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Uncertainty Propagation and Nonlinear Filtering for Space Navigation using Differential Algebra

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Outline

- **Motivations of the Work**
- **Introduction to Space Navigation**
- **DA-based higher order Taylor series approach**
  - Nonlinear Uncertainty Propagation
  - Nonlinear Filtering
- **Simulations and Results**
- **Conclusions and Development Axes**

Motivations and Background

- **Increase of interplanetary/deep space/sample and return exploration missions**

- Long duration
- Large heliocentric/geocentric distances
- Unknown and hostile environments
- Complex dynamics

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➡ **Strong need to study the navigation problem in space**

A **navigation system** is a state estimation filter that, starting from sensors measurements, can estimate the spacecraft state variables

- The state of a space vehicle can include a large number of parameters, first of all satellite orbital position and velocity

Introduction

- **The orbit model can be written in the following general equation form:**

$$\mathbf{x_{k+1}} = \mathbf{f}(\mathbf{x_k}, \mathbf{t_k}) + \mathbf{w_k}$$

$$\mathbf{y_k} = \mathbf{h}(\mathbf{x_k}, \mathbf{t_k}) + \mathbf{v_k}$$

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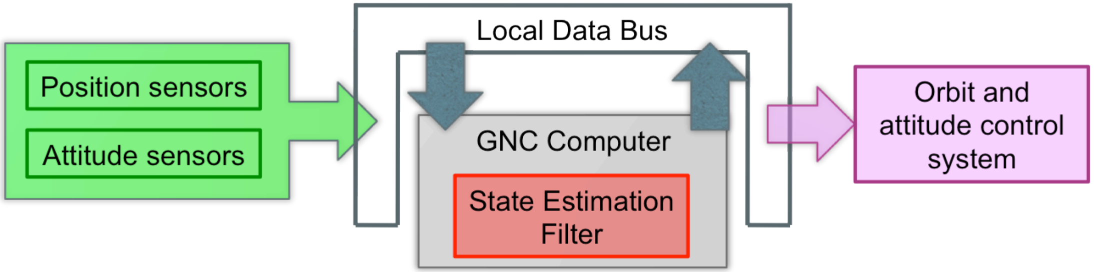
- **The on-board navigation system is made up by two main steps:**

1. PREDICTION STEP

Use of the dynamics equation to predict the state of the vehicle in a future time

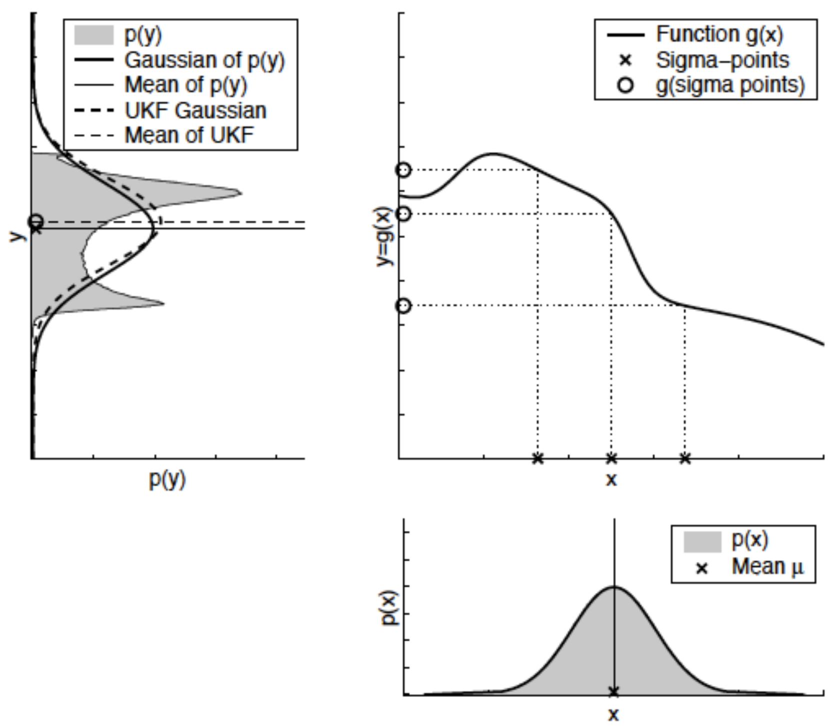
2. CORRECTION STEP

Use the measurements from the on-board sensors (autonomous navigation) or from the ground station (ground-based navigation) to correct the predicted state



Introduction

- **Autonomous navigation system requirements**
  - Real time estimation
  - High accuracy
  - Need to fully account for nonlinearities in the system



Introduction

- **Present day orbit uncertainty propagation and filtering**
  - Linearized models
  - Full nonlinear Monte Carlo simulations
  - **Higher order Taylor series approaches**

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CONSISTENT COMPUTATIONAL COST

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  CONSISTENT COMPUTATIONAL COST

➡ **Differential Algebra can provide powerful tools to face the problem**

- Possibility to **consider nonlinearities**
- Strong **reduction of computational costs**
- Opportunity to **directly evaluate the impact of changes in some physical variables** on the estimated state
  - Impact on the estimated state of errors in the initial state
  - Impact on the estimated state of measurement noises (sensors architecture definition)

