Center-Focus and Smale-Pugh problems for Abel equation: why to study them?

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Background

Abel differential equation

$$y' = p(x)\,y^2 + q(x)\,y^3 \qquad (*)$$

- $x \in [a,b]$
- p,q are:

  - real, complex polynomials of bounded degrees

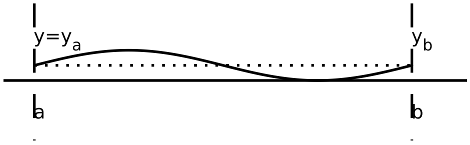
  - trigonometric, Laurent polynomials of bounded degrees

  - piecewise-linear functions ...

Smale-Pugh problem

$$y' = p(x)\,y^2 + q(x)\,y^3 \qquad (*)$$

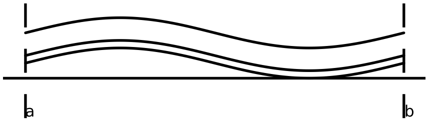
Find a uniform (in p, q in a given class) upper bound on the number of closed periodic solutions $y = y(x)$ such that $y(a) = y(b)$.



Center-Focus problem

$$y' = p(x)\,y^2 + q(x)\,y^3 \qquad (*)$$

Find conditions on p, q (in a given class) for all solutions to be periodic, i.e. for $(*)$ to have a center.



Relation to the classical problems

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -y + F(x,y) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + G(x,y) \end{cases} \qquad (**)$$

$$\Downarrow \text{Cherkas transform}$$

$$y' = p(x)\,y^2 + q(x)\,y^3 \qquad (*)$$

Hilbert's 16th problem, second part

Given a polynomial vector field $(**)$ of a given degree find a uniform (in F, G) upper bound for the number of isolated closed trajectories (limit cycles).

Poincaré's Center-Focus problem

Given a polynomial vector field $(**)$ of a given degree find conditions for all the trajectories near the origin to be closed.

Why Abel equation?

1. Closely related to the corresponding classical problems.

2. (Arguably) the simplest case where these problems remain non-trivial.

3. A lot of encouraging results on both the above problems have been obtained, starting with Lins Neto, Lloyd, Alwash ...

4. Powerful (and partially new in this context) algebraic-analytic tools are applicable.

Analytic - algebraic tools

- Classical and generalized moments, iterated integrals

- Composition algebra

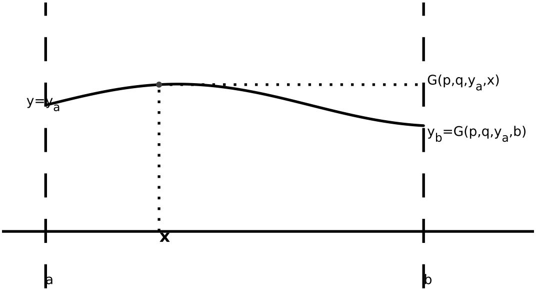
- Algebraic geometry

- Analytic continuation (the main topic of this talk),
i.e. reading out the global properties of $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *from its Taylor coefficients* a_k.

New tools: a short review

First return map

$$y' = p(x)y^2 + q(x)y^3 \qquad (*)$$



$$G(p,q,a,b,y_a) = y_a + \sum_{k=2}^{\infty} v_k(p,q,a,b)\,y_a^k$$

Smale-Pugh: count zeros of $G(y) - y$.

Center-Focus: give conditions for $v_k \equiv 0$ for $k = 2, 3, \ldots,$

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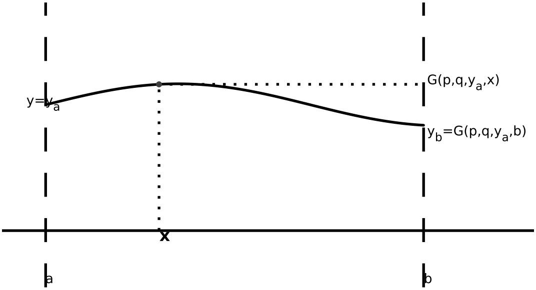

