High-Order Taylor Model Enclosures of Invariant Manifolds of ODEs

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Invariant Sets and Manifolds in ODEs

- Vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ has hyperbolic critical point at origin
- $\Phi_t(x)$ is flow of F such that $\Phi_0(x) = x$.
- Stable set: $M_s \subset \mathbb{R}^n$

$$\forall x \in M_s \ \lim_{t\to\infty} \Phi_t(x) = 0.$$

- Unstable set: $M_u \subset \mathbb{R}^n$

$$\forall x \in M_u \ \lim_{t\to\infty} \Phi_{-t}(x) = 0.$$

Invariant Sets and Manifolds in ODEs

Invariant set of ODE is identical to invariant set of time 1 map $\Phi_1(x)$.

- Stable set: $M_s \subset \mathbb{R}^n$

$$\forall x \in M_s \ \lim_{k\to\infty} \Phi_1^k(x) = \Phi_k(x) = 0.$$

- Unstable set: $M_u \subset \mathbb{R}^n$

$$\forall x \in M_u \ \lim_{k\to\infty} \Phi_{-1}^k(x) = \Phi_{-k}(x) = 0.$$

Invariant Sets and Manifolds in ODEs

Theorem (Invariant Manifold Theorem)

If $F(0) = 0$, *and*

$$DF(0) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with $\lambda_1, \ldots, \lambda_k > 0$ *and* $\lambda_{k+1}, \ldots, \lambda_n < 0$ *then the invariant sets are manifolds of dimension* k *and* $n - k$*.*
These manifolds are tangent to the linear space spanned by $x_1, \ldots, x_k$ *and* $x_{k+1}, \ldots, x_n$ *at the origin.*

Goal

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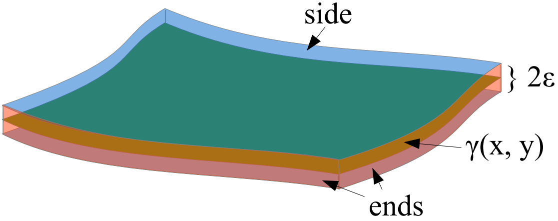
We want to obtain sharp enclosures of the invariant manifolds.

What exactly does that mean?



1. local manifold must not intersect "sides"
2. local manifold must intersect all "ends"

Goal

Goal

We want to obtain sharp enclosures of the invariant manifolds.

General recipe:

1. Construct polynomial approximation of invariant manifold
2. Add thin, heuristic error bound to approximation
3. Verify that no manifold sticks out at the sides
4. Verify that it does come out at the ends

Invariant Manifold Approximation

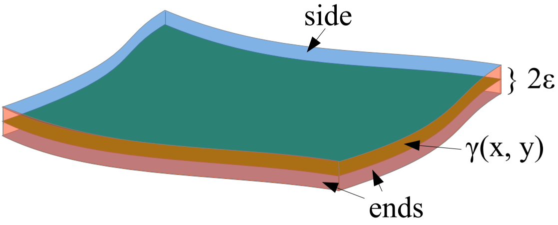
Polynomial approximation is obtained in an order-by-order construction:

1. Taylor expansion of flow around 0
2. Insert small time $t_0 \Rightarrow$ time t_0 map $\Phi_{t_0}(x)$
3. Construct invariant polynomial $\gamma(s) : \mathbb{R}^k \to \mathbb{R}^n$, mapping polynomial $P : \mathbb{R}^k \to \mathbb{R}^k$
   Condition:
   $$\Phi_{t_0}(\gamma(s)) = \gamma(P(s))$$
   P is chosen such that γ on stable/unstable subspace $\mathbb{R}^k$ is identity.

All steps non-verified, can do whatever you want (almost).

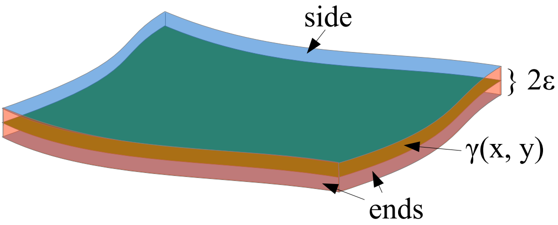
Unstable Manifold Verification for 2D manifold in 3D