Differential Algebra

- **Given any function f of v variables, DA delivers its Taylor expansion up to the arbitrary order n**

  - Any integration scheme is based on algebraic operations, involving the evaluation of f at several integration points

  ➡ Replacing x_0 with $[x_0] = x_0 + \delta x_0$ and carrying out all the operations within the DA framework → Taylor expansion of the ODE flow

$$[x_k] = x_k + \mathcal{M}(\delta x_0, \delta\alpha, \delta\beta, \delta\gamma, \ldots)$$

→ **Pointwise integration can be replaced by fast evaluation of polynomials**

→ **The derivatives of the function up to the order of the Taylor expansions are available**

Higher order Taylor series approaches

- **Nonlinear uncertainty propagation**
- Nonlinear filtering

Nonlinear Mapping of the System Dynamics

- Consider the spacecraft dynamics governed by the **equations of motion**:

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{f}_i[t, \mathbf{x}(t)] \\ \mathbf{x}_i(t^0) = \mathbf{x}_i^0 \end{cases}$$

Park R. and Scheeres D., "Nonlinear mapping of Gaussian state covariance and orbit uncertainties", AAS Paper 05-170, January, 2005.

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- Performing a **Taylor series expansion** of the deviation of the current state from the nominal trajectory in terms of the initial deviation:

$$\delta \mathbf{x_i}(\mathbf{t}) = \sum_{\mathbf{p=1}}^{\mathbf{m}} \mathbf{\Phi_{i,k_1 \ldots k_p}} \delta \mathbf{x^0_{k_1}} \ldots \delta \mathbf{x^0_{k_p}}$$

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- The **current mean and covariance matrix** can be written as:

$$\delta \mathbf{m_i}(\mathbf{t}) = \sum_{\mathbf{p}=1}^{\mathbf{m}} \frac{\mathbf{1}}{\mathbf{p}!} \mathbf{\Phi}_{\mathbf{i},\mathbf{k_1}\ldots\mathbf{k_p}} \mathbf{E}[\delta \mathbf{x}^{\mathbf{0}}_{\mathbf{k_1}} \ldots \delta \mathbf{x}^{\mathbf{0}}_{\mathbf{k_p}}]$$

$$\mathbf{P_{ij}}(\mathbf{t}) = \sum_{\mathbf{p}=1}^{\mathbf{m}} \sum_{\mathbf{q}=1}^{\mathbf{m}} \frac{\mathbf{1}}{\mathbf{p}!\mathbf{q}!} \mathbf{\Phi}_{\mathbf{i},\mathbf{k_1}\ldots\mathbf{k_p}} \mathbf{\Phi}_{\mathbf{j},\gamma_1\ldots\gamma_\mathbf{q}} \mathbf{E}[\delta \mathbf{x}^{\mathbf{0}}_{\mathbf{k_1}} \ldots \delta \mathbf{x}^{\mathbf{0}}_{\mathbf{k_p}} \delta \mathbf{x}^{\mathbf{0}}_{\gamma_1} \ldots \delta \mathbf{x}^{\mathbf{0}}_{\gamma_\mathbf{q}}]$$

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Nonlinear Mapping of the System Dynamics

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- The higher order derivatives can be directly extracted from the expansion map
  - ➡ The integration of this system is substituted with ONE integration in DA
  - ➡ More flexible approach
  - ➡ Very general approach (applying nonlinear transformations to mean and covariance estimates)

Nonlinear Mapping of the System Dynamics

- The second numerical operation is the **higher-order moment computation**

$$\delta \mathbf{m}_i(t) = \sum_{p=1}^{m} \frac{1}{p!} \mathbf{\Phi}_{i,k_1 \dots k_p} \underline{E[\delta \mathbf{x}^0_{k_1} \dots \mathbf{x}^0_{k_p}]}$$

$$\mathbf{P}_{ij}(t) = \sum_{p=1}^{m} \sum_{q=1}^{m} \frac{1}{p!q!} \mathbf{\Phi}_{i,k_1 \dots k_p} \mathbf{\Phi}_{j,\gamma_1 \dots \gamma_q} \underline{E[\delta \mathbf{x}^0_{k_1} \dots \mathbf{x}^0_{k_p} \delta \mathbf{x}^0_{\gamma_1} \dots \mathbf{x}^0_{\gamma_q}]} - \delta \mathbf{m}_i(t) \mathbf{m}_j(t)$$

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→ Calculated using the **Joint Characteristic Function** of the initial distribution:

$$E[x_i] = m_i$$

$$E[x_i x_j] = m_i m_j + P_{ij}$$

$$E[x_i x_j x_k] = m_i m_j m_k + (m_i P_{jk} + m_j P_{ik} + m_k P_{ij})$$

$$E[x_i x_j x_k x_l] = m_i m_j m_k m_l + (m_i m_j P_{kl} + m_i m_k P_{jl} + m_j m_k P_{il} + m_i m_l P_{jk} + m_j m_l P_{ik} + m_k m_l P_{ij}) + P_{ij}P_{kl} + P_{ik}P_{jl} + P_{il}P_{jk}$$

<u>NOTE</u>: If the initial mean is zero → Further semplification: **all the odd moment are zero**

