Consistent Beam Phase Space Measure for Matter-Dominated Lattices

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Muon Collider I
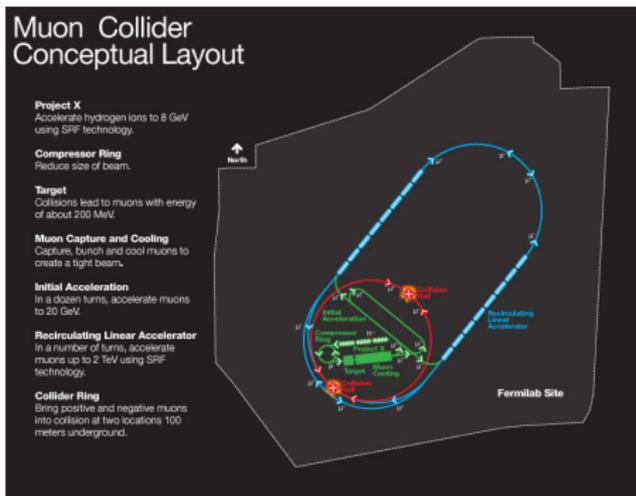

Muon Collider Conceptual Layout

Project X
Accelerate hydrogen ions to 8 GeV using SRF technology.

Compressor Ring
Reduce size of beam.

Target
Collisions lead to muons with energy of about 200 MeV.

Muon Capture and Cooling
Capture, bunch and cool muons to create a tight beam.

Initial Acceleration
In a dozen turns, accelerate muons to 20 GeV.

Recirculating Linear Accelerator
In a number of turns, accelerate muons up to 2 TeV using SRF technology.

Collider Ring
Bring positive and negative muons into collision at two locations 100 meters underground.

- Muon Accelerator Program (MAP) strives to design and build the next-generation accelerator facility for fundamental research: a Muon Collider (MC).
- Muons have certain advantages over electrons and protons.
- MC is very compact compared to other proposal such as CLIC or VLHC.

Muon Collider II

- On the other hand, there is a clear challenge: muons are unstable (decay after $\tau = 2.2~\mu$s at rest), so particle capture and acceleration should be rapid.
- Muon Collider is a tertiary beam machine: protons $\Rightarrow$ pions $\Rightarrow$ muons.
- The initial muon beam has a phase space that is too large to fit in the downstream components; need to reduce the size of the beam.
- As a result of the beam size reduction the particles deviate less from the reference particle, this is somewhat similar to reducing the thermodynamical temperature; hence, the term "muon cooling".
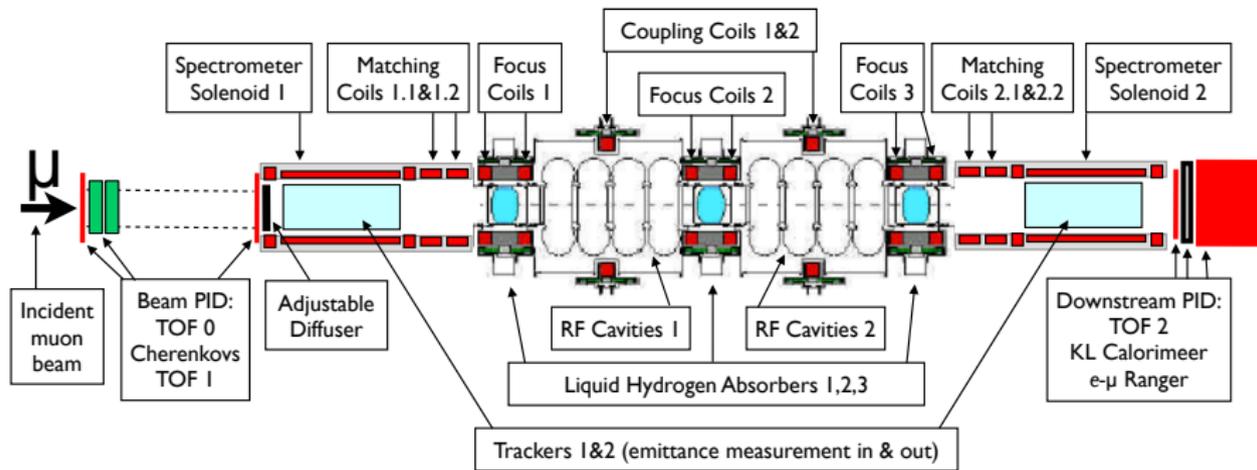
Ionization cooling
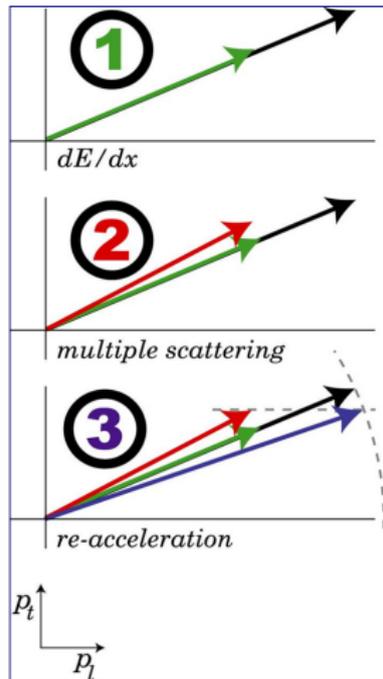
- Ionization cooling is the only efficient technique to cool within a muon lifetime.
- Passing through material reduces all three components of momentum, only the longitudinal is restored, $\Rightarrow$ net transverse cooling (=size reduction).
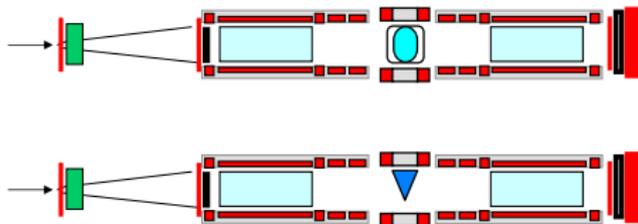- Need emittance exchange to cool in 6D.
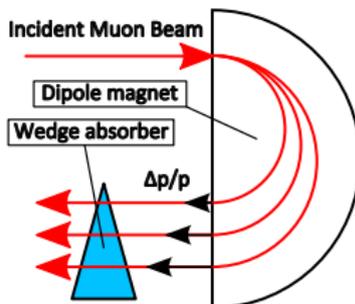
MICE: the ionization cooling experiment

