The Existence of Chaos in the Lorenz System

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Outline

The concept of Chaos in Smooth Systems

Chaos in Smooth Systems

Usually taken to mean some sort of complicated orbit structure.

- Existence of complicated invariant sets
- Compact invariant sets containing uncountably many dense orbits,
- Infinitely many distinct periodic orbits

How does one find such sets?

In general require some topological, analytic, or geometric information.
A useful concept is that of topological entropy.

- This is an invariant associated to any Dynamical System.
- When it is positive, there is some sort of chaos in the system.

Topological Entropy $h(f)$ of a map $f : X \to X$:
Let $n \in \mathbf{N}$, $x \in X$.
An $n-orbit$ $O(x,n)$ is a sequence $x, fx, \dots, f^{n-1}x$
For $\epsilon > 0$, the $n-$orbits $O(x,n), O(y,n)$ are $\epsilon-$different if there is a $j \in [0, n-1)$ such that

$$d(f^j x, f^j y) > \epsilon$$

Let $r(n,\epsilon,f)$ = maximum number of $\epsilon-$different $n-$orbits. $(\leq e^{\alpha n} \; \exists \alpha)$
Set

$$h(\epsilon, f) = \limsup_{n\to\infty} \frac{1}{n} \log r(n,\epsilon,f)$$

(entropy of size ϵ)
and

$$h(f) = \lim_{n\to\infty} h(\epsilon, f) = \sup_{\epsilon>0} h(\epsilon, f)$$

(topological entropy of f) [ϵ small $\Longrightarrow$ f has $\sim e^{h(f)n}$ $\epsilon-$ different orbits]

Properties of Topological Entropy

- Dynamical Invariant: $f \sim g \Longrightarrow h(f) = h(g)$
- Monotonicity of sets and maps:
  - $\Lambda \subset X, f(\Lambda) \subset \Lambda, \Longrightarrow \text{h(f,}\Lambda) \leq h(f)$
  - (g, Y) a factor of f: $\exists \pi : X \to Y$ with $g\pi = \pi f \Longrightarrow h(f) \geq h(g)$
- Power property: $h(f^n) = nh(f)$ for $N \in \mathbf{N}$.
  $h(f^t) = \mid t \mid h(f^1)$ for flows
- $f : M \to M$ C^∞ map $\Longrightarrow$
  $h(f) =$ maximum volume growth of smooth disks in M
- $h : \mathcal{D}^\infty(M^2) \to R$ is continuous (in general usc for C^∞ maps)
- Variational Principle:

$$h(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f)$$

Examples of Calculation of Topological Entropy

Topological Markov Chains TMC (subshifts of finite type SFT)

First, the full $N - shift$:

Let $J = \{1, \dots, N\}$ be the first N integers, and let

$$\Sigma_N = J^{\mathbf{Z}} = \{\mathbf{a} = (\dots, a_{-1}a_0a_1\dots), \; a_i \in J\}$$

with metric

$$d(\mathbf{a}, \mathbf{b}) = \sum_{i \in \mathbf{Z}} \frac{| a_i - b_i |}{2^{| i |}}$$

This is a compact zero dimensional space (homeomorphic to a Cantor set)

Define the left shift by

$$\sigma(\mathbf{a})_i = a_{i+1}$$

This is a homeomorphism and $h(\sigma) = \log N$.

Chaos for 3 Dimensional Vector Fields

Chaos = Positive Topological Entropy

For a vector field X, with flow $\phi(t, x)$, define

$$h_{top}(X) = h_{top}(\phi_1) = \sup_{\text{compact invariant } \Lambda} h_{top}(\phi_1 \mid \Lambda)$$

Basic Facts for C^∞ vector fields in dimension 3:

- $X \to h_{top}(X)$ is continuous (N, Katok, Yomdin)
- $h_{top}(X)$ is the maximum length growth of smooth curves (N, Yomdin)
- $h_{top}(X)$ is the supremum of h_{top} on suspensions of subshifts (Katok) implies existence of compact topologically transitive sets with infinitely many saddle type periodic orbits
- If $P : \Sigma \to \Sigma$ is $n - th$ iterate of the Poincaré map to a cross-section Σ and the return times of P are bounded above by $T > 0$, then

$$h_{top}(X) > \frac{h_{top}(P)}{nT}$$

Pictures of maps f to guarantee $h_{top}(f) > 0$:

- Lorenz Markov returns

The Lorenz Equations

Consider the Lorenz system: $\dot{u} = L_{\sigma,\rho,\beta}(u), \ u = (x, y, z)$,

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= (\rho - z)x - y \\ \dot{z} &= xy - \beta z\end{aligned}$$

Main Reference: Colin Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, 1981
Basic Properties, Detailed Numerical Study, Many Conjectures, Mostly Unsolved

Known Results for large ρ

Robbins (1979):
—For $\beta = 1$, $\sigma = 5$, large ρ, there is a unique stable periodic orbit

Sparrow Conjecture: For $\sigma = 10, \beta = 8/3$, $\rho >> 1 \Longrightarrow L$ is Morse-Smale.

X. Chen: (1996) $\sigma, \beta > 0$. There exists a homoclinic orbit for some $\rho \in (0, \infty)$ iff $\sigma > \frac{2\beta+1}{3}$

X. Chen: ($\sim$ 1996 (not published)) For every $\beta > 0$, there exist $\sigma > 0$, and large $\rho > 0$ such that the corresponding Lorenz system exhibits chaos.