Unstable manifold in x-y plane:
Thicken in z (stable) direction by $\varepsilon > 0$ (typically $\varepsilon \approx 10^{-12}$)



$$\Gamma(x, y, t) = \begin{pmatrix} x \\ y \\ \gamma_z(x, y) + \varepsilon t \end{pmatrix}$$

Unstable Manifold Verification for 2D manifold in 3D



Theorem

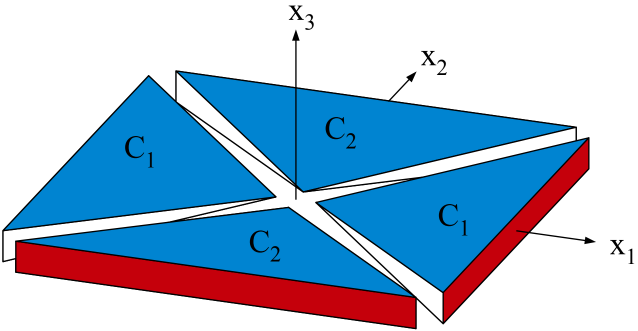
Local unstable manifold W^u_{loc} does not intersect sides of Γ if F on sides points inwards.

For stable manifold just use $-F$

Unstable Manifold Verification for 2D manifold in 3D

Does manifold leave Γ?

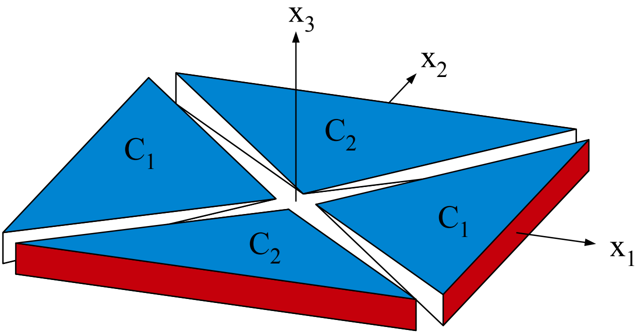
Need different enclosure C to show this.



- Thickness of red ends $\approx 0.1 \Rightarrow$ very rough enclosure
- Contains open neighborhood of local invariant manifold around origin

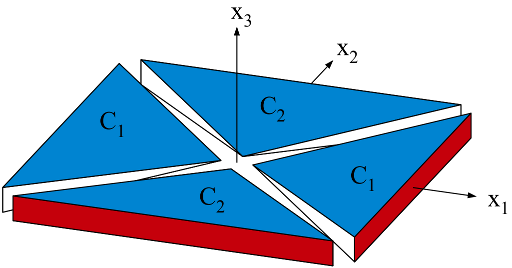
Unstable Manifold Verification for 2D manifold in 3D

Want to show: Every point in $C\backslash\{0\}$ leaves C.



- Can't use Γ, because Γ contains stable manifold!
- $\Rightarrow$ Must avoid local stable manifold in C.

Unstable Manifold Verification for 2D manifold in 3D



If

1. F points into C on sides (blue)
2. F points out of C on ends (red)
3. F has positive radial component in x-y plane in $C \backslash \{0\}$:

$$F_x(\vec{p}) \cdot p_x + F_y(\vec{p}) \cdot p_y > 0 \qquad \forall \vec{p} \in C \backslash \{0\}$$

then every point in $C \backslash \{0\}$ leaves C through the ends.

Unstable Manifold Verification for 2D manifold in 3D

Numerical Problem

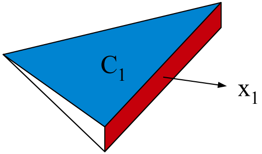
Since $F(0) = 0$ is continuous

$$F_x(\vec{p}) \cdot p_x + F_y(\vec{p}) \cdot p_y \to 0$$

as $\vec{p} \to 0$.
Simple bounding not possible to show $F_x(\vec{p}) \cdot p_x + F_y(\vec{p}) \cdot p_y > 0$.