Nonlinear Mapping of the System Dynamics

- Consider the two-body problem of a Earth graviting satellite
- Initial state defined as DA variable
  (lengths units scaled by the semimajor axis and time units by the factor $\sqrt{\frac{a^3}{\mu}}$)

$$\mathbf{x}_0 = \begin{pmatrix} \mathbf{r}_0 \\ \mathbf{v}_0 \end{pmatrix} = \begin{pmatrix} -0.68787 + \delta x \\ -0.39713 + \delta y \\ +0.28448 + \delta z \\ -0.51331 + \delta v_x \\ +0.98266 + \delta v_y \\ +0.37611 + \delta v_z \end{pmatrix} \qquad \mathbf{P} = \begin{bmatrix} 0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10^{-4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^{-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 10^{-4} \end{bmatrix}$$

- Carry out the integration of the motion in differential algebra
- Calculate the estimation of the mean and covariance

Nonlinear Mapping of the System Dynamics

	DA-1st order $\Delta\mathbf{r}[\%]$	DA-2nd order $\Delta\mathbf{r}[\%]$	DA-3rd order $\Delta\mathbf{r}[\%]$
0.1 orbit	0.021	0.017	0.017
0.5 orbit	0.559	0.111	0.111
0.8 orbit	3.299	0.007	0.007
1 orbit	6.197	0.092	0.092

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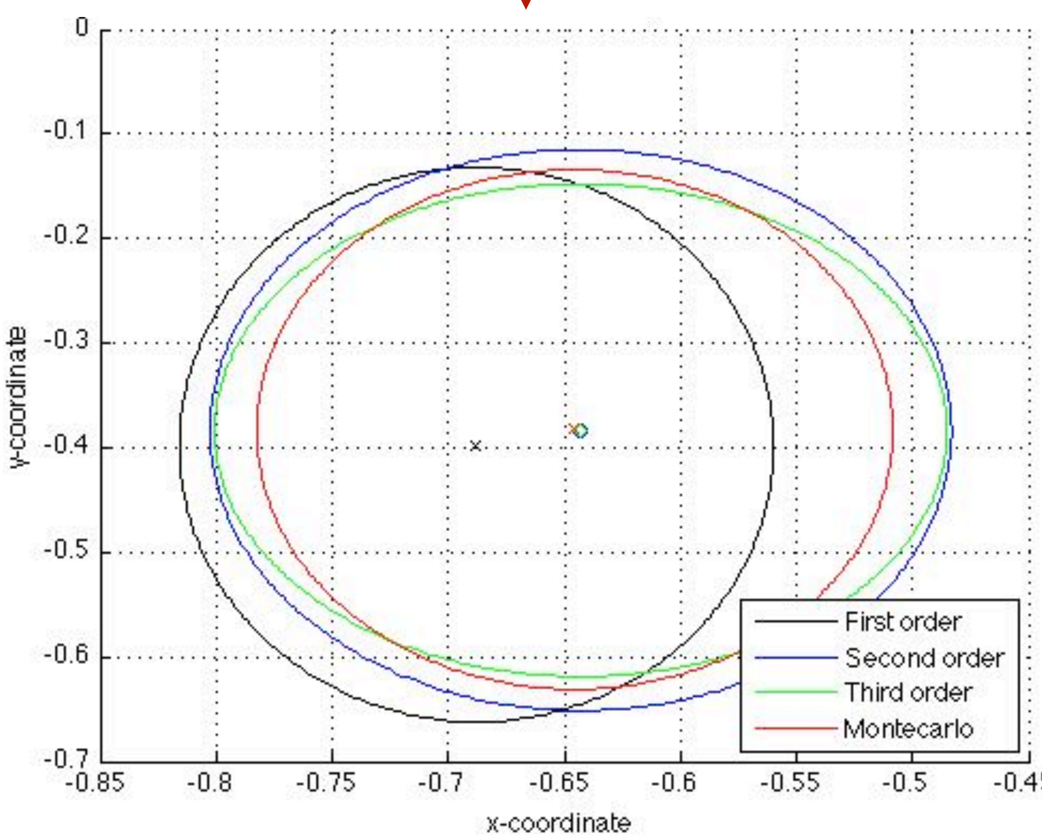

Higher order Taylor series approaches

- Nonlinear uncertainty propagation
- **Nonlinear filtering**

Higher Order Taylor Series Filtering Methods

- The application of higher order Taylor series methods to nonlinear filtering is the Higher-Order Numerical Extended Kalman Filter (HNEKF) *[Park&Scheeres2007]*

