Rigorous Fixed Point Enclosures and an Application to Beam Transfer Maps

Alexander Wittig

Department of Physics and Astronomy, Michigan State University
Department of Mathematics, Michigan State University
East Lansing, MI, 48824

7th Taylor Model Workshop, Key West 2011

Sheldon's conjecture

The Hénon map:

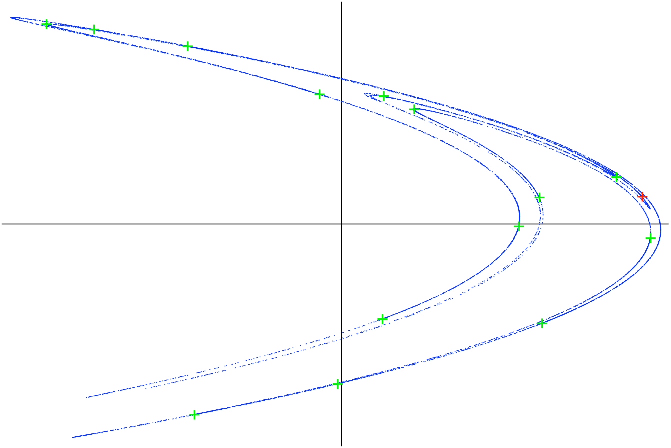
$$\mathcal{H} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 + y - Ax^2 \\ Bx \end{pmatrix}$$

$A = 1.4$ and $B = 0.3$ in standard Hénon map.

Sheldon says...

For $A = 1.422$ and $B = 0.3$ there's an attracting periodic point near $x = -0.0146599433606556$ and $y = -0.294827857184849$.

Sheldon's conjecture



Hénon Map with $A = 1.422$, $B = 0.3$. Periodic point of order 15 (non-verified).

Sheldon's conjecture

What order is it?

f : Hénon map with $A = 1.422$ and $B = 0.3$

$f^{(00)}$:	$x = -0.0869282203452\underline{939}$	$y = 0.23915367507167\underline{47}$
$f^{(15)}$:	$x = -0.0869282203454\underline{442}$	$y = 0.2391536750716\underline{964}$
$f^{(30)}$:	$x = -0.0869282203452\underline{939}$	$y = 0.23915367507167\underline{47}$
$f^{(45)}$:	$x = -0.0869282203454\underline{442}$	$y = 0.2391536750716\underline{964}$
$f^{(60)}$:	$x = -0.0869282203452\underline{939}$	$y = 0.23915367507167\underline{47}$
$\vdots$	$\vdots$	$\vdots$

Sheldon's conjecture

Theorem

In the Henon map

$$\mathcal{H} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 + y - Ax^2 \\ Bx \end{pmatrix}$$

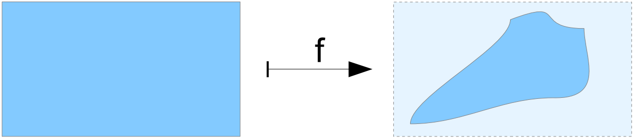
with $A = 1.422$ *and* $B = 0.3$ *there is a periodic point of order 15 in*

$$\begin{aligned} x &= 1.19576936506755033604110098396554 89 \\ &\quad 352337235594806801053003707350839 68^{32853}_{10139}, \\ y &= 0.05050761649556464888828848017561 61 \\ &\quad 0168414268082837062814105551657822 9^{4397960}_{1531331}. \end{aligned}$$

Attractive Fixed Point

Theorem (Schauder Fixed Point Theorem)

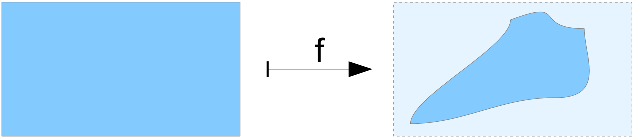
Let $K \subset \mathbb{R}^n$ be a non-empty, compact, and convex set. Then, any continuous map $f : K \to K$ has a xed point in K.