MICE is an experiment at Rutherford Appleton Laboratory, UK that will demonstrate muon cooling.

Wedge absorber in MICE Step IV

- Top: MICE Step IV with a liquid hydrogen absorber. MICE is a 4D cooling experiment: transverse emittance is reduced while longitudinal emittance stays the same or increases slightly due to stochastic processes in the energy loss.

- Bottom: LH_2 absorber is replaced with a solid wedge absorber. This way emittance exchange can be observed if the beam is properly matched (dispersion is introduced).
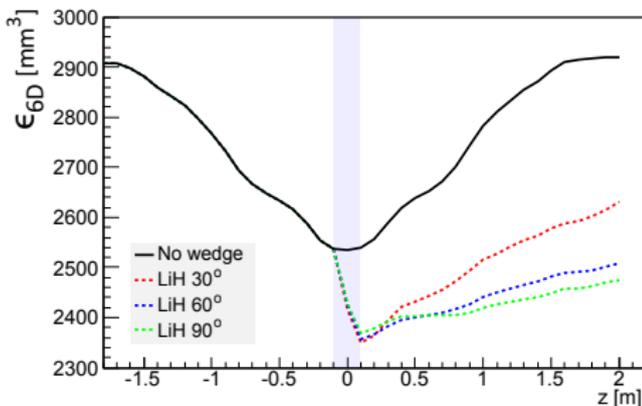
Cooling performance
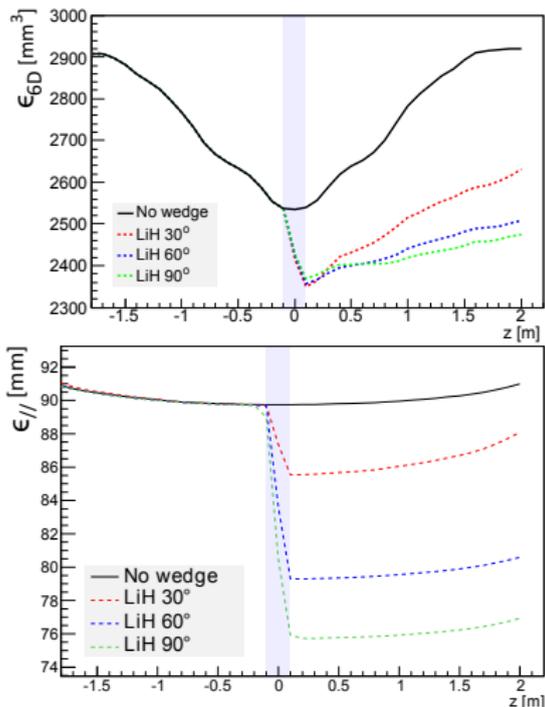
- Cooling effect observed for different wedge absorber opening angles (red – $30°$, blue – $60°$, green – $90°$)

