Verification of a rigorous integrator using an eighth-order Runge-Kutta

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Abstract. The behavior of nonlinear (chaotic) dynamical systems is understood by calculating flows in phase space. Stable points can emerge after iterating these flows many times, revealing significant information about the system. Calculating these flows is computationally expensive but it is important to be precise, since, due to the nonlinear behavior of the system, small variations are exacerbated with each iteration. A rigorous integrator has been developed using Taylor models and implemented in the code COSY INFINITY which integrates ODEs and PDEs rigorously. Presented are examples of these integrations of various point densities using an eighth-order Runge-Kutta with automatic step size control using reverse communication.

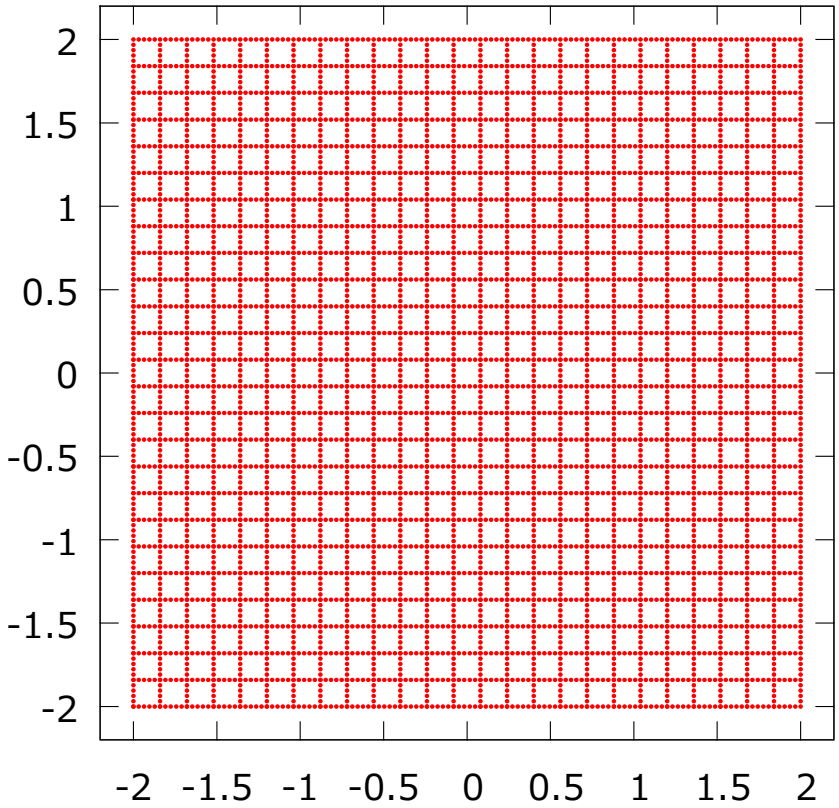
Two systems were examined:

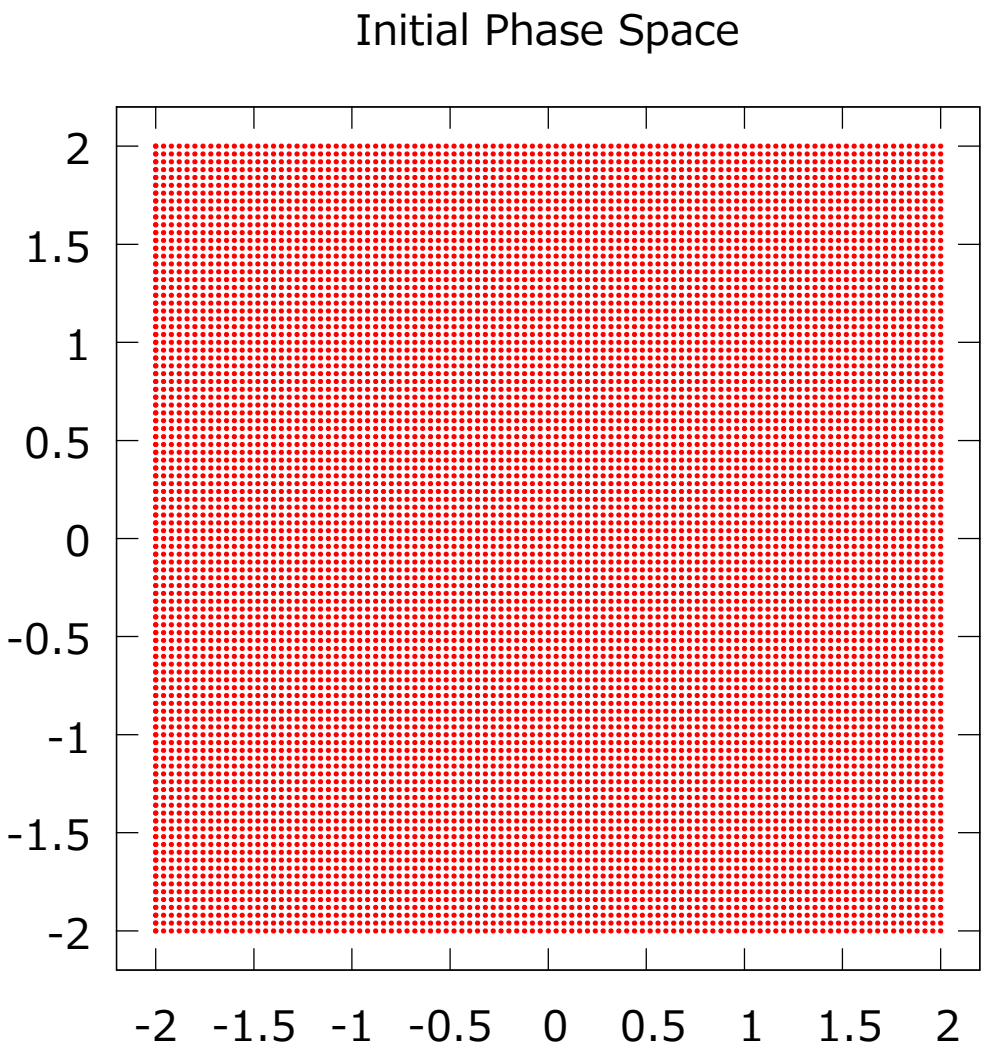
1. For the Duffing Equation, the standard parameters used were $\delta = 0.25; \gamma = 0.3$ from $t = 0$ to $t = \pi$ with the initial condition box $[-2, 2]$x$[-2, 2]$.

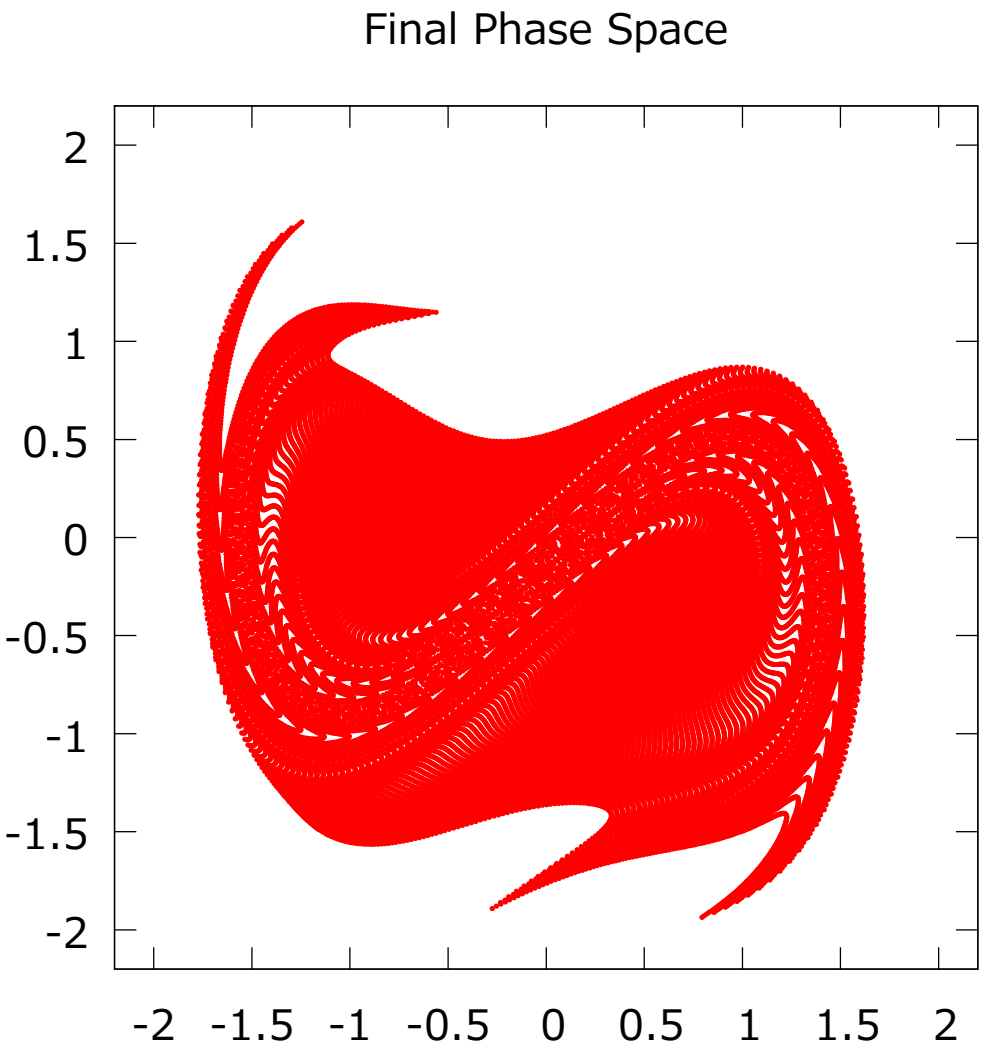
2. For the Lorenz Equation, $\sigma = 10; \beta = \frac{8}{3}; \rho = 28$ and initial condition set $[-40, 40]$x$[-50, 50]$x$[-25, 75]$ were integrated from $t = 0$ until $t = 0.1$