Attractive Fixed Point

Theorem (Schauder Fixed Point Theorem)

Let $K \subset \mathbb{R}^n$ be a non-empty, compact, and convex set. Then, any continuous map $f : K \to K$ has a xed point in K.



1. Approximate attractive fixed point,
2. place initial box K around it,
3. show $\mathcal{H}^{15}(K) \subset K$.

Banach's Theorem

Theorem

Let K be a non-empty, complete metric space. Then, any contraction $f : K \to K$ has a unique xed point in K.

Reminder: A map f is a contraction if $\exists\ 0 \leqslant K < 1$ such that

$$|f(x) - f(y)| < K\,|x - y|$$

$\forall\ x, y \in K$

Combining both theorems:

Corollary

Let $K \subset \mathbb{R}^n$ be a convex, compact set, and $f : K \mapsto K$ be continuously di erentiable with $|Df(x)| < 1 \; \forall x \in K$.
Then f has a unique xed point in K.

Need to verify numerically:

1. Find small box B around fixed point and show it is mapped into itself.
2. Bound $|Df|$ over B and show it is less than 1.

Attractive Fixed Point: Results

Taylor Model Enclosure

$$x = 1.1957^{80721557596}_{58008577504}$$

$$y = 0.050^{52194963414509}_{49328335698421}$$

	Taylor Models	HP Taylor Models	HP Intervals
Halfwidth	10^{-5}	10^{-60}	10^{-70}
Precision	16	75	75
Boxes	1	1	$70,000,000$
Time	<1 sec.	~ 1 sec.	130 min.

Attractive Fixed Point: Results

High Precision Taylor Model Enclosure

$$x = 1.195769365067550336041100983965548935233723559480680105300 3^{7188}_{6812}$$

$$y = 0.050507616495564648888288480175616101684142680828370628141 05^{797}_{403}$$

	Taylor Models	HP Taylor Models	HP Intervals
Halfwidth	10^{-5}	10^{-60}	10^{-70}
Precision	16	75	75
Boxes	1	1	$70,000,000$
Time	<1 sec.	~ 1 sec.	130 min.

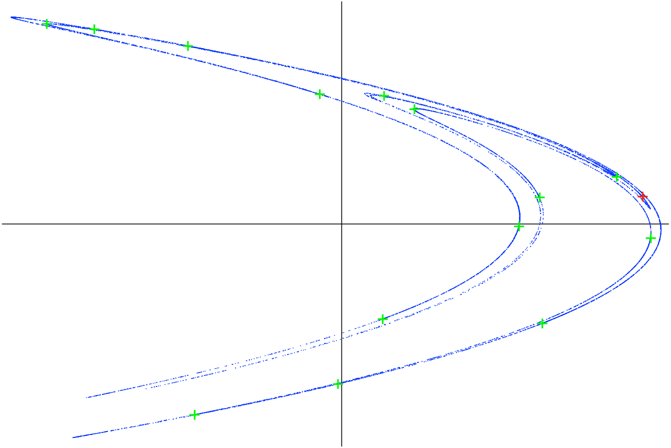
Attractive Fixed Point: Results

High Precision Interval Enclosure

$$
\begin{aligned}
x &= 1.1957693650675503360411009839655489 \\
&\quad 352337235594806801053003707350839 68^{32853}_{10139} \\
y &= 0.0505076164955646488882884801756161 \\
&\quad 016841426808283706281410555165782 29^{4397960}_{1531331}
\end{aligned}
$$

	Taylor Models	HP Taylor Models	HP Intervals
Halfwidth	10^{-5}	10^{-60}	10^{-70}
Precision	16	75	75
Boxes	1	1	$70,000,000$
Time	<1 sec.	~ 1 sec.	130 min.

Result



Hénon Map with $A = 1.422$, $B = 0.3$. Periodic point of order 15 (verified).

Goals

Global fixed point finder should

1. work for all types of fixed points (repelling, hyperbolic, elliptic, as well as attracting)
2. find verified fixed point enclosures automatically
3. find *all* fixed points in given area

Rigorous Root & Fixed Point Finder

For any regular matrix A:

$$f(x) = x \iff A \cdot f(x) = A \cdot x$$

As fixed point problem:

$$f(x) = x \iff A \cdot (f(x) - x) + x = x$$

Idea

Choose A so $A \cdot (f(x) - x) + x$ has a strongly contracting fixed point x_0.

