exact solution of the optimal control problem is first identified by solving the associated TPBVP using a simple shooting technique. The result is reported in Fig. 7, Fig. 8, and Fig. 9 in terms of position, velocity, and control profile, respectively. As can be seen from Fig. 7 and Fig. 8, the exact solution (solid lines) takes the chaser to the target state in the assigned time. The exact solution is then compared with those achieved by the DA-based optimal feedback control algorithm introduced in Sect. 5 using different expansion orders. As can be seen, the low accuracy of the 1-st order correction is significantly improved using 4-th and 6-th order expansions.

The main advantage of the high order opti-

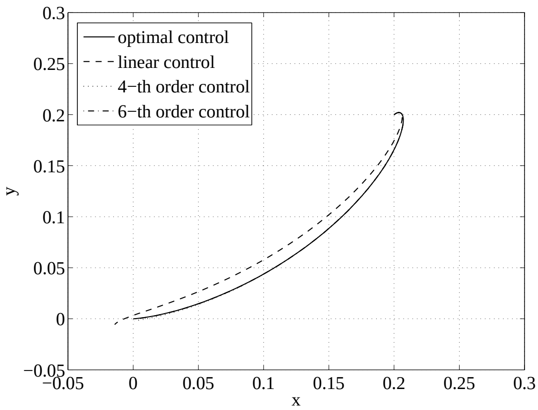

Figure 7: Rendezvous: position.

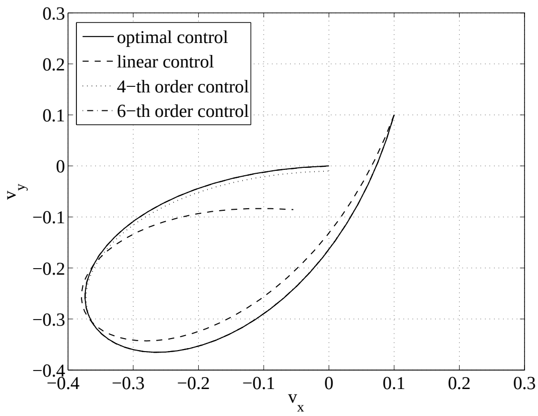

Figure 8: Rendezvous: velocity.

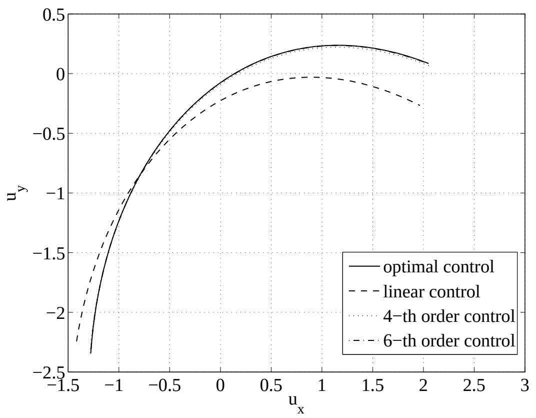

Figure 9: Rendezvous: control.

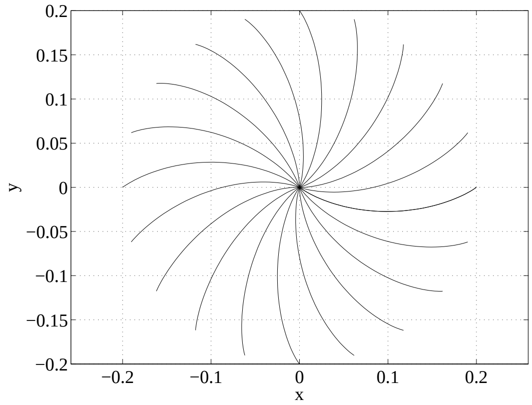

Figure 10: Rendezvous: trajectories corresponding to displaced initial positions lying on a circle of radius 0.2.

mal feedback control algorithm is that, for any initial offset of the chaser with respect to the target, the same polynomial map is evaluated to compute the corresponding optimal control law. This means that the high order map in Eq. (25) must be computed only once for all possible offsets, and the optimal control laws are then obtained through the mere evaluation of polynomials. This feature is exploited in Fig. 10: a set of displaced positions distributed over a circle of radius 0.2 in the rotating frame is selected. For each sample, a 6-th order correction is computed using the polynomial map of Eq. (25). As can be seen, the chaser is always moved to the origin of the reference frame.

8 Conclusion

A method for the computation of optimal feedback control laws based on differential algebra has been introduced, with applications to lunar landing and rendezvous maneuvers. The method relies on the high order expansion of the solution of the optimal control problem about a reference trajectory. Thus, it improves the results of classical techniques based on the linearization of the dynamics. Moreover, the method reduces the computation of new optimal control laws to the mere evaluation of polynomials. This is a valuable advantage over the conventional nonlinear optimal control strategies, which are mainly based on iterative procedures. This work focused on the problem of transferring a spacecraft from an initial fixed state to a final fixed state. Further developments will investigate the imposition of soft constraints, as well as the minimization of alternative performance criteria. Moreover, path constraints on the controls will be included.

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