$$\epsilon_{6D} = \frac{c}{m}\sqrt{\det(\mathrm{cov}(\mathrm{ct}, \mathrm{E}, \mathrm{x}, \mathrm{p_x}, \mathrm{y}, \mathrm{p_y}))},$$

$$\epsilon_{\parallel} = \frac{c}{m^3}\sqrt{\det(\mathrm{cov}(\mathrm{ct}, \mathrm{E}))},$$

Concern: 6D emittance change with no material

- With no material in the channel the system is Hamiltonian.
- According to the graph, the 6D emittance changes.
- Is Liouville safe? What is the conserved quantity?

Emittance

Emittance definition I

- The concept of beam emittance is a most useful one, but is often abused in practice.
- There is a number of closely related but distinct quantities all referred to as "emittance".
- The conditions under which these quantities are conserved should be understood, and the degree to which these conditions are satisfied in a given application should be considered when quoting an emittance.

Emittance definition II

Consider a one-dimensional case.

- The so-called normalized or invariant emittance can be defined in terms of Γ^x, the area occupied by the beam in a two-dimensional phase space (x, p_x), as $\epsilon_n^x = \Gamma^x/\pi m_0 c$.
- Another definition is the geometric emittance defined in terms of the area A^x occupied by the beam in the trace space (x, x') as $\epsilon^x = A^x/\pi$; $\epsilon_n^x = \gamma\beta\epsilon^x$.
- Emittance is often approximated by

$$\epsilon^x = \sqrt{\det \Sigma} = \sqrt{\det \left(\begin{array}{cc} \sigma_{xx} & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'x'} \end{array} \right)},$$

where Σ is the matrix of the second moments of the distribution. For a Gaussian beam this corresponds to $1/\pi$ times the area occupied by the elliptical contour containing 39% of the particles.
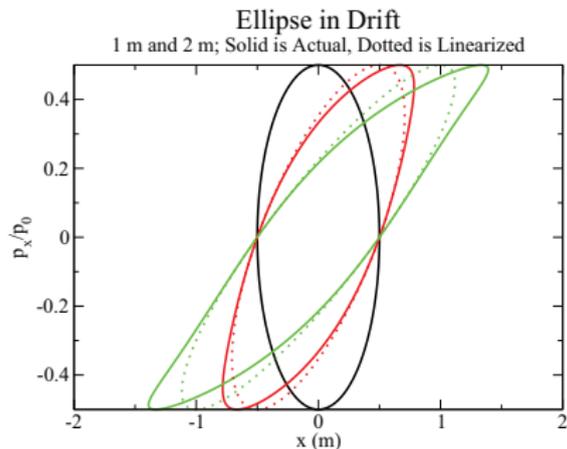
Validity of the approximation I

- The particles do not interact with one another. Coulomb interactions lead to "space-charge" growth of the emittance.

- The beam transport does not couple the various two-dimensional projections of the six-dimensional phase space. Certain beamline elements such as sextupole magnets and rf kickers in fact provide coupling between the subspaces.