Rigorous Root & Fixed Point Finder

$$A = -\left(Df\left(x_0\right) - I\right)^{-1}$$

Derivative of

$$A \cdot \left(f\left(x\right) - x\right) + x$$

at fixed point x_0:

$$A \cdot \left(Df\left(x_0\right) - I\right) + I = -\left(Df\left(x_0\right) - I\right)^{-1} \cdot \left(Df\left(x_0\right) - I\right) + I = -I + I = 0$$

Rigorous Root & Fixed Point Finder

$$A = -\left(Df\left(x_0\right) - I\right)^{-1}$$

Derivative of

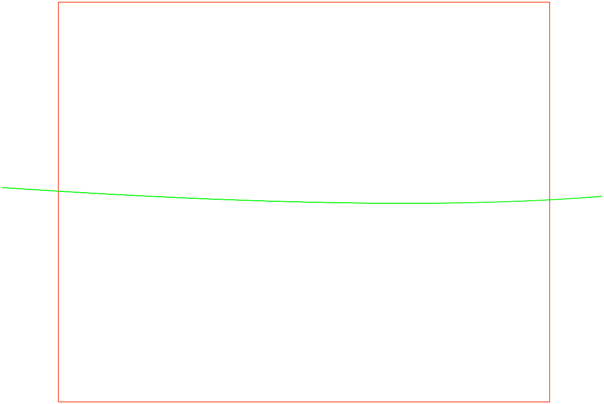
$$A \cdot \left(f\left(x\right) - x\right) + x$$

at fixed point x_0:

$$A \cdot \left(Df\left(x_0\right) - I\right) + I = -\left(Df\left(x_0\right) - I\right)^{-1} \cdot \left(Df\left(x_0\right) - I\right) + I = -I + I = 0$$

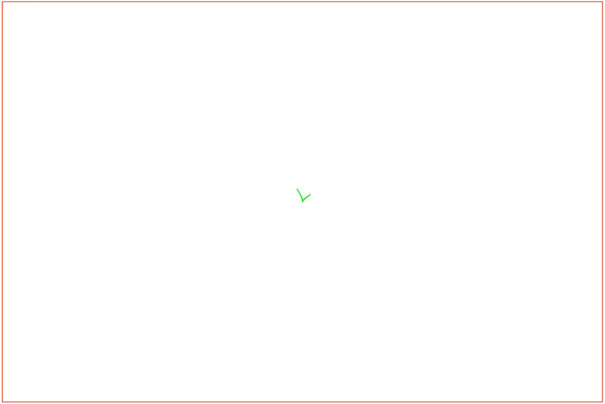
- Similar to Newton method applied to $f(x) - x$,
- A does not have to be rigorous, any regular A will do,
- A above is best choice yielding superlinear contraction.

Effect of Preconditioning



Red box: $[-10^{-4}, 10^{-4}]^2$ around fixed point. Green box: image of red box under H^{15}.

Effect of Preconditioning



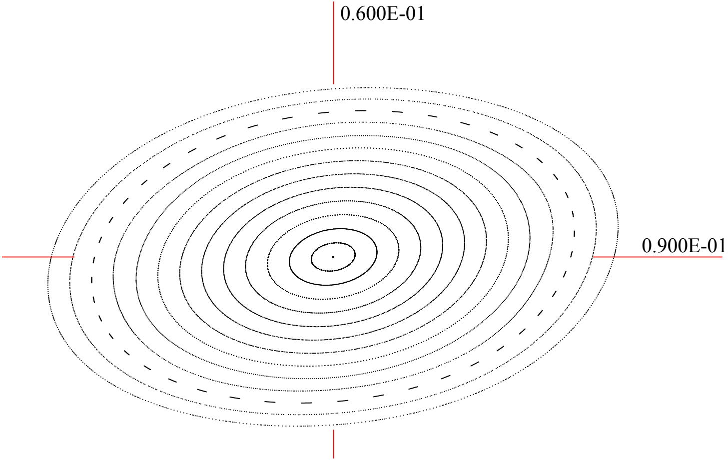
Red box: $[-10^{-4}, 10^{-4}]^2$ around fixed point. Green box: image of red box under preconditioned H^{15}.