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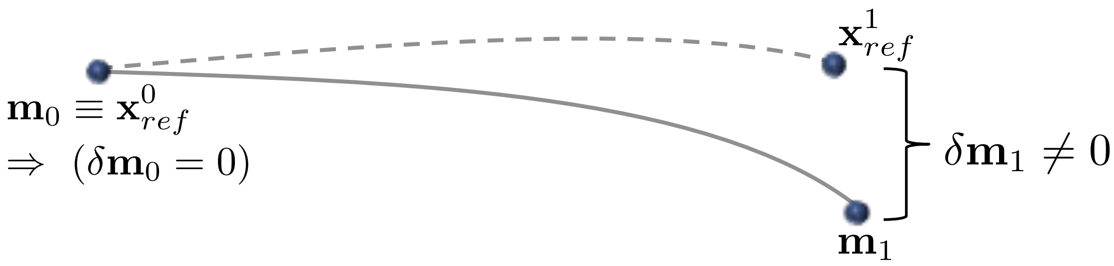
HNEKF PREDICTION EQUATION

$$(m_{k+1}^-)^i = E[\Phi^i(t_{k+1};\mathbf{m}_k^+ + \delta\mathbf{x}_k, t_k) + w_k^i] = \Phi^i(t_{k+1};\mathbf{m}_k^+, t_k) + \sum_{p=1}^{m} \frac{1}{p!}\Phi^{i,\gamma_1\ldots\gamma_p}_{(t_{k+1},t_k)} E[\delta x_k^{\gamma_1}\ldots\delta x_k^{\gamma_p}]$$



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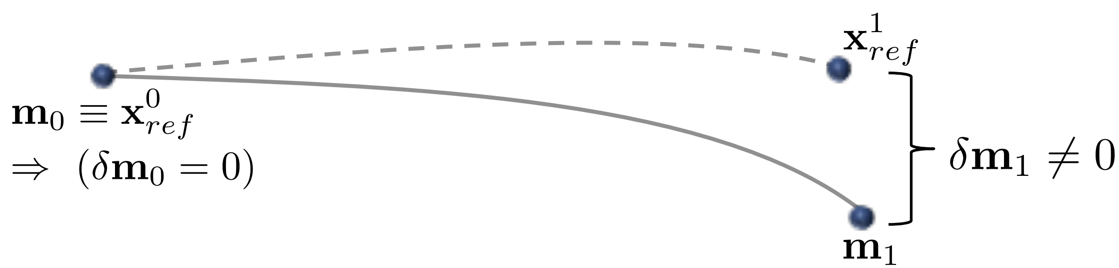

- $$(P_{k+1}^{-})^{ij} = (\sum_{p=1}^{m} \sum_{q=1}^{m} \frac{1}{p!q!} \Phi^{i,\gamma_1 \dots \gamma_p}_{(t_{k+1},t_k)} \Phi^{j,\zeta_1 \dots \zeta_q}_{(t_{k+1},t_k)} E[\delta x_k^{\gamma_1} \dots x_k^{\gamma_p} x_k^{\zeta 1} \dots x_k^{\zeta q}]) - \delta m_{k+1}^{i}(\delta\mathbf{x}_k) \delta m_{k+1}^{j}(\delta\mathbf{x}_k) + Q_k^{ij}$$
- $$(n_{k+1}^{-})^{i} = E[h^{i}(t_{k+1}; \mathbf{m}_k^{+} + \delta\mathbf{x}_k, t_k) + v_{k+1}^{i}] = h^{i}(t_{k+1}; \mathbf{m}_k^{+}, t_k) + \sum_{p=1}^{m} \frac{1}{p!} h^{i,\gamma_1 \dots \gamma_p}_{t_{k+1},t_k} E[\delta x_k^{\gamma_1} \dots \delta x_k^{\gamma_p}]$$

HNEKF UPDATE EQUATION

- $$(P^{zz}_{k+1})^{ij} = R^{ij}_{k+1} + \sum_{p=1}^{m}\sum_{q=1}^{m}\frac{1}{p!q!}h^{i,\gamma_1\dots\gamma_p}_{t_{k+1},t_k}h^{i,\zeta_1\dots\zeta_q}_{t_{k+1},t_k}E[\delta x^{\gamma_1}_k\dots\delta x^{\gamma_p}_k\delta x^{\zeta_1}_k\dots\delta x^{\zeta_q}_k] - (\delta n^-_{k+1})^i(\delta n^-_{k+1})^j$$
- $$(P^{xz}_{k+1})^{ij} = \sum_{p=1}^{m}\sum_{q=1}^{m}\frac{1}{p!q!}\Phi^{i,\gamma_1\dots\gamma_p}_{t_{k+1},t_k}h^{i,\zeta_1\dots\zeta_q}_{t_{k+1},t_k}E[\delta x^{\gamma_1}_k\dots\delta x^{\gamma_p}_k\delta x^{\zeta_1}_k\dots\delta x^{\zeta_q}_k] - (\delta m^-_{k+1})^i(\delta n^-_{k+1})^j$$
- $$\mathbf{K}_{k+1} = \mathbf{P}^{xz}_{k+1}(\mathbf{P}^{zz}_{k+1})^{-1}$$
- $$\mathbf{m}^+_{k+1} = \mathbf{m}^-_{k+1} + \mathbf{K}_{k+1}(\mathbf{z}_{k+1} - \mathbf{n}^-_{k+1})$$
- $$\mathbf{P}^+_{k+1} = \mathbf{P}^-_{k+1} - \mathbf{K}_{k+1}\mathbf{P}^{zz}_{k+1}\mathbf{K}^T_{k+1}$$

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- The **state** and the **measurements** have been defined as **DA variables**

$$[\mathbf{x}_{k+1}] = \bar{\mathbf{x}}_{k+1} + \mathcal{M}(\delta \mathbf{x}_k)$$

$$[\mathbf{z}_{k+1}] = \bar{\mathbf{z}}_{k+1} + \mathcal{M}(\delta \mathbf{x}_{k+1}) = \bar{\mathbf{z}}_{k+1} + \mathcal{M}(\delta \mathbf{x}_k)$$