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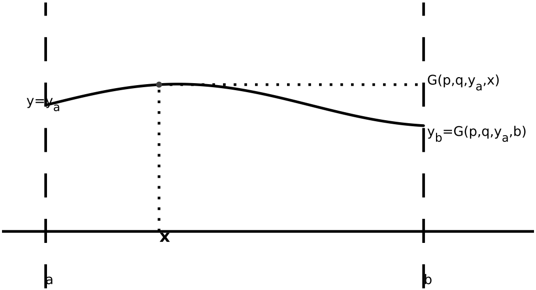

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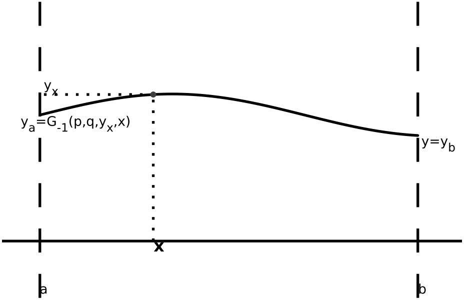
Inverse Poincaré map



$$y_a = G_{-1}(y_x) = y_x + \sum_{k=2}^{\infty} \psi_k(p,q,x)\, y_x^k$$

Poincaré coefficients

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$$y' = p(x)\, y^2 + q(x)\, y^3 \qquad (*)$$

$$\begin{cases} \psi_0(x) & \equiv 0 \\ \psi_1(x) & \equiv 1 \\ \psi_n(0) & = 0 \\ \psi_n'(x) & = -(n-1)\, p(x)\, \psi_{n-1}(x) - (n-2)\, q(x)\, \psi_{n-2}(x) \end{cases}$$

$$\psi_n(x) = \sum \alpha \int p \int q \cdots \int p \cdots \int q$$

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Composition Algebra

$$y' = p(x)y^2 + q(x)y^3 \qquad (*)$$

Composition condition (Alwash-Lloyd, ...)

$P = \int p$ and $Q = \int q$ are said to satisfy *Composition condition* on $[a,b]$ if $\exists W(x)$ with $W(a) = W(b)$ and $\tilde{P}(x), \tilde{Q}(x)$ such that

$$P(x) = \tilde{P}(W(x)),\ Q(x) = \tilde{Q}(W(x))$$

Theorem

Composition $\Longrightarrow$ *Center.*

Conjecture

For p, q - polynomials $\qquad$ *Composition* $\Longleftarrow$ *Center.*

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Current status of Composition conjecture

1. True for small degrees of p and q (Alwash, Lloyd, Blinov, ...,)

2. True for some specific families (Alwash, Llibre, Zoladek, Briskin-Francoise-Yomdin, Brudnyi, ...,)

3. True in rather general situations "up to small correction" (see below)

4. A general result strongly supporting the conjecture has been recently announced by H. Zoladek

Iterated integrals

Poincaré coefficients are linear combinations of iterated integrals

$$\psi_n(x) = \sum \alpha \int p \int q \cdots \int p \cdots \int q$$

Recently a classical Chen's theory of iterated integrals has been applied to the study of the Center conditions for Abel equation.

In particular, the notions of the "universal center" and the "tree composition condition" have been studied (A. Brudnyi, Gine-Grau-Llibre, Brudnyi - Yomdin).

Generalized Moments

$$y' = p(x)\,y^2 + \varepsilon q(x)\,y^3$$

$$G^{-1}(y_b,\varepsilon) = y_b + \sum_{k=2}^{\infty} \psi_k(p,q,b,\varepsilon)\,y_b^k$$

Theorem

$$J(y) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} G^{-1}(y,\varepsilon)\Big|_{\varepsilon=0} = \sum_{k=3}^{\infty} m_k(p,q)\,y^k$$

where the coefficients m_k *are the generalized moments*

$$m_k = \int_a^b P^k(x)\,q(x)\,\mathrm{d}x, \qquad P = \int p$$

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Infinitesimal Smale-Pugh and C-F problems

Infinitesimal Smale-Pugh problem

Count the number of zeros of $J(y)$.