Unstable Manifold Verification for 2D manifold in 3D

$$C(s, t_1, t_2) = s \cdot \begin{pmatrix} 1 \\ t_1 \\ \vartheta t_2 \end{pmatrix}$$



$$\begin{aligned} F(C(s,t_1,t_2)) &= F\left(s \cdot \begin{pmatrix} 1 \\ t_1 \\ \vartheta t_2 \end{pmatrix}\right) \\ &= \int_0^s DF\left(\hat{s} \cdot \begin{pmatrix} 1 \\ t_1 \\ \vartheta t_2 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 \\ t_1 \\ \vartheta t_2 \end{pmatrix} \, d\hat{s} + \underbrace{F(C(0,t_1,t_2))}_{=0} \end{aligned}$$

Unstable Manifold Verification for 2D manifold in 3D

For $s \neq 0$:

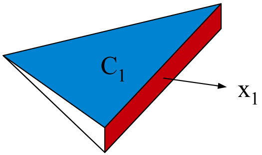
$$\begin{aligned} \frac{1}{s} F(C(s,t_1,t_2)) &= \frac{1}{s}\int_0^s DF\left(\hat{s}\cdot\begin{pmatrix}1\\ t_1\\ \vartheta t_2\end{pmatrix}\right)\cdot\begin{pmatrix}1\\ t_1\\ \vartheta t_2\end{pmatrix}\, d\hat{s} \\ &\approx \frac{1}{s}\int_0^s DF(0)\cdot\begin{pmatrix}1\\ t_1\\ \vartheta t_2\end{pmatrix}\, d\hat{s} \\ &\approx \frac{1}{s}\int_0^s \begin{pmatrix}\lambda_1\\ t_1\lambda_2\\ \vartheta t_2\lambda_3\end{pmatrix}\, d\hat{s} \\ &\approx \begin{pmatrix}\lambda_1\\ t_1\lambda_2\\ \vartheta t_2\lambda_3\end{pmatrix} \end{aligned}$$

Can be evaluated directly in Taylor Model arithmetic!

(The exact expression, not the approximation!)

Unstable Manifold Verification for 2D manifold in 3D

$$C(s, t_1, t_2) = s \cdot \begin{pmatrix} 1 \\ t_1 \\ \vartheta t_2 \end{pmatrix}$$



For $s \neq 0$:

$$\begin{aligned} & F_x(\vec{p}) \cdot p_x + F_y(\vec{p}) \cdot p_y \\ = \quad & F_x(C(s,t_1,t_2)) \cdot C_x(s,t_1,t_2) + F_y(C(s,t_1,t_2)) \cdot C_y(s,t_1,t_2) \\ = \quad & s^2 \underbrace{\left(\frac{1}{s} F_x(C(s,t_1,t_2)) + \frac{1}{s} F_y(C(s,t_1,t_2)) \cdot t_1 \right)}_{>K>0} \\ \approx \quad & s^2 \left(\lambda_1 + \lambda_2 \cdot t_1^2 \right) \end{aligned}$$

The Lorenz Equation

The Lorenz vector field F is given by the equations

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= (\rho - x_3)x_1 - x_2 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \end{aligned}$$

where ρ, σ, β are parameters. In the classical Lorenz equations, $\rho = 28, \sigma = 10, \beta = 8/3$.

The Lorenz Equation

At the origin, the Lorenz System has a hyperbolic critical point with eigenvalues

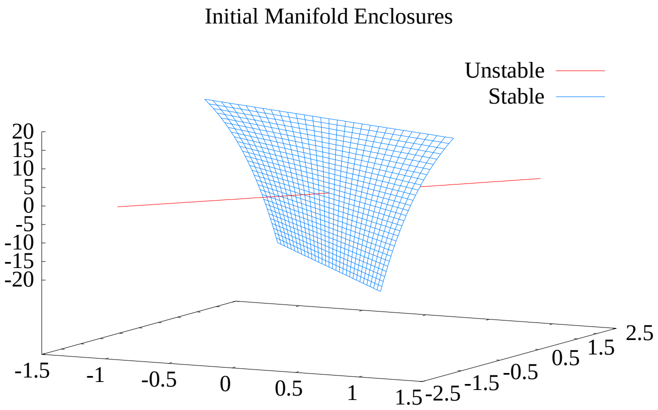
$$\lambda_1 \approx 11.8$$

$$\lambda_2 \approx -22.8$$

$$\lambda_3 (= \beta = 8/3) \approx -2.67$$

Big difference between λ_2 and λ_3!