Duffing Equation	Lorenz Equation
$\frac{dx_1}{dt} = x_2$ $\frac{dx_2}{dt} = x_1 - \delta x_2 - x_1^3 + \gamma cos(t)$	$\frac{dx_1}{dt} = \sigma(x_2 - x_1)$ $\frac{dx_2}{dt} = x_1(\rho - x_3) - x_2$ $\frac{dx_3}{dt} = x_1 x_2 - \beta x_3$

To compute a map, the initial phase space is divided into a numbered grid of cells. The Runge-Kutta (RK) then integrates each cell individually as a grid of points and determines the cells mapped to.




Initial Phase Space
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


Final Phase Space
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

The phase space was divided into varying cell dimensions. For Duffing Equation, the space varied from $192x192$ to $1536x1536$ cells, and the Lorenz from $16x10x10$ to $64x80x80$.

The maps are saved in a two column list in the format:

(initial cell 1) (final cell 1)
(initial cell 1) (final cell 2)
(initial cell 1) (final cell 3)
(initial cell 2) (final cell 1)
(initial cell 2) (final cell 2)
(initial cell 2) (final cell 3)
...

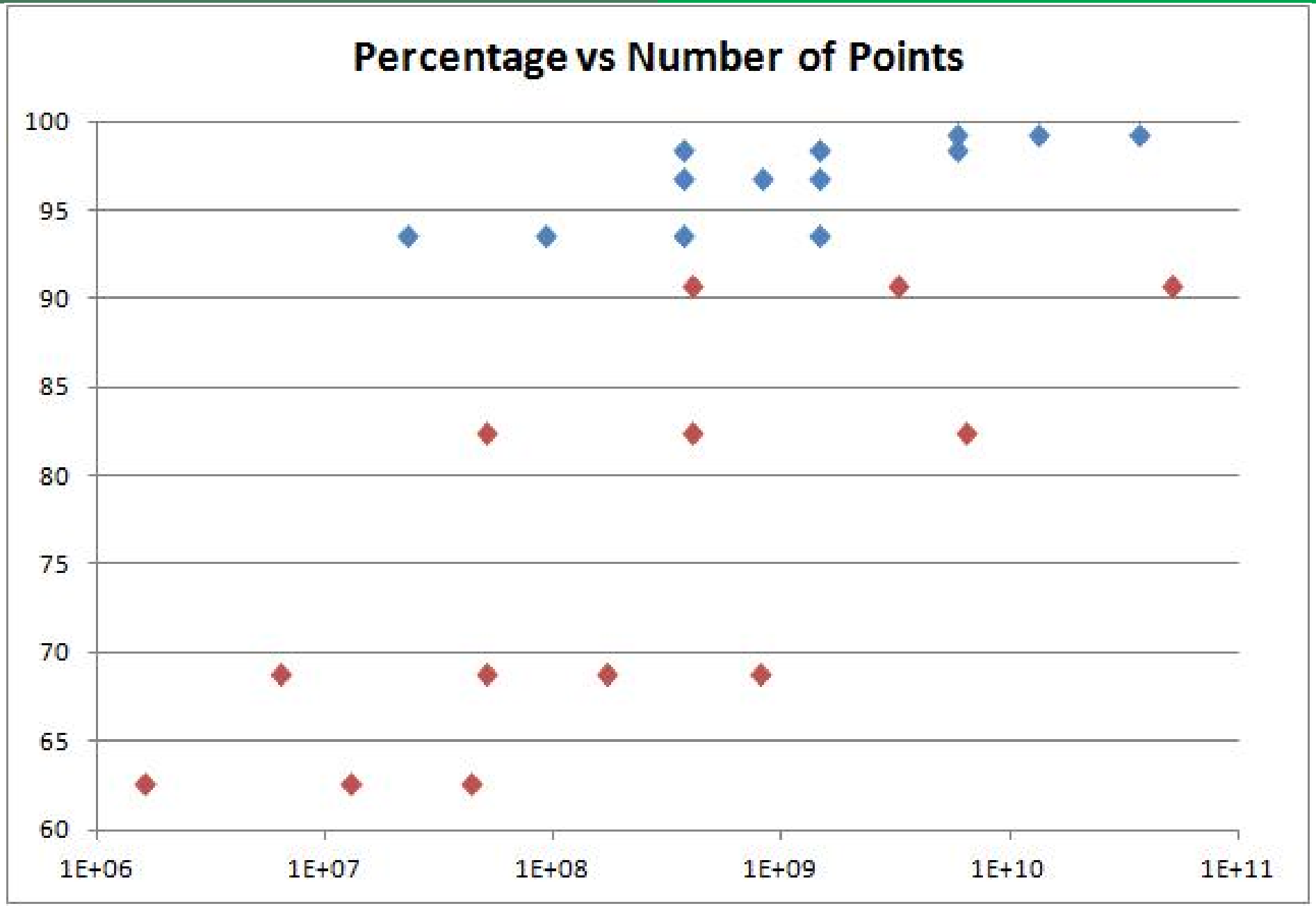
The rigorous and RK maps were compared line by line for agreement. The differences were sufficiently large to warrant listing the RK map file length as a percentage of the rigorous map length, as a function of the total number of points in the phase space, as shown in the following tables.