The answer can be obtained by many methods. In particular, the "Petrov trick" works (L. Gavrilov), as well as the Taylor domination method described below.

Infinitesimal Center-Focus problem

Give conditions on p, q, a, b for $m_k \equiv 0,\ k = 0, 1, \ldots$

For p, q polynomials - completely solved by Pakovich and Muzichuk. Difficult result, but the answer is "close to Composition Condition".

Algebraic Geometry applied to C-F

$$y' = p(x)\, y^2 + q(x)\, y^3 \qquad (*)$$

P - projective completion of the space of coefficients p and q, $H \subset P$ the infinite hyperplane.

Theorem

Center equations $\Psi_k = 0$ at infinity (i.e. restricted to H) reduce to the moment equations $m_k = 0$.

Pakovich results + some Algebraic Geometry (study of singularities near infinity) $\Longrightarrow$

Composition set is a "skeleton" of the Center set

A sample specific result

Theorem ([Briskin et al.(2010)])

Assume

1. $q(x)$ *with* $\deg q = d$ *is fixed;*
2. $p = \alpha_m x^m + \alpha_{m+1} x^{m+1} + \cdots + \alpha_n x^n$*;*
3. $[m+1, n+1]$ *does not contain nontrivial multiples of prime divisors of* $d+1$*.*

If Abel equation $(*)$ *has a center then either:*

1. p, q *satisfy Composition Condition; or*
2. p *equals one of the finite number of polynomials* $p_1, \ldots p_s$ *(depending on* q*).*

Analytic continuation

Goal

$$G_{-1}(p,q,y) - y = \sum_{k=2}^{\infty} \psi_k(p,q)\, y^k \qquad \left(\psi_k = \sum \alpha \int p \int q \cdots \int p \cdots \int q \right)$$

$$I(y) = \sum_{k=0}^{\infty} m_k(p,q)\, y^k \qquad \left(m_k = \int_a^b P^k(x)\, q(x)\, \mathrm{d}x \right)$$

Ultimate Goal

Estimate the number of zeros of the function $G_{-1}(y) - y$, based on the properties of its Taylor coefficients.

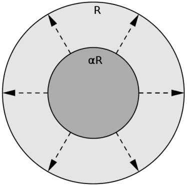
Intermediate goal

The same for the function $I(y)$.

Bernstein classes

Definition

f analytic in D_R and continuous in $\overline{D_R}$ belongs to the *first Bernstein class* $B^1_{K,\alpha,R}$ *if*



$$\frac{\max_{D_R} |f|}{\max_{D_{\alpha R}} |f|} \leq K$$

Theorem ([Van der Poorten(1977)])

The number of zeros of $f \in B^1_{K,\alpha,R}$ *in* $D_{\alpha R}$ *is at most*

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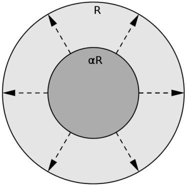

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Definition

$f(x) = \sum_{i=0}^{\infty} a_i x^i$ belongs to the *Bernstein class* $B^2_{C,N,R}$ *if*

$|a_k| R^k \le C \max_{i=0,\dots,N} |a_i| R^i$ (*(N,R,C)- Taylor domination property*)

Theorem (Biernacki, 1932)

If f is p-valent in D_R, i.e. the number of solutions in D_R of $f(z) = c$ for any c does not exceed p, then for $k > p$

$$|a_k| R^k \le (Ak/p)^{2p} \max_{i=0,\dots,p} |a_i| R^i.$$

For $p = 1, a_0 = 0, R = 1$ $|a_k| \le k|a_1|$ (De Branges)

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Bernstein classes

Partial inverse:

Theorem (See, e.g. [Roytwarf and Yomdin(1997)])

If $f \in B^2_{C,N,R}$ then for every $\alpha < 1$ and $R' < R$, $f \in B^1_{K,\alpha,R'}$ with $K = K\left(C, \alpha, \frac{R'}{R}, N\right)$. If $f \in B^1$ then it belongs also to B^2 with appropriate N, C, R.