- Higher moments than the second are not needed to characterize the density in phase space. This is true so long as the effects of the beamline elements are accurately described by linear transformations (Gaussian optics). Nonlinear elements lead to distortions of the phase volume that do not violate Liouville's theorem but render a second-moment description inadequate. (The increase in the emittance is often a good measure of the nonlinearity of a transport system.)
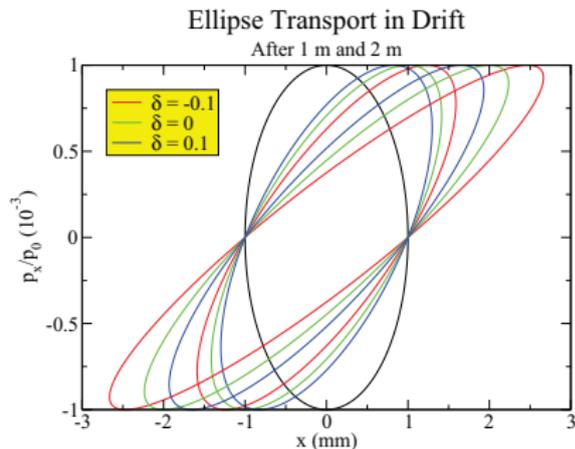
Validity of approximation II

- There are situations in which the normalized emittance is conserved but the geometric emittance is not, and vice versa.
- x and x' are canonically conjugate variables (resulting in the conservation of geometric emittance) only if
    - the transverse components of the vector potential, A_x and A_y, are zero (violated in many magnetic beamline elements),
    - the beam energy remains constant (violated during acceleration, while the invariant emittance remains the same).
- The reverse case occurs when the beam has a nonzero energy spread. Even in the absence of magnetic fields or acceleration, the achromaticity of a beam will cause the normalized emittance to increase during a simple drift (propagation through free space), while the geometric emittance is unaffected.

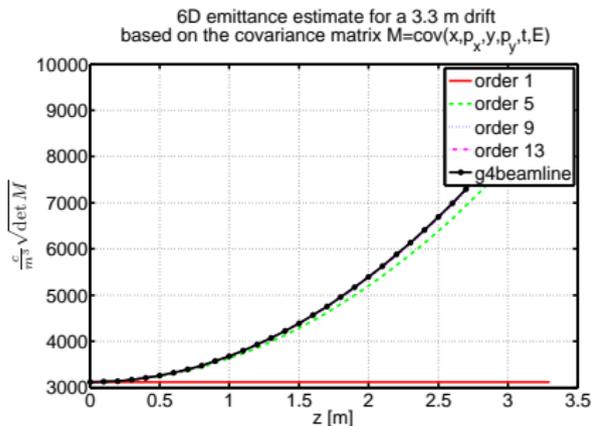# Invariant emittance growth

Ellipse in Drift
1 m and 2 m; Solid is Actual, Dotted is Linearized

Ellipse Transport in Drift
After 1 m and 2 m

- Ellipse distortion in drift due to angular spread.

- Ellipse parameter change due to energy spread.

Emittance growth with nonlinearity order

6D emittance estimate for a 3.3 m drift
based on the covariance matrix M=cov(x,p_x,y,p_y,t,E)

- COSY allows to increase the order of nonlinearities gradually and see how the emittance growth rate change.
- The result is compared with a simulation carried out using another beamline code, g4beamline.

16

Alternative measures of the phase space volume change

- In addition to the commonly used $\epsilon_{6D} = \frac{c}{m}\sqrt{\det(\mathrm{cov}(\mathrm{ct}, \mathrm{E}, \mathrm{x}, \mathrm{p_x}, \mathrm{y}, \mathrm{p_y}))}$, which is a generalization of the two-dimensional normalized emittance introduced above, there are other quantities invariant under certain conditions.

- For symplectic systems $\mathrm{Jac}(\mathcal{M}) = 1$, and when the particles pass through matter $\mathrm{Jac}(\mathcal{M})$ is a good measure of the phase space volume change.

- The experiment MICE described above is a single-particle experiment, so a "single-particle emittance" can be introduced to estimate the amount of cooling (=beam size reduction).

Phase volume conservation

- Hamiltonian system: $\det(\mathrm{Jac}(\mathcal{M})) = 1$, where $\mathrm{Jac}(\mathcal{M})$ is the Jacobian of the transformation of phase space: $\mathcal{M}(\vec{q}, \vec{p}) = (\vec{Q}, \vec{P})$, (q, p) are phase space coordinates before the transformation, (Q, P)—after the transformation.

- Let S_1 be a subset of phase space, $S_2 = \mathcal{M}(S_1)$.

- Then,
  $V_2 = \int_{S_2} d^n \vec{Q} d^n \vec{P} = \int_{S_1} \det(\mathrm{Jac}(\mathcal{M})) d^n \vec{q} d^n \vec{p} = \int_{S_1} d^n \vec{q} d^n \vec{p} = V_1$.