Initial Manifold Enclosures



$\eta_1 = \eta_2 = 1, \eta_3 = 20$

Initial Stable Manifold

$\eta_2 \backslash \eta_3$	1	5	10	15	20
1	$5 \cdot 10^{-14}$	$1 \cdot 10^{-13}$	$1 \cdot 10^{-12}$	$1 \cdot 10^{-9}$	$1 \cdot 10^{-7}$
0.5	$1 \cdot 10^{-14}$	$5 \cdot 10^{-14}$	$1 \cdot 10^{-13}$	$5 \cdot 10^{-11}$	$5 \cdot 10^{-8}$
0.2	$5 \cdot 10^{-15}$	$1 \cdot 10^{-14}$	$5 \cdot 10^{-14}$	$1 \cdot 10^{-11}$	$6 \cdot 10^{-9}$

Table: Thickening of the manifold enclosure (in diagonalized coordinates) for which verification is successful for various values of η_2 and η_3.

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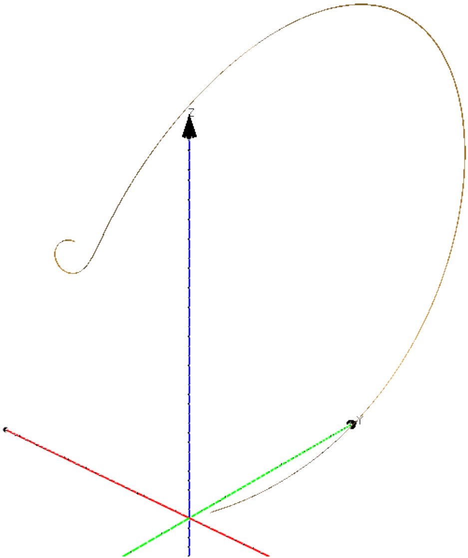
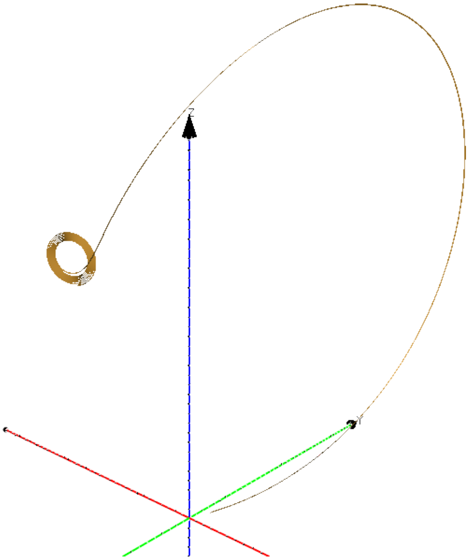
Unstable Manifold