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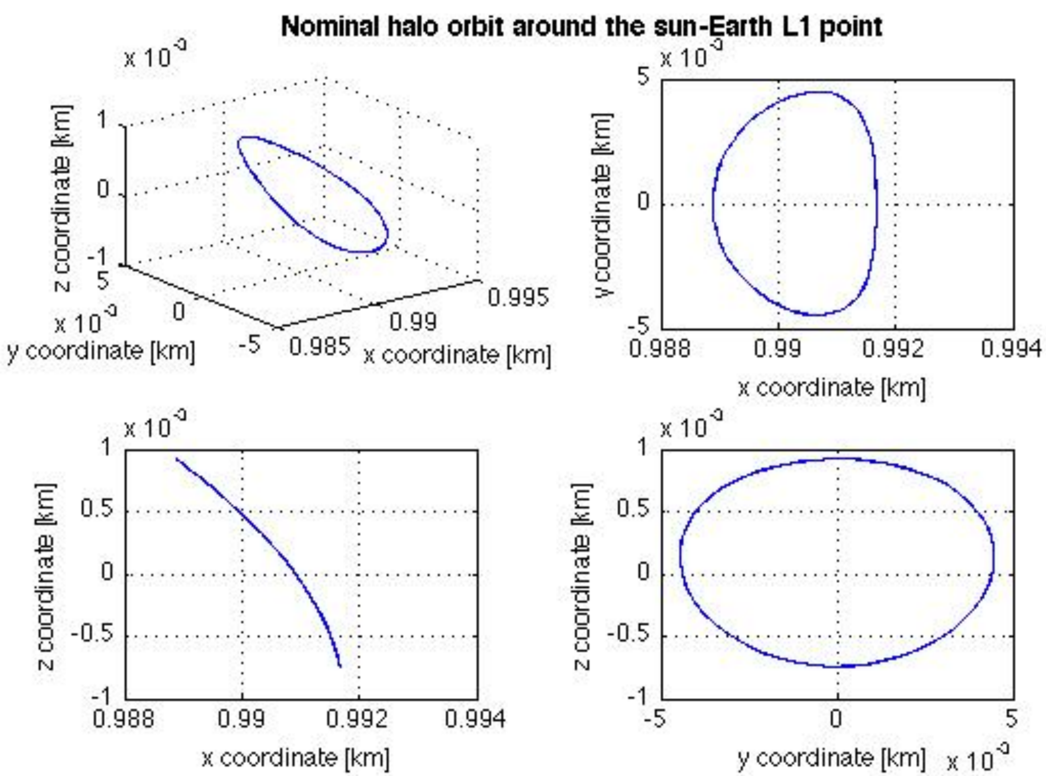
→ Higher order derivatives are **provided from one DA integration of the state** and **measurement computation**

HNEKF: Halo orbit about the Sun-Earth L_1 point

- **State:** position and velocity
- **Measurement model:** linear
- **Initial guess:** off by 100 km and 0.1 m/s from the true state (boundary of initial ellipsoid)
- **No process noise:** errors in the initial state and in the measurements