Rigorous Root & Fixed Point Finder

Basic algorithm for Global Fixed Point Finder:

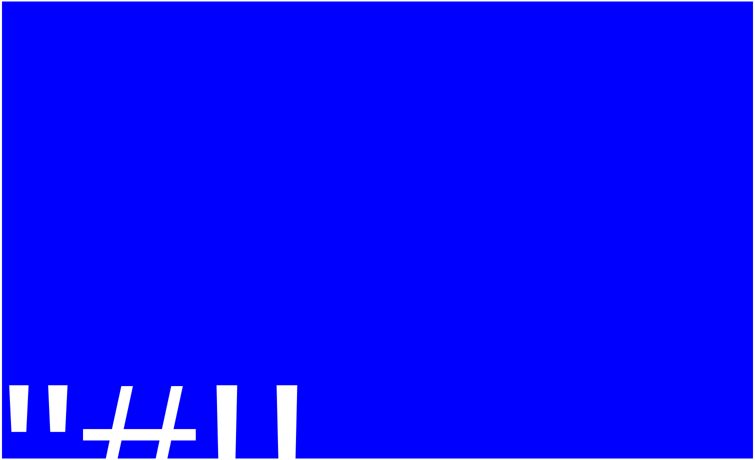
1. Start with region of interest on stack.
2. Test top box on stack for fixed point.
   - No FP: discard,
   - FP found: keep box as result (or split if enclosure too big),
   - Unknown: split box.
3. Yields verified enclosures of all fixed points in area of interest.

Fixed Points in Beam Physics



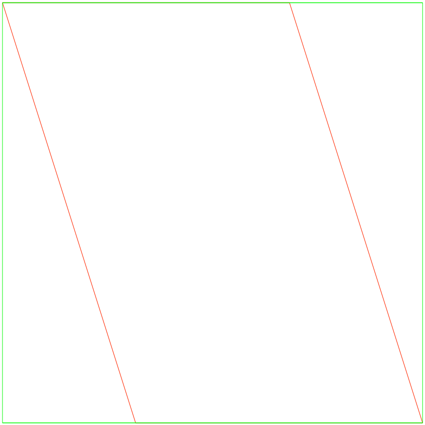
x-a tracking picture of Tevatron map

Fixed Points in Beam Physics



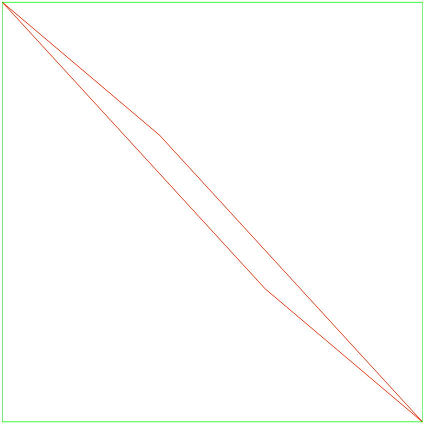
x-a tracking picture of Tevatron map

Taylor Models vs. Intervals



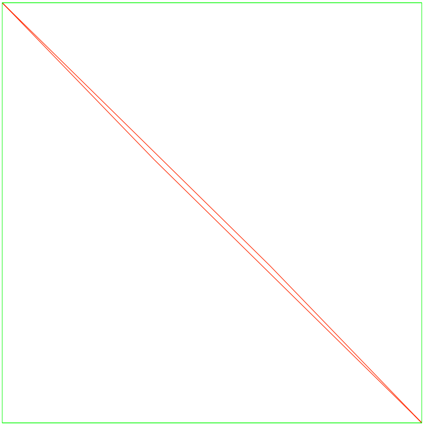
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