- $V_2 = V_1 = \mathrm{const}$.

- Need to check that the determinant in question is indeed equal to 1.

COSY Infinity analysis I

- COSY Infinity DA tools allow to conveniently obtain $\det(\mathrm{Jac}(\mathcal{M}))$ for the whole phase space volume of interest:
- Start with a high-order transfer map $\mathcal{M}$.
- Use DA operations on the map to obtain the components of

$$
\mathrm{Jac}(\mathcal{M}) = \begin{pmatrix} \frac{\partial \mathcal{M}_1(\vec{z})}{\partial z_1} & \cdots & \frac{\partial \mathcal{M}_1(\vec{z})}{\partial z_6} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{M}_6(\vec{z})}{\partial z_1} & \cdots & \frac{\partial \mathcal{M}_6(\vec{z})}{\partial z_6} \end{pmatrix}
$$

as polynomials of order $n - 1$.

- Use DA vector operations to obtain $\det(\mathrm{Jac}(\mathcal{M}))$.

COSY Infinity analysis II

- Implemented MICE magnets in COSY, compared to g4beamline, very good agreement.
- Calculated a high-order transformation map.
- Obtained the determinant of the Jacobian as a high-order polynomial of particle optical coordinates.
- $\det(\mathrm{Jac}(M)) = 1$, deviation from 1 in (x, y) is shown on right.

Magnetic field comparison

Deviation of det(Jac(M)) from 1, ninth order
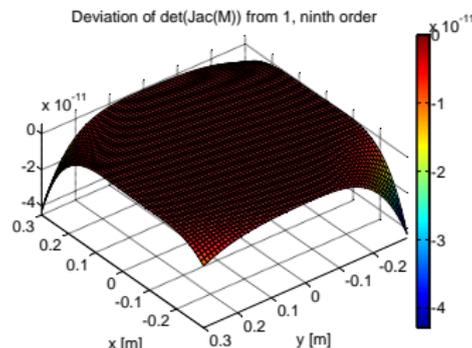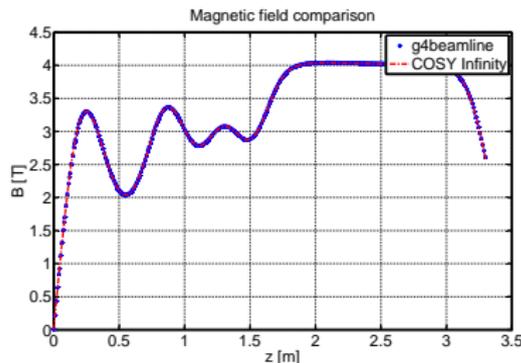
COSY analysis summary

- 6D phase space volume is conserved when there is no material in the way of the beam, as it should be.
- Change in emittance observed is due to approximation: $\epsilon_{6D} = \frac{c}{m}\sqrt{\det(\mathrm{cov}(\mathrm{ct}, \mathrm{E}, \mathrm{x}, \mathrm{p_x}, \mathrm{y}, \mathrm{p_y}))}$,
- When particles pass through matter $\det(\mathrm{Jac}(M))$ always gives consistent results while $\epsilon_{6D} = \frac{c}{m}\sqrt{\det(\mathrm{cov}(\mathrm{ct}, \mathrm{E}, \mathrm{x}, \mathrm{p_x}, \mathrm{y}, \mathrm{p_y}))}$ often shows artificial growth.
- $\det(\mathrm{Jac}(M))$ is clearly a better 6D phase space volume change measure.

Conclusions

- $\epsilon_{6D} = \frac{c}{m}\sqrt{\det(\text{cov}(\text{ct}, \text{E}, \text{x}, \text{p}_x, \text{y}, \text{p}_y))}$ is not applicable for strongly nonlinear systems and for beams with large energy/angular spread.

- $\det(\text{Jac}(M))$ is a consistent phase space volume measure, both with and without material in the beamline.

- Issue: in the presence of a magnetic field one has to take into account that the mechanical momentum is not the canonical momentum anymore, magnetic vector potential has to be accounted for.

- Issue: practicality/applicability. While the second moment matrix can be obtained relatively easily from measurements, this is not the case for $\det(\text{Jac}(M))$.

- Issue: hard to explain the concept to the peers.