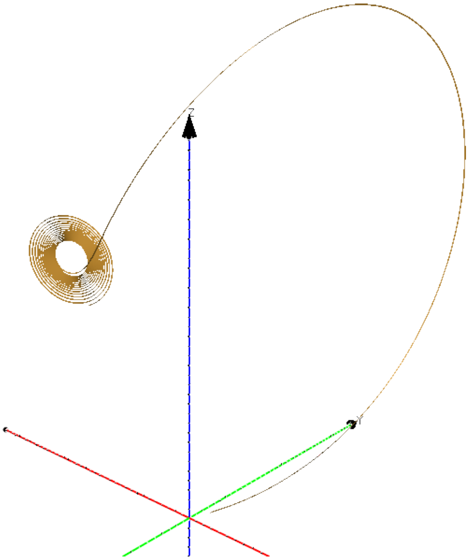
Time $t = 1$

Unstable Manifold



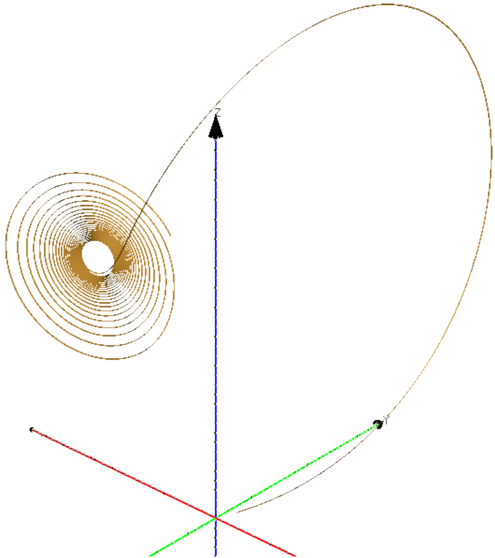
Time $t = 5$

Unstable Manifold



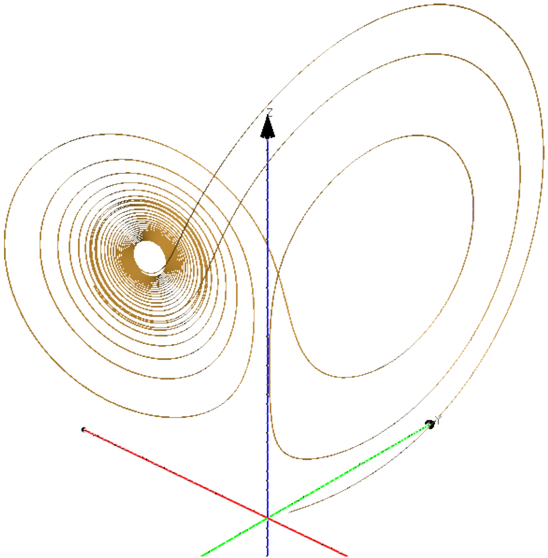
Time $t = 10$

Unstable Manifold



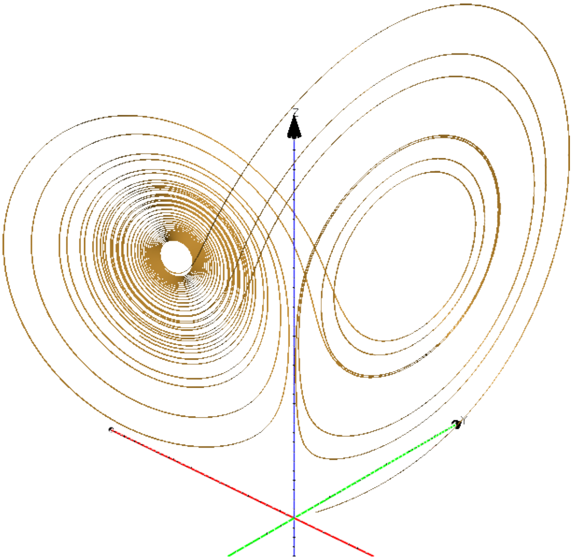
Time $t = 15$

Unstable Manifold



Time $t = 20$

Unstable Manifold



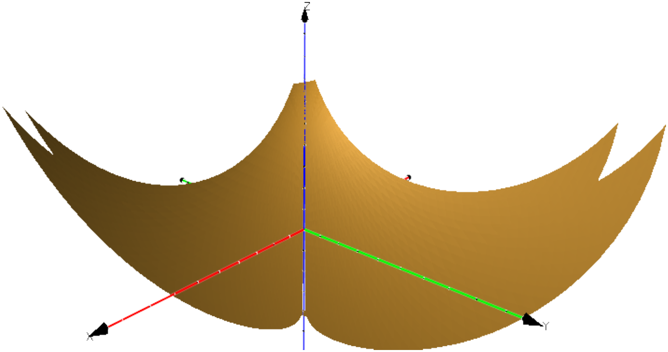
Time $t = 28$

Unstable Manifold

t	1	2	3	4	5
Approximate Length	126	144	166	189	214
Maximum Error	$2 \cdot 10^{-11}$	$3 \cdot 10^{-11}$	$5 \cdot 10^{-11}$	$1 \cdot 10^{-10}$	$2 \cdot 10^{-10}$
# of Taylor Models	44	72	106	140	175
t	10	15	20	25	28
Approximate Length	390	710	1156	1592	1879
Maximum Error	$2 \cdot 10^{-9}$	$7 \cdot 10^{-8}$	$5 \cdot 10^{-5}$	$7 \cdot 10^{-4}$	$2 \cdot 10^{-2}$
# of Taylor Models	353	537	718	1091	9196

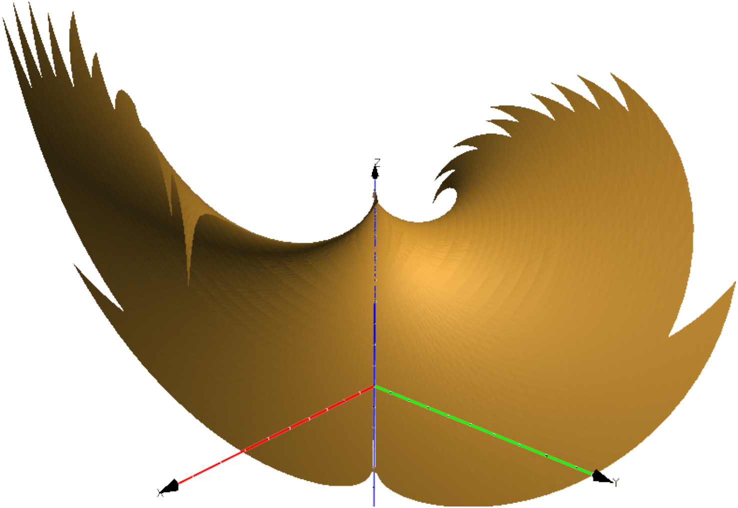
Table: Approximate length, maximum error, and number of Taylor Models covering the unstable manifold after propagating for the given time t.