Duffing Equation

Grid Size	Number of Points per Cell	Percentage	Integration Time
192x192	25x25	93.6	125 s
	50x50	93.6	535 s
	100x100	93.6	2784 s
	200x200	93.6	12456 s
384x384	50x50	96.8	2150 s
	75x75	96.8	6147 s
	100x100	96.8	11424 s
768x768	25x25	98.4	1950 s
	50x50	98.4	7176 s
	100x100	98.4	27842 s
1536x1536	50x50	99.2	30000 s
	75x75	99.2	80951 s
	125x125	99.2	311506 s

Lorenz Equation

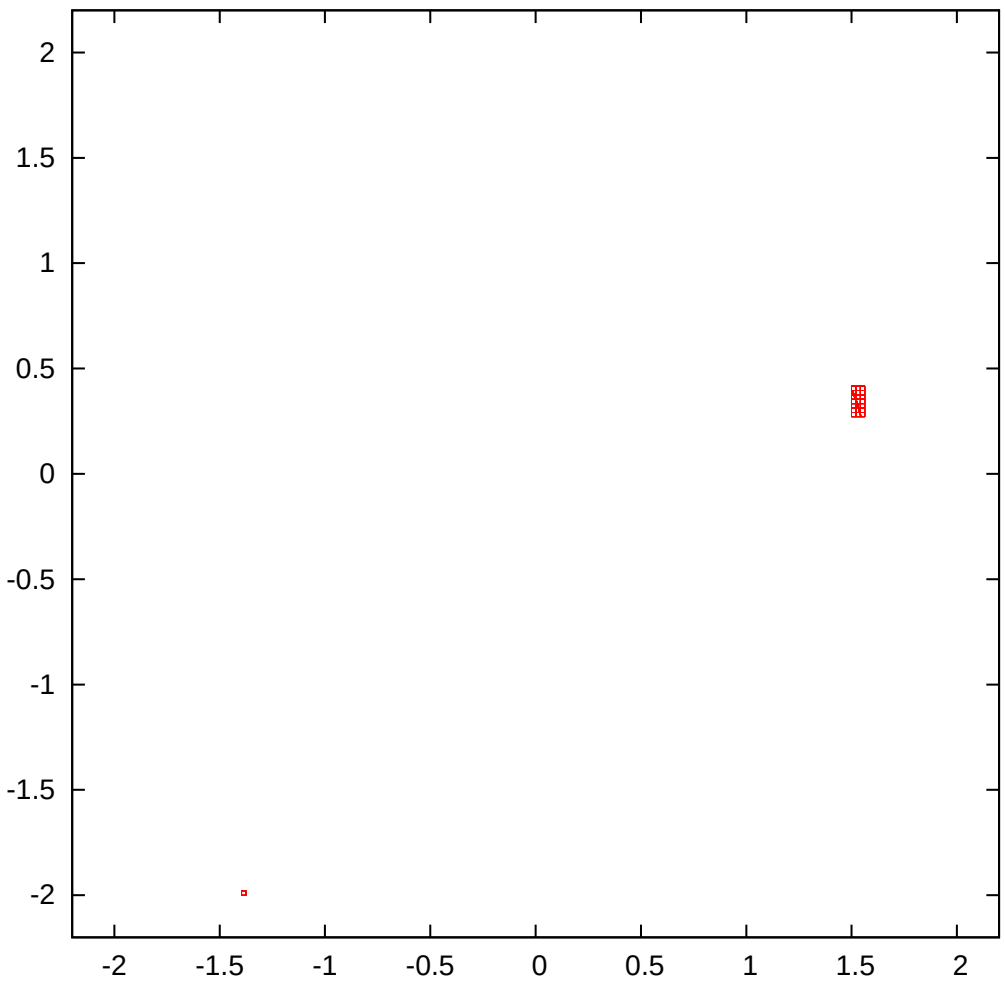
Grid Size	Number of Points per Cell	Percentage	Integration Time
16x10x10	10x10x10	62.5	33 s
	20x20x20	62.5	266 s
	30x30x30	62.5	1304 s
16x20x20	10x10x10	68.8	70 s
	20x20x20	68.8	552 s
	30x30x30	68.8	2708 s
	50x50x50	68.8	45309 s
32x40x40	10x10x10	82.4	294 s
	20x20x20	82.4	2493 s
	50x50x50	82.4	196101 s
64x80x80	10x10x10	90.7	2370 s
	20x20x20	90.7	203101 s
	50x50x50	90.7	1036498 s