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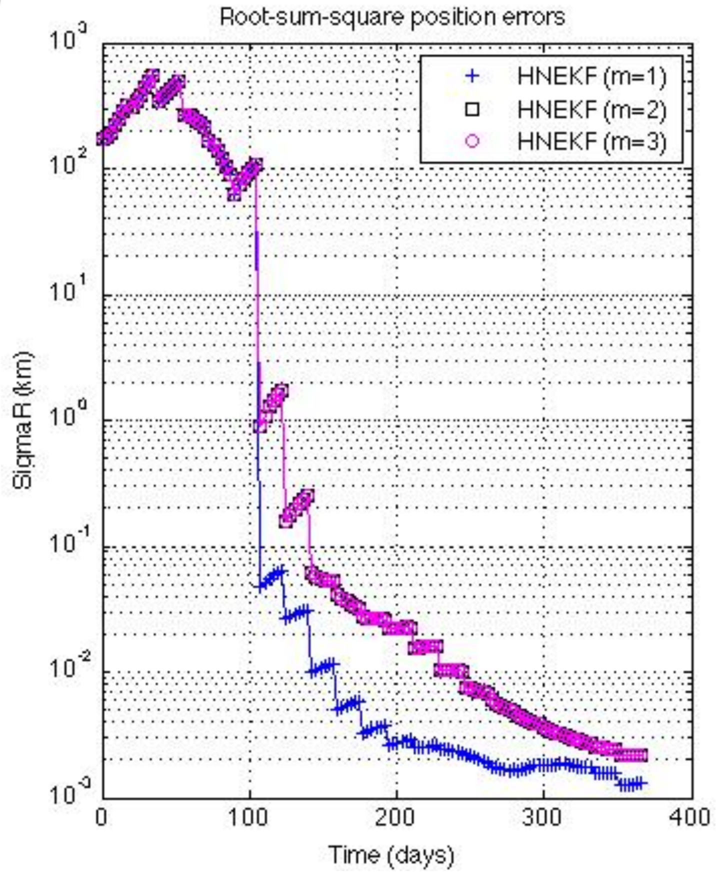
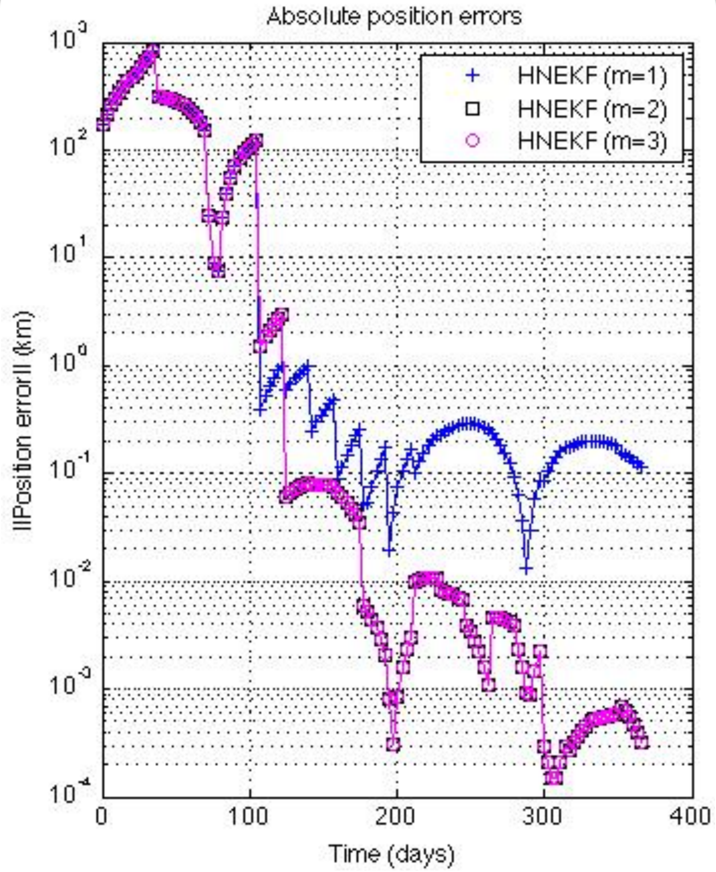
- **No process noise:** errors in the initial state and in the measurements





HNEKF: Halo orbit about the Sun-Earth L_1 point

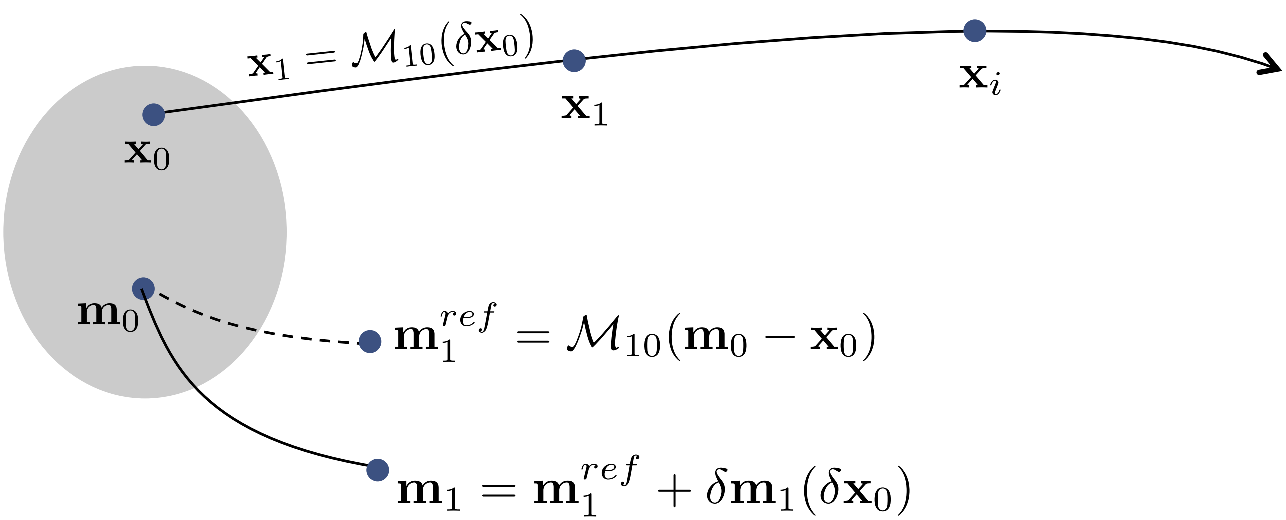
- **Without using DA:**
  - Two system of differential equations up to the m^{th} (order of the Taylor expansion) order must be solved
  - If something changes in the dynamical or measurement equations the whole solving system must be rewritten

- **Using DA:**
  - At each time step only one integration in DA is required
  - More flexibility
  - The same accuracy of the standard HNEKF is guaranteed