Stable Manifold



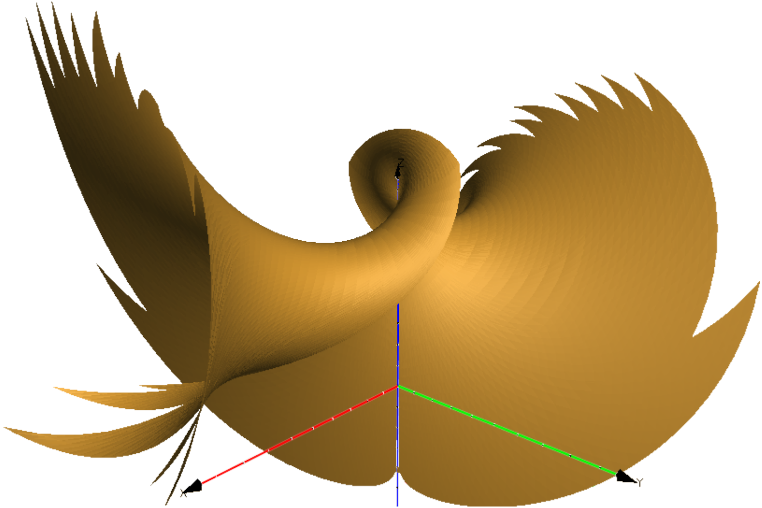
Time $t = 0.2$

Stable Manifold



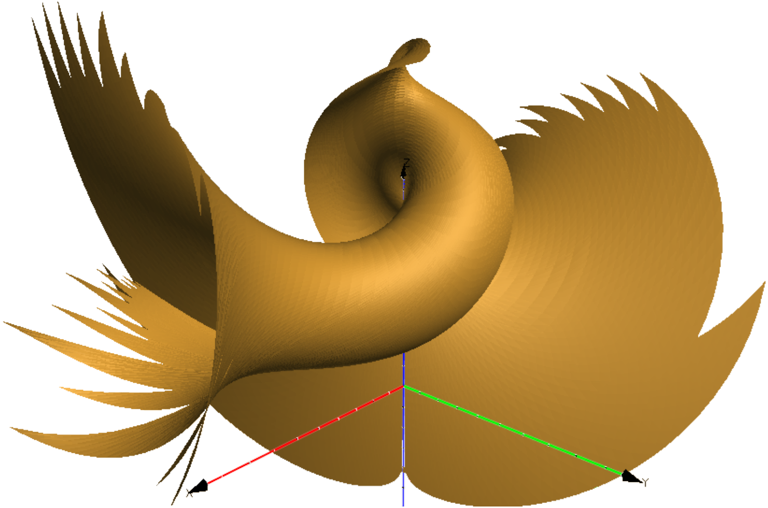
Time $t = 0.3$

Stable Manifold



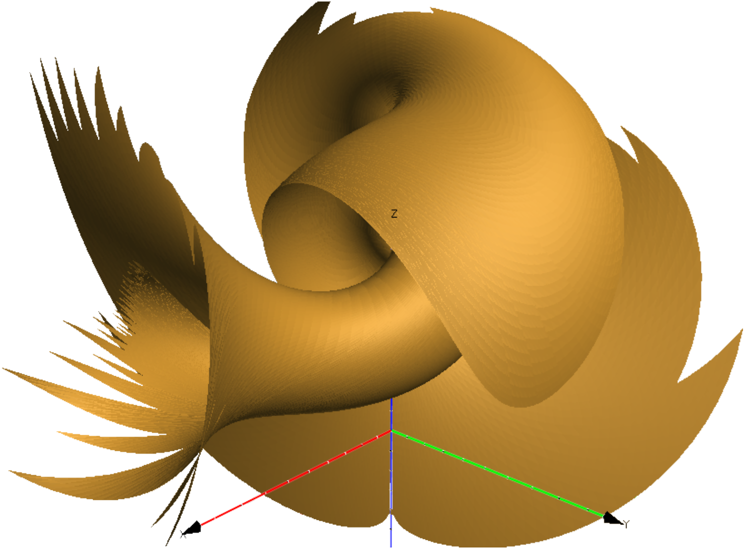
Time $t = 0.4$

Stable Manifold



Time $t = 0.5$

Stable Manifold



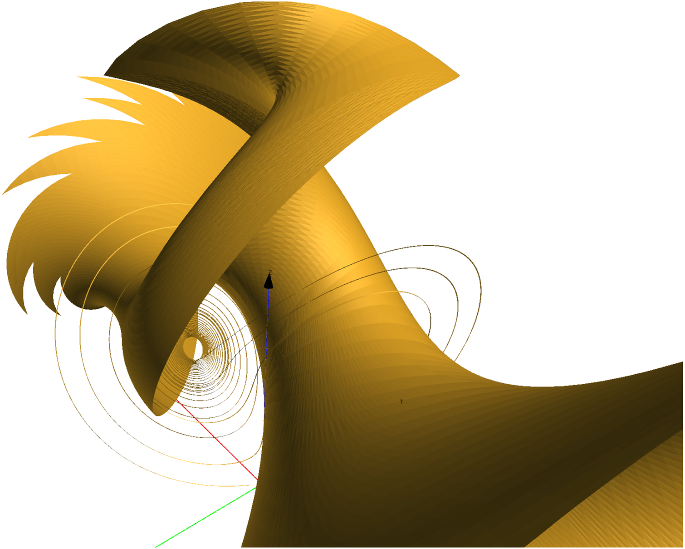
Time $t = 0.6$

Stable Manifold

t	0.2	0.3	0.4	0.5	0.6
Approximate Area A	2360	6307	8112	10192	14862
Maximum Error e_{max}	$3.7 \cdot 10^{-7}$	$3.7 \cdot 10^{-7}$	$3.7 \cdot 10^{-7}$	$1.5 \cdot 10^{-6}$	$7.9 \cdot 10^{-6}$
Volume V	$8.7 \cdot 10^{-4}$	$2.3 \cdot 10^{-3}$	$3.0 \cdot 10^{-3}$	$1.5 \cdot 10^{-2}$	$1.2 \cdot 10^{-1}$