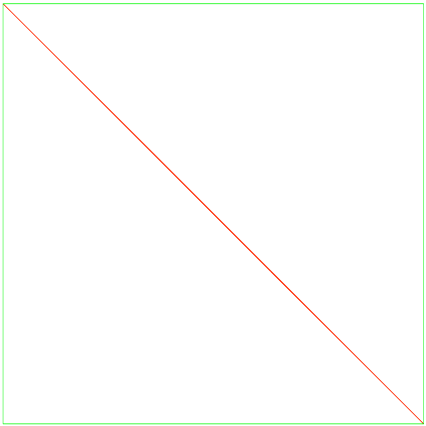
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



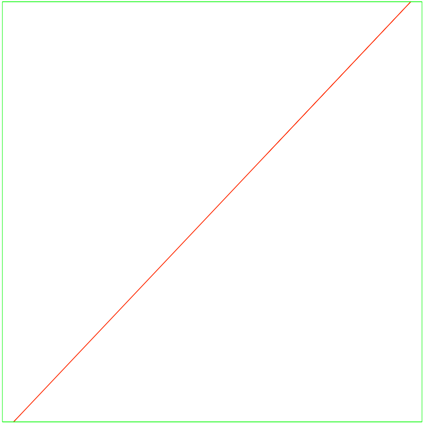
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



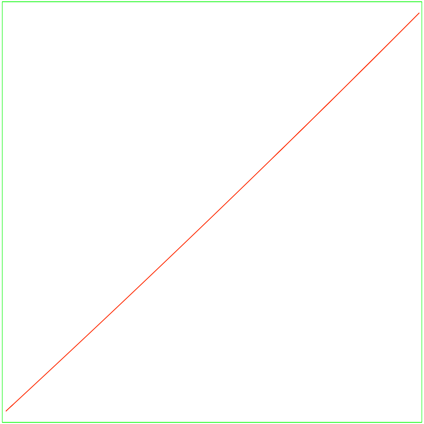
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



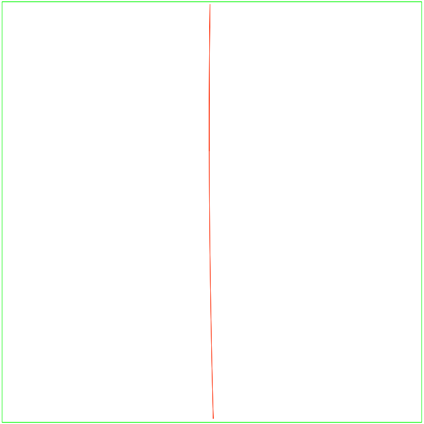
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



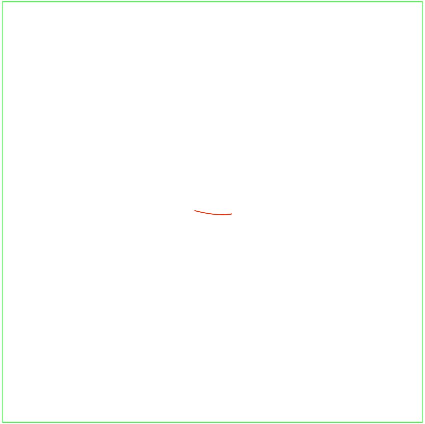
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



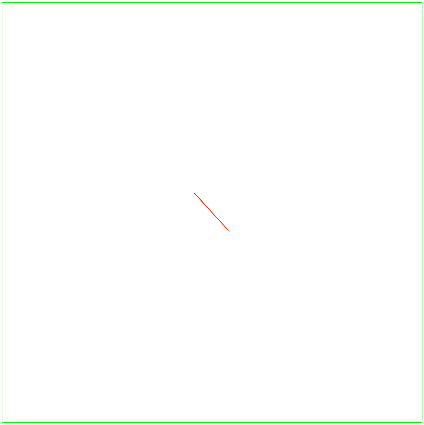
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