Corollary

Let $f \in B^2_{C,N,R}$. Then for any $R' < R$, f has at most $M = M\left(N, \frac{R'}{R}, C\right)$ zeros in $D_{R'}$.

Problem: bound zeroes beyond the disk of convergence.

Uniform Taylor domination

$$f_\lambda(z) = \sum_{k=0}^{\infty} a_k(\lambda) z^k, \qquad a_k(\lambda) \in \mathbb{C}[\lambda], \ \lambda \in \mathbb{C}^n$$

For our original problems $\lambda = (p, q, a, b)$ comprises the set of the coefficients of p, q and the end-points a, b. The position of singularities (and hence the radius of convergence $R(\lambda)$) of G_{-1} and of I depend on λ.

Uniform Taylor domination

Characterize families $f_\lambda(z)$ for which

$$|a_k(\lambda)| R^k(\lambda) \leq C \max_{i=0,\dots,N} |a_i(\lambda)| R^i(\lambda)$$

with N and C not depending on λ.

Uniform Taylor domination

Uniform Taylor domination implies a uniform in λ bound on zeroes in any disk $D_{\alpha R(\lambda)}$ for any fixed $\alpha < 1$.

Hope (at present works only in toy examples):

If we control the singularities (for example, for solutions of linear polynomial ODE's) we can cover all the plane with a finite number of such concentric disks, and so to get a global bound on zeroes uniform in λ.

Analytic continuation

Recurrence relations and Taylor domination

Bautin's approach to Taylor domination

Taylor domination and Remez inequalities

Finite determinacy

$$|a_k| R^k \le C \max_{i=0,\dots,N} |a_i| R^i \qquad \text{((N,R,C)- Taylor domination property)}$$

This is an "infinite" condition: have to use ALL the Taylor coefficients. But if these coefficients are produced by a recurrence relation with a finite number of parameters, the problem becomes "finitely determined".

Basic recurrence relations

1. Rational functions
$f(x) = \sum_{k=0}^{\infty} a_k z^k = \frac{P(z)}{Q(z)},\ Q(z) = z^d + c_1 z^{d-1} + \cdots + c_d$. Then for each $k > degP$ we have

$$a_k = \sum_{j=1}^{d} c_j a_{k-j}.$$

2. Solutions of linear ODE's with polynomial coefficients
$Q(z)y^r = P_1(z)y^{r-1} + \cdots + P_r(z)y$, $Q(z)$ as above. Then for each k we have

$$a_k = \sum_{j=1}^{d} [c_j + S_j(\frac{1}{k})] a_{k-j}.$$

Basic recurrence relations

3. Finally, for the Poincaré function we have
$\psi_n'(x) = -(n-1)p(x)\psi_{n-1}(x) - (n-2)q(x)\psi_{n-2}(x)$.

For the first case (Rational functions) - there is $(d, C(d))$ Taylor domination ("Turan's lemma").

For the second case (solutions of Fuchsian ODE's) - there is $(N, C(d))$ Taylor domination, where
$N = C_1(d)\max(\|P_1\|, \dots, \|P_r\|)$. (Very recent result).

For the third case ???

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Generalized Bautin's method

$$f_{\lambda}(x) = \sum_{k=0}^{\infty} a_k(\lambda) x^k, \qquad a_k(\lambda) \in \mathbb{C}[\lambda], \lambda \in \mathbb{C}^n$$

Theorem (Bautin, 1939)

The Bautin ideal $\{a_0(\lambda), \ldots, a_N(\lambda), \ldots\}$ *stabilizes at index* d $\implies$ *for each* λ, $f_{\lambda}(x)$ *has at most* d *zeros in a small neighborhood of the origin.*

Question

Can one explicitly estimate the size of the neighborhood via Taylor domination?