# of Taylor Models	70	145	233	473	2469
# of Intervals	$1.7 \cdot 10^{16}$	$4.6 \cdot 10^{16}$	$5.9 \cdot 10^{16}$	$4.5 \cdot 10^{15}$	$2.4 \cdot 10^{14}$

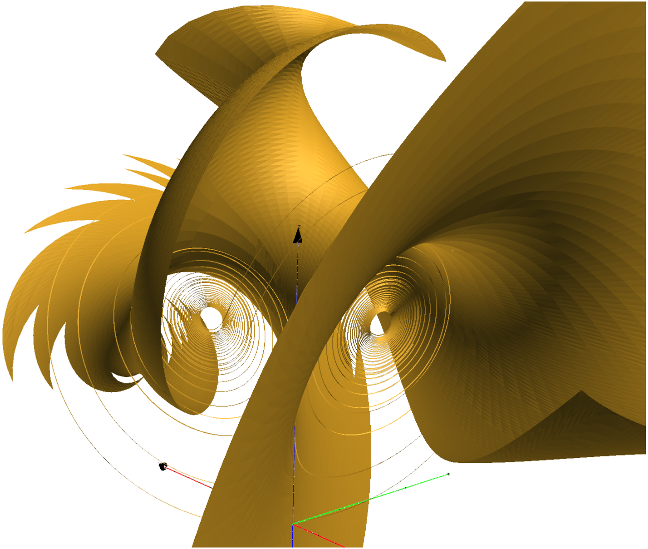
Table: Approximate area, maximum error, volume, and number of Taylor Models covering the integrated half of the stable manifold within the box $B = [-50, 50] \times [-50, 50] \times [-20, 90]$ after propagating for the given time t backwards through the Lorenz.

Both Manifolds



Accuracy: $5 \cdot 10^{-5}$

Both Manifolds



Accuracy: $5 \cdot 10^{-5}$

Thank You.

Thank you for your attention.

Questions?