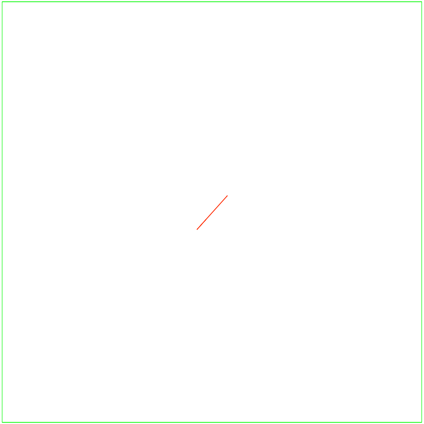
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



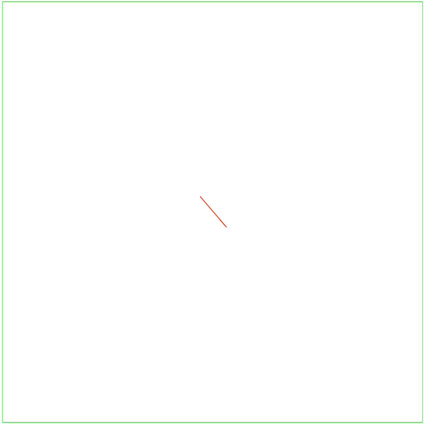
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



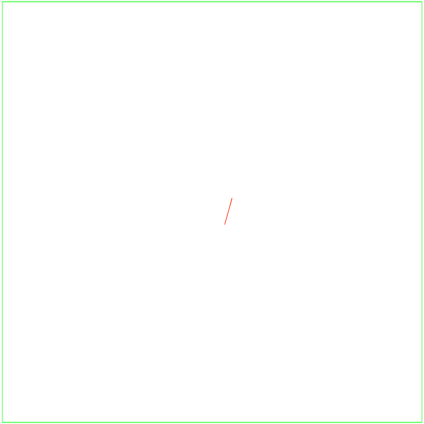
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



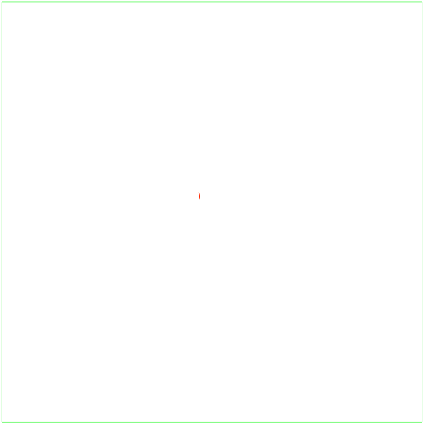
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



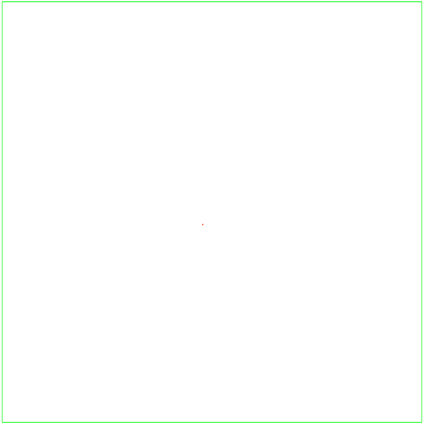
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



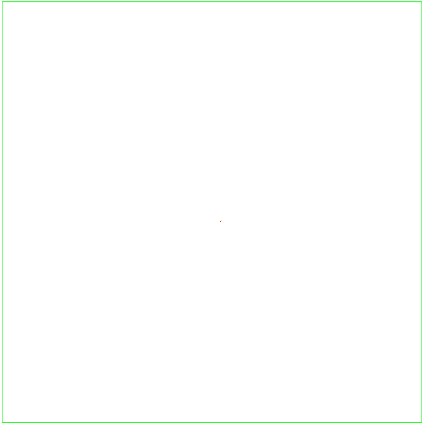
Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval

Taylor Models vs. Intervals



Iterates of initial box of width $2 \cdot 10^{-4}$. Red: Taylor Model, Green: Interval