Hastings, Troy (1994), There is a homoclinic orbit for $(0, 0)$, $\sigma \sim 10, \beta \sim 1, \rho = 1000)$

Remark All of above require very large ρ and give small positive topological entropy

In Sparrow, numerical calculations suggest that a homoclinic orbit (for $(0, 0)$ exists for $\sigma = 10, \beta = 8/3$ and $\rho \sim 13.94$

The arrival of Computer Assisted (CA) proofs for the Lorenz system

Hassard, Zhang (1994), There is a homoclinic orbit for $(0, 0)$ $\sigma = 10, \beta = 8/3$ and $13.9625 < \rho < 13.967$. —Computer assisted using Interval Analysis

Some current plots: Figure: $\rho = 13.9265$ Figure: $\rho = 13.9266$

In connection with the Sparrow conjecture on Morse-Smale for large ρ partial result (Computer Assisted) by Zou and Wittig: long stable periodic orbit for $\rho = 350, \sigma = 10, \beta = 8/3$

Figure: Lorenz-350-periodic

Figure: Lorenz-350-periodic-plus

Previous CA Proofs of Chaos in the Lorenz system

Mischaikow-Mrozek-Szymczak: (1995+)For (σ, ρ, β) in small neighborhoods of (10,28,8/3), (10,60,8/3), (10,54, 45), the Poincare maps to the plane $z = \rho - 1$ have factors which are SFT with positive entropies.

Galias-Zgliczynski: (1998)For (σ, ρ, β) in a small neighborhood of (10,28,8/3) the square of the Poincare map, P^2 has an invariant set conjugate to the full two-shift.

Tucker: (2001) For (σ, ρ, β) in a small neighborhood of (10,28,8/3), the Poincare map to the plane $z = \rho - 1$ has a chaotic attractor.

These results are all computer assisted and make use of Interval Analysis and Verified Integrators

The computer codes are very specifically created for the particular parameter values apparently found by experimentation.

Main Theorem. There is an open neighborhood U of the line segment

$$\sigma = 10,\ \beta = 8/3,\ 25 \leq \rho \leq 95$$

in parameter space such that if $(\sigma, \rho, \beta) \in U$ then $L_{\sigma,\rho,\beta}$ has topological entropy greater than

$$\frac{\log(2)}{4}$$

In fact, the square of the Poincare map to $z = \rho - 1$ has an invariant subset which factors onto the full 2-shift and the return time is less than 2

- Further, there is an (non-rigorous, easy to implement, computational) technique to suggest the existence of positive entropy (based on growth of lengths of curves)!

- The proof is computer assisted and makes use of a verified integrator (Berz-Makino) based on Taylor Models

Good News: There is a proof.
Bad News: It takes a lot of computer resources
To describe the main ideas of the proof, we need some basic facts.

• Lorenz system is invariant under symmetry $(x, y, z) -> (-x, -y, z)$
For $\sigma > 0, \beta > 0, \rho > 1$ and $\alpha = \sqrt{\beta(\rho - 1)}$
• There are three critical points, C_1, C_2, C_3

$$C_1 = (\alpha, \alpha, \rho - 1), C_1 = (-\alpha, -\alpha, \rho - 1), C_3 = (0, 0, 0)$$

Change of Coordinates $x = \alpha x_1, y = \alpha y_1, z = (\rho - 1)z_1,\ \alpha = \sqrt{\beta(\rho - 1)}$, transforms the system to

$$\begin{aligned} \dot{x_1} &= \sigma(y_1 - x_1) \\ \dot{y_1} &= (\rho - (\rho - 1)z_1)x_1 - y_1 \\ \dot{z_1} &= \beta(x_1 y_1 - z_1) \end{aligned}$$

Moves the critical points to $(1, 1, 1), (-1, -1, 1), (0, 0, 0) = C_1, C_2, C_3$
(Unit Critical Points)

— Lorenz: Orbits are forward bounded
(there is a quadratic Lyapunov function decreasing along orbits outside an ellipsoid)

— Eigenvalues at the critical points: C_1, C_2, C_3:

For $\sigma = 10, \beta = 8/3, \rho > \frac{470}{19} \approx 24.74$:

— Eigenvalues at $C_3 = (0,0,0)$ are real: $\lambda_{31} < -\beta < 0 < \lambda_{32}$
two-dimensional stable manifold and one dimensional unstable manifold

— Eigenvalues at C_1, C_2: $\lambda_{11} < 0,\ \lambda_{12} = a + bi,\ a > 0,\ b \neq 0,\ \bar{\lambda}_{12}$
-unstable spirals

— unstable eigenspaces at C_1, C_2 are transverse to the plane $z_1 = 1$

Main Ideas

- Study stable and unstable manifolds of critical points

Manifolds: rho=28 rho=60 rho=95

- Get horseshoe type sets for second iterate
  of the Poincare map to the $z = 1$ plane

In $z = 1$, take a line from $[-1, -1, 1]$ to $[0, 0, 0]$ and take its first and second images.

line and images, rho=28 line, box and images, rho=28

- Use numerical tools to "guess" the proper behavior

How do we get the boxes for the returns?

Plot the return times, Non-verified Runge-Kutta 7-8

- Lorenz return times

- Take local maxima of the return times to give approximate vertical boundaries

- Do this for $\rho = 25, 25.5, 30, 30.5, \ldots, 95$ —- to get candidate boxes

- Squeeze vertical boundaries closer together, expand horizontally bondaries across unstable manifolds

– gives candidate boxes for discrete set of $\rho's$.

- Linearly interpolate in between to get boxes for all $25 \leq \rho \leq 95$

- Use verified integrator to prove desired return pictures
- Some verified pictures

Future Work

- Develop tools to shorten the computation time in the proof.
- Estimation of topological entropy upper and lower bounds
- Higher dimensional invariant manifolds
- proofs of hyperbolicity in various systems
- rigorous descriptions of other 3d systems, e.g. Lorenz (for many parameters) and Rossler

- Software package (like Yorke's Dynamics Program or Guckenheimer's dstool) which does rigorous calculations.