Generalized Bautin's method

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Generalized Bautin's method

$$f_\lambda(x) = \sum_{k=0}^{\infty} a_k(\lambda) x^k, \qquad a_k(\lambda) \in \mathbb{C}[\lambda]$$

$$k > d \Rightarrow a_k(\lambda) = \sum_{i=0}^{d} \varphi_i^k(\lambda) a_i(\lambda)$$

- Estimate $\|\varphi_i^k\|$ in terms of $\|a_k\| \Longrightarrow$ Taylor domination.
- Was done in [Francoise and Yomdin(1997)] based on Hironaka's division theorem.
- Problem: non-uniform! While the radius of convergence $R(\lambda)$ is $\sim \frac{C}{|\lambda|^{K_1}}$, we can bound zeros only in $D_{R'(\lambda)}$ with $R' \sim \frac{1}{|\lambda|^{K_2}}$, $K_2 > K_1$.

Example

$$I_\lambda(y) = \sum_{k=0}^{\infty} m_k(\lambda)\, y^k \qquad \left(m_k(\lambda) = \int_a^b P^k(x)\, q(x)\, \mathrm{d}x \right)$$

Theorem ([Briskin and Yomdin(2005)])

Let $P(x)$ and the degree d of q be fixed, and let R be the radius of convergence of $I_\lambda(y)$. Let $N(P,d,a,b)$ be the Bautin index. Then

$$j > N \Longrightarrow m_j = \sum_{i=0}^{N} c_i^j m_i, \text{ s.t.} \qquad \left| c_i^j \right| \leq C(P,d,a,b) \frac{1}{R^j}$$

Corollary

In this case, for any $R_1 < R$, $I_\lambda(y)$ has at most $Z = Z\left(C, N, \frac{R_1}{R},\right)$ zeros in D_{R_1}. (But C depends on P!)

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Open questions

$$f_\lambda(z) = \sum_{k=0}^{\infty} a_k(\lambda) z^k, \qquad a_k(\lambda) \in \mathbb{C}[\lambda]$$

- Identify "natural" families $f_\lambda(z)$ for which the global analytic continuation is feasible
- Find the radius of convergence $R(\lambda)$
- Find positions and types of singularities
- Give conditions for a uniform Taylor domination

Conjecture

The answers can be given in "algebraic terms", through certain "Bautin-type" ideals (see [Yomdin(1998)] for some very initial results).

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Remez-type inequalities

Theorem (Remez, 1936)

Let $p(x)$ be a real polynomial of degree d, $I \subset \mathbb{R}$ an interval and $B \subseteq I$ a set of positive measure. Then

$$\max_I |p(x)| \le \left(\frac{4\mu(I)}{\mu(B)}\right)^d \max_B |p(x)|$$

Theorem (Turan-Nazarov inequality)

Let $p(x) = \sum_{i=1}^d a_i \mathrm{e}^{\lambda_i x}$ with $\lambda_i \in \mathbb{C}$. Then

$$\max_I |p(x)| \le \mathrm{e}^{\mu(I)\max_i |\Re\lambda_i|} \left(\frac{c \cdot \mu(I)}{\mu(B)}\right)^{d-1} \max_B |p(x)|$$

Both can be extended to discrete and finite sets B ([Yomdin(2011), Friedland and Yomdin(2011)]).

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$$\max_I |p(x)| \leq \left(\frac{4\mu(I)}{\mu(B)}\right)^d \max_B |p(x)|$$

Theorem (Turan-Nazarov inequality)

Let $p(x) = \sum_{i=1}^d a_i \mathrm{e}^{\lambda_i x}$ with $\lambda_i \in \mathbb{C}$. Then

$$\max_I |p(x)| \leq \mathrm{e}^{\mu(I)\max_i |\Re\lambda_i|} \left(\frac{c \cdot \mu(I)}{\mu(B)}\right)^{d-1} \max_B |p(x)|$$

Both can be extended to discrete and finite sets B ([Yomdin(2011), Friedland and Yomdin(2011)]).