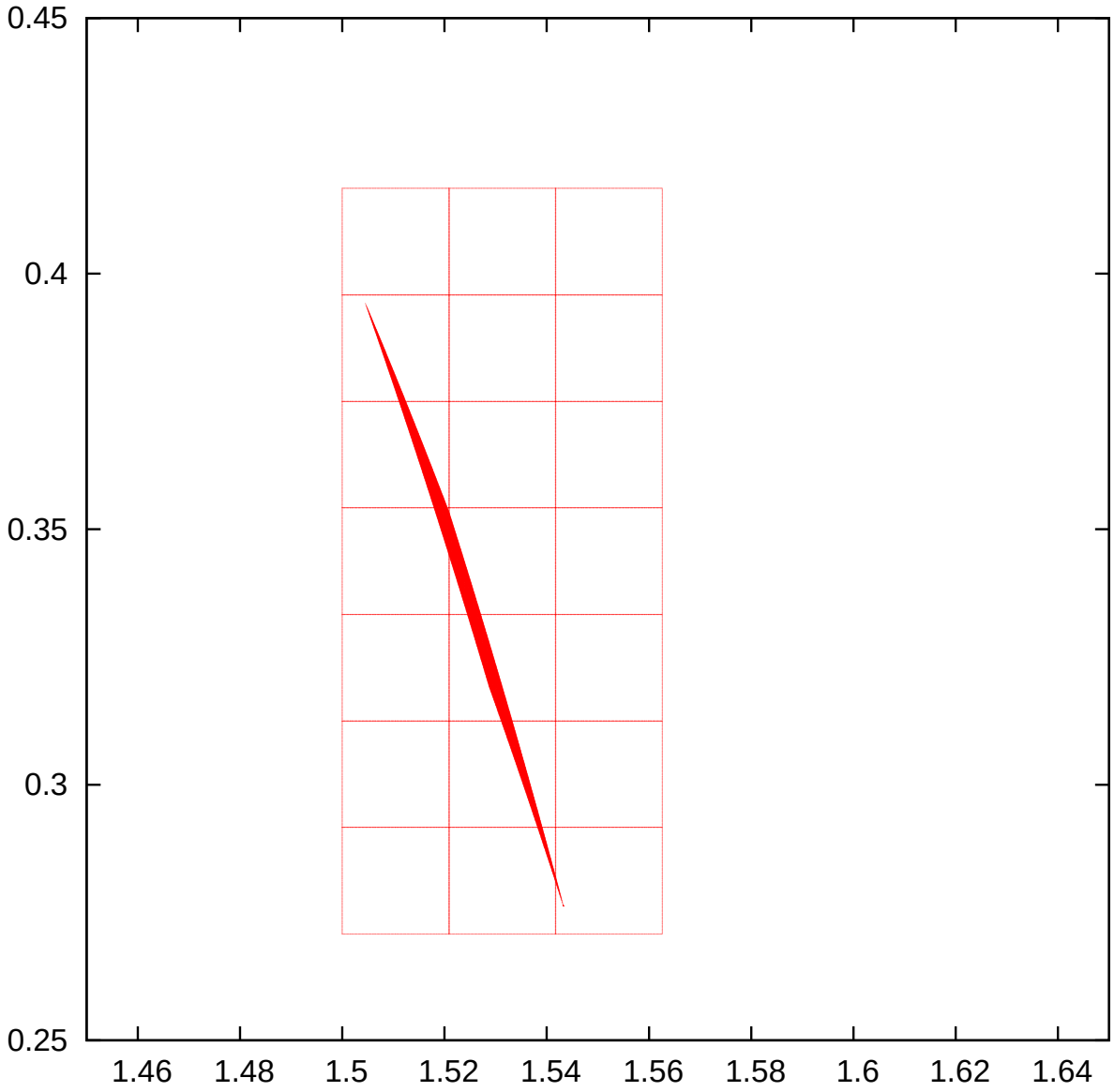

Percentage vs Number of Points
100
95
90
85
80
75
70
65
60
1E+06
1E+07
1E+08
1E+09
1E+10
1E+11

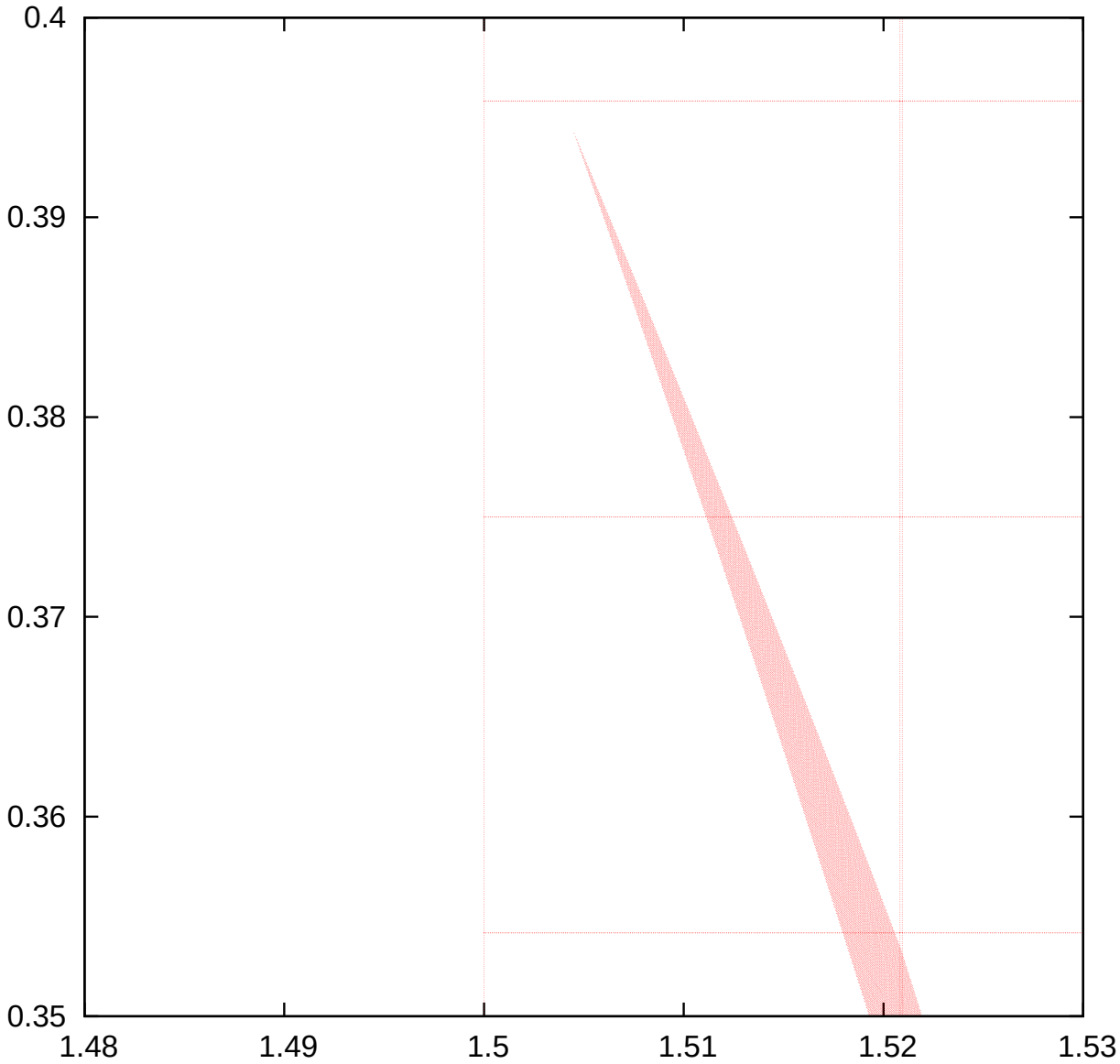
To understand the differences in data files, cells which did not agree between the RK and rigorous map were integrated individually. For example, for the case of cell dimension 192x192, cell #29 was the first to disagree:

Runge-Kutta		Rigorous	
29	21096	29	21096
29	21097	29	21097
29	21098	29	21098
29	21288	29	21288
29	21289	29	21289
29	21290	29	21290
...	...	...	...
29	22056	29	22056
29	22057	29	22057
29	22058	29	22058
		29	22248
		29	22249
		29	22250

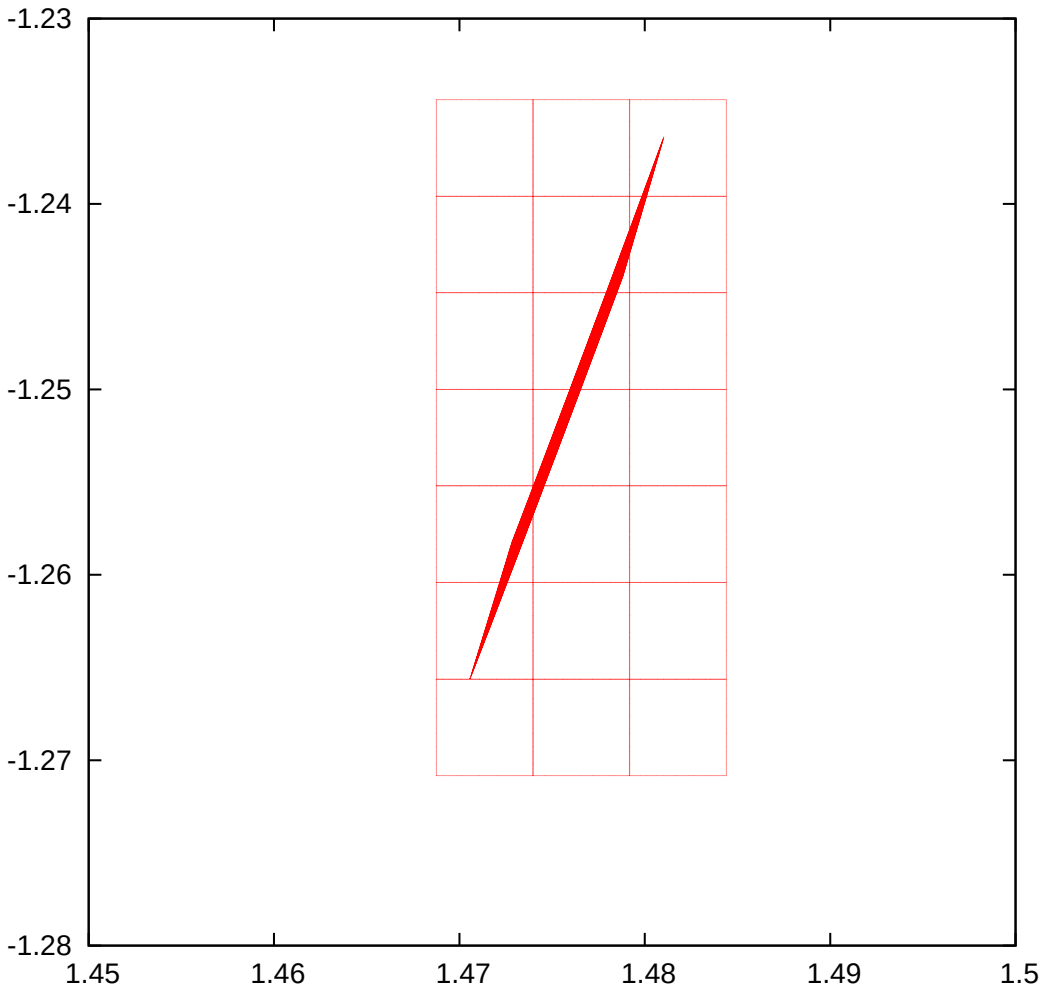
192x192, cell 29

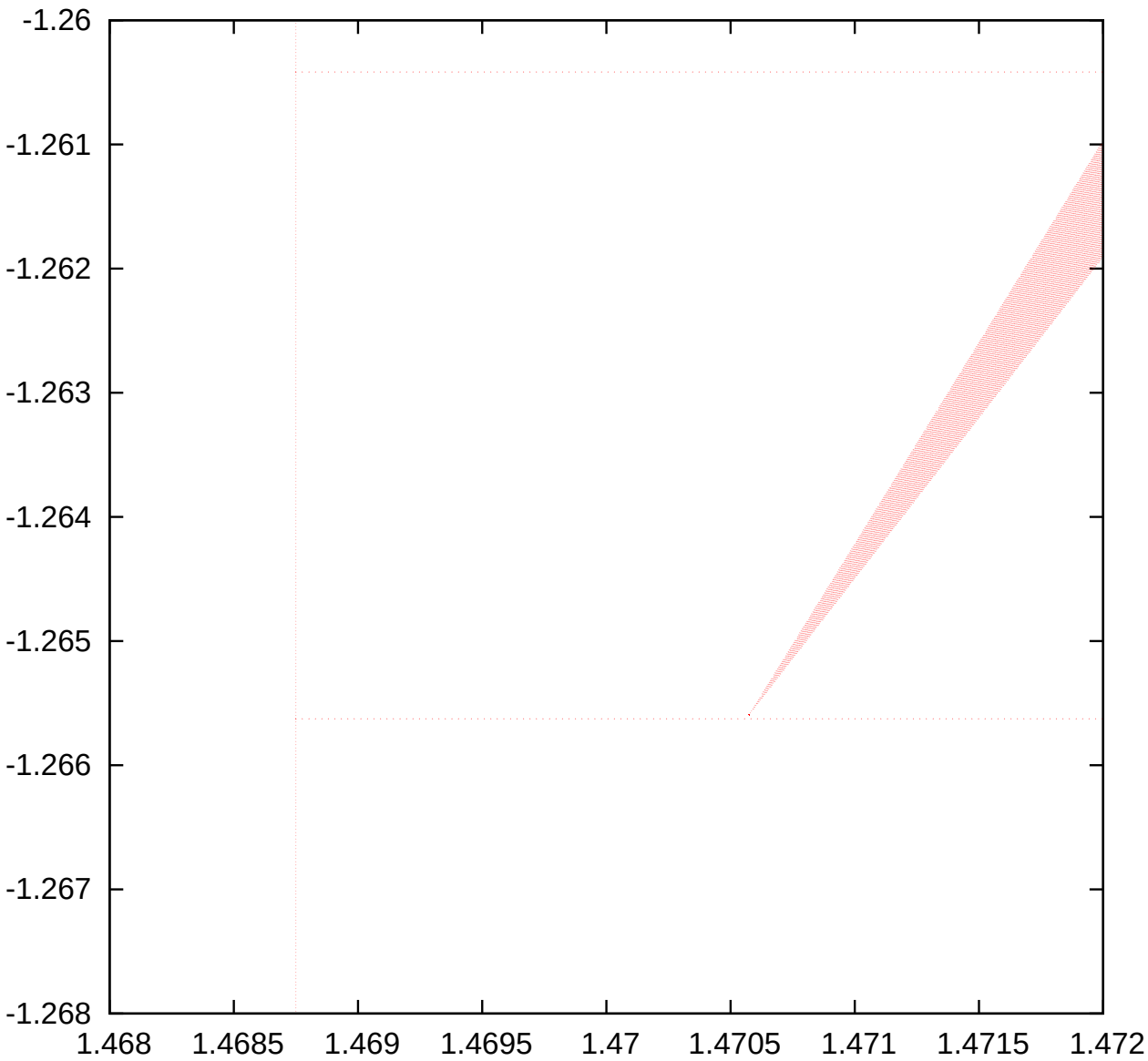



0.45
0.4
0.35
0.3
0.25
1.46
1.48
1.5
1.52
1.54
1.56
1.58
1.6
1.62
1.64


0.4
0.39
0.38
0.37
0.36
0.35
1.48
1.49
1.5
1.51
1.52
1.53

768x768 cell 54




-1.26
-1.261
-1.262
-1.263
-1.264
-1.265
-1.266
-1.267
-1.268
1.468
1.4685
1.469
1.4695
1.47
1.4705
1.471
1.4715
1.472

Since the map is determined by taking the minimum and maximum values in each variable, the critical corners which would allow the RK and rigorous maps to agree was recorded for increasing point resolution.

Dim - Cell	Point Dim.		
192 - 29		MIN(X)	MAX(Y)
	48x48	1.50451174017975	0.394202198889612
	480x480	1.50451174017985	0.394202198891457
	1920x1920	1.50451174017985	0.394202198891473
768 - 54		MIN(X)	MIN(Y)
	48x48	1.47057481186014	-1.26559403273587
	480x480	1.47057481186222	-1.26559403273073
	1920x1920	1.47057481186224	-1.26559403273069

It appears that as the point resolution increases, the key points that determine the map converge and do not approach the boundary to the next row of cells.

To conclude, the RK provides a lower bound to the map size, while the rigorous provides an upper bound, possibly overestimating the some individual cell mappings.

As the number of cells per dimension increases, the RK map approaches the rigorous map basically independent of the number of points per cell.

Most importantly, the RK never predicted a final cell that the rigorous did not for any individual cell.