Higher Order Taylor Series Filtering Methods

- Missions with predetermined reference trajectories

→ compute the reference trajectory over a time span before filtering

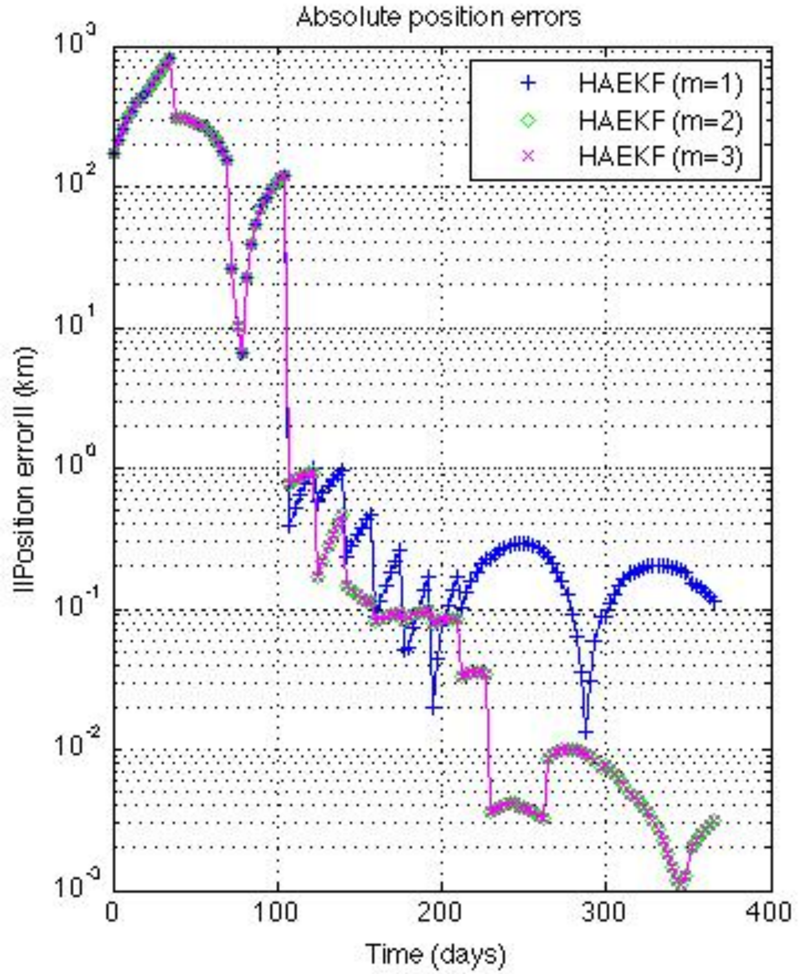


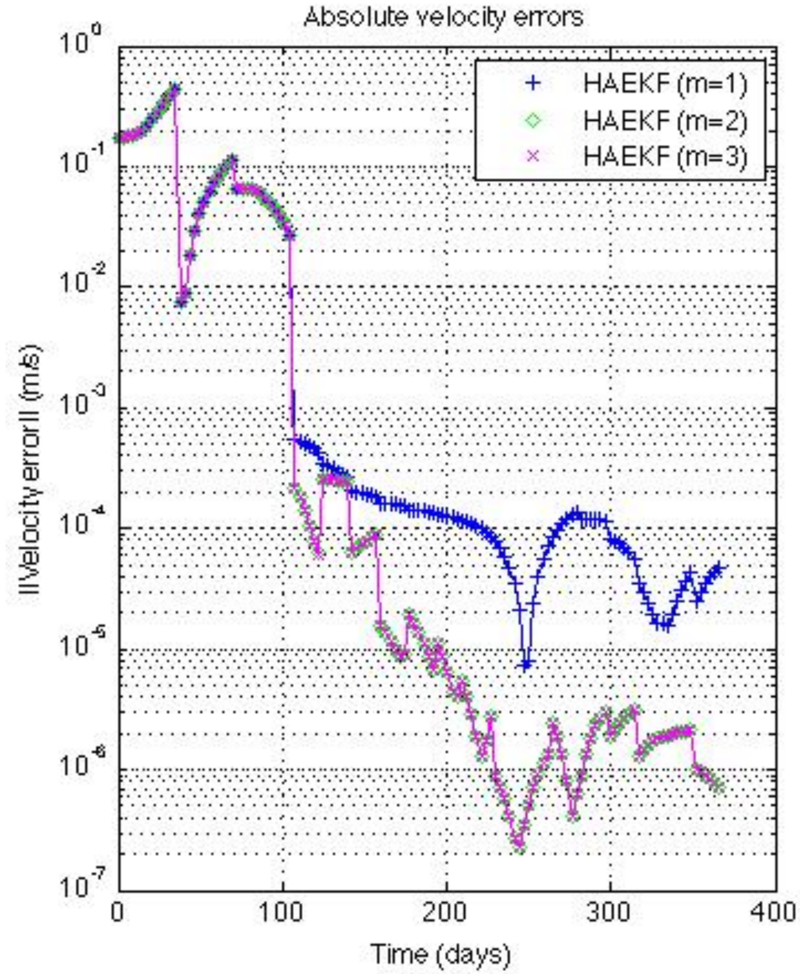
Predetermined trajectory

→ integrated offline

→ DA with high accuracy

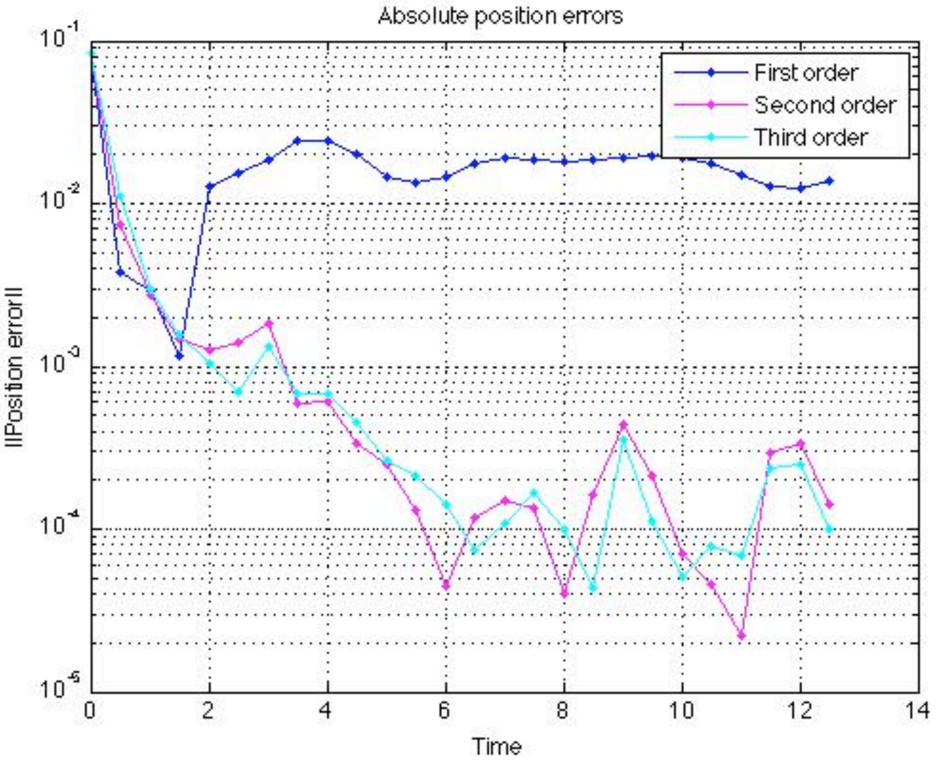
HAEKF: Halo orbit about the Sun-Earth L_1 point

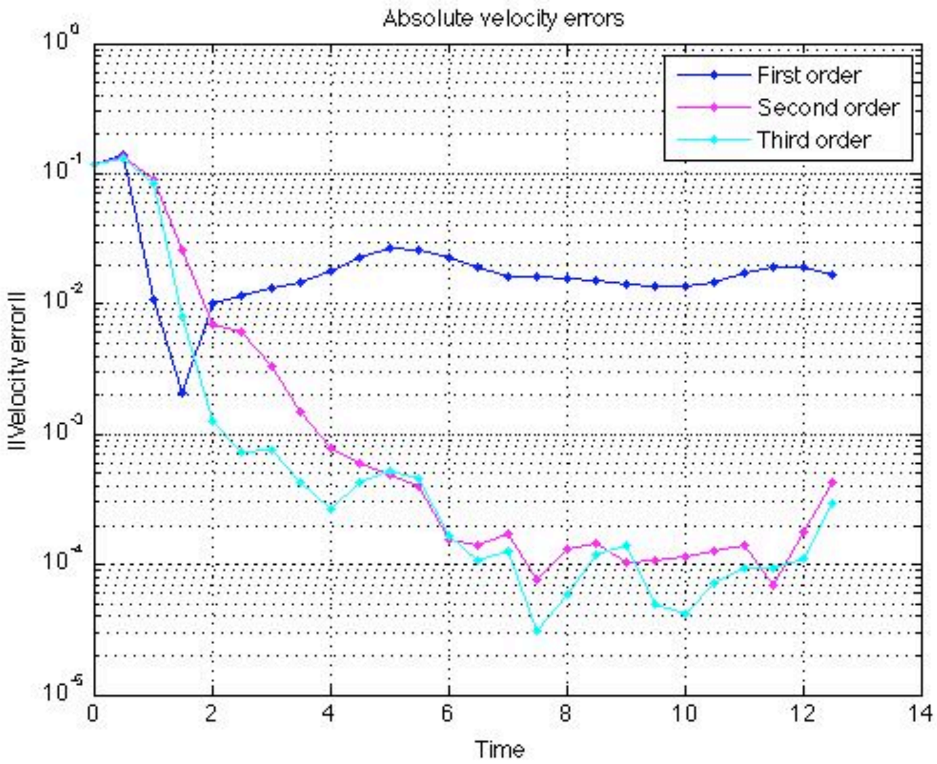




HNEKF: 2-body Problem - Nonlinear Measurements

- **DA-based HNEKF behavior in case of <u>low frequency estimation</u> and <u>nonlinear measurements</u>**
  - Measuring the velocity from S/C to Earth, the right ascension and declination of the Earth
  - Better performance of higher orders wrt first order filter





Conclusions and Development Axes

CONCLUSIONS

- New methods for nonlinear uncertainty propagation and filtering have been presented
- The methods are based on Taylor differential algebra implemented in COSY-Infinity
- The methods have good performance in terms of computational costs and flexibility

FURTHER DEVELOPMENTS

- Study and development of Monte Carlo-based filters using DA
- Use of DA to define a sensors architecture that increase estimation precision and reduce the computational load

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