Remez-type inequalities

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Main result

$$m_k(\alpha) = \int_0^\alpha x^k f(x)\, \mathrm{d}x$$

Theorem

Assume that $f(x)$ has at most d sign changes and satisfies

$$\max_{[0,\alpha]} |f(x)| \le K \left(\frac{\alpha}{\mu(\Omega)} \right)^d \max_{\Omega} |f(x)|$$

for any measurable $\Omega \subset [0,\alpha]$. Then

$$\max_{[0,\alpha]} |f(x)| \le \frac{1}{\alpha} K \cdot C(d) \max_{i=0,\ldots,d} |m_i| \, \alpha^{-i}$$

Main result

$$\max_{[0,\alpha]} |f(x)| \leq \frac{1}{\alpha} K \cdot C_1(d) \max_{i=0,\dots,d} |m_i| \alpha^{-i}$$

Integrating with x^k we get immediately

$$m_k(\alpha) = \int_0^\alpha x^k f(x)\,\mathrm{d}x \leq \int_0^\alpha x^k\,\mathrm{d}x \frac{1}{\alpha} K \cdot C_1(d) \max_{i=0,\dots,d} |m_i| \alpha^{-i} =$$

$$= \alpha^k C(K,d) \max_{i=0,\dots,d} |m_i| \alpha^{-i}.$$

Corollary

The sequence $\{m_k\}$ has the domination property with $R = \alpha^{-1}$, $N = d$ and C depending only on K and d.

Main result

Given a family $f_\beta(x)$ with the same number of sign changes d and the same Remez constant K for each β, put $\lambda = (\alpha, \beta)$.

$$g_\lambda(y) = \sum_{k=0}^{\infty} m_k(\lambda) y^k, \quad m_k(\lambda) = \int_0^\alpha x^k f_\beta(x)\,\mathrm{d}x$$

(The radius of convergence $R = \alpha^{-1}$).

Theorem

The family $g_\lambda(y)$ has the uniform Taylor domination property with $R = \alpha^{-1}$, $N = d$ and C depending only on d, K.

Reformulation

Number of zeros of g_λ inside its disk of convergence can be uniformly in λ bounded in terms of d, K.

Another point of view

$$\max_{[0,\lambda]} |f(x)| \leq \frac{1}{\lambda} K \cdot C(d) \max_{i=0,\ldots,d} |m_i| \lambda^{-i}$$

$$M_f(s) = \int_a^b x^s f(x)\, \mathrm{d}x \qquad \text{Mellin transform}$$

Corollary

The Mellin transform satisfies a “discrete Remez-type inequality”

$$\left| M_f(s) \right| \leq b^s \cdot C^* \cdot K \cdot \max_{s_i \in \{0,1,\ldots,d\}} \left| M_f(s_i) \right|$$

Proof idea

- Build an auxiliary polynomial $P(x)$ with the same sign pattern as $f(x)$
- Consider the integral $\int Pf$
  - $\int Pf \leq C_1 \cdot \lambda^d \cdot \max_{0,\dots,d} |m_i| R^i$
  - Find a "big enough" $\Omega \subset [0, \lambda]$ on which f is small
- Apply Remez inequality for f

Infinitesimal Smale-Pugh

$$I_\lambda(y) = \sum_{k=0}^{\infty} m_k(\lambda) y^k \qquad \left(m_k(\lambda) = \int_0^\lambda P^k(x) q(x) \, \mathrm{d}x \right)$$

$$m_k(\lambda) = \int_\gamma s^{k-1} g(s) \, \mathrm{d}s$$

$$g(s) = \sum_{\text{branches of } P^{-1}} q\left(P^{-1}(s)\right) \qquad \text{semi-algebraic, no poles}$$

Fact

# of sign changes of $g(s) \le d = d(\deg P, \deg q)$.

Conjecture

$g(s)$ satisfies Remez-type inequality with K depending only on $\deg P, \deg q$ (OK if g is a polynomial).

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