Map Making
Cartography In Beam Physics

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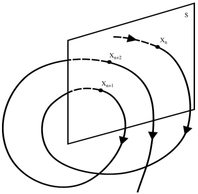
General Idea

Plan: Generate a High Order Map for an arbitrary field
Restrictions: Beam Element

- Surface Field Method
- Cylindrical Symmetry
- In Free Space

Focus on Quadrupole fields since we have data from RIKEN.
Need a few DA and TM based tools to do this properly to high order.

Poincaré Section



- Transversal to the flow
- Generally thought of for Periodic Dynamical Systems
- Differs from Recurrance Map - Spatial relation and not Time
- To generate a section for a whole flow instead of just a point

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Inverse Mapping

To find the inverse of a function, we decompose it into it's linear and non-linear parts.

- $f = m + n$ Assuming f^{-1} exists
- $f \circ f^{-1} = I \Rightarrow m \circ f^{-1} = I - n \circ f^{-1}$
- So we have $f^{-1} = m^{-1} \circ (I - n \circ f^{-1})$ where the right hand side can be shown to be contracting.

Constraint Satisfaction.

- Given a function $F(x, y, z)$, and constraint equation $g(x, y, z) = 0$, and wishing to find the subsurface of F where g is true, we do the following:
- Set up $F(x, y, z) = (g(x, y, z), y, z)$
- Find $x = F^{-1}(g(x, y, z), y, z)$
- Substitute x into F

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Poincaré Algorithm

- Obtain ODE flow including time expansion $f(x_0, t)$
- Choose a suitable Poincaré section $\sigma = 0$ - Transverse to the flow to provide invertibility
- Create an Auxiliary Function $\psi_k = x_k$ for v variables and $\psi_{v+1} = \sigma(f(x_0, t))$
- Invert to perform our constraint satisfaction to obtain an expansion $\psi^{-1}(x_0)$ so that $\sigma(f(x_0, t(x_0))) = 0$
- $t(x_0) = \psi^{-1}_{v+1}(x_0, 0)$
- Evaluate $f(x_0, t(x_0))$ for our Poincaré Section

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ODEs

The ODEs we wish to integrate through are the suitably scaled familiar ones developed for stable numerical integration by He Zhang and Martin Berz:

- $\frac{d\vec{x}}{d\tau} = \vec{\beta} = \frac{\vec{p}}{m\sqrt{1+\frac{p^2}{m^2}}} = \frac{\vec{p}}{\gamma m}$
- $\frac{d\vec{p}}{d\tau} = q\vec{E} + qc\frac{\vec{p}}{m\sqrt{1+\frac{p^2}{m^2}}} \times \vec{B} = q(\vec{E} + c\vec{\beta} \times \vec{B})$

Currently we use a simple Picard-Lindelof flow expansion operator:
$\vec{x} = \vec{A}(\vec{x}) = \vec{x} + \int_{t_i}^{t} \vec{f}(\vec{x}, t')dt'$

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Particle Optical Coordinates

$$
\begin{aligned}
r_1 = \quad x &= \quad x \\
r_2 = \quad a &= \quad p_x/p_0 \\
r_3 = \quad y &= \quad y \\
r_4 = \quad b &= \quad p_y/p_0 \\
r_5 = \quad l &= \quad -v_0\gamma(t-t_0)/(1+\gamma) \\
r_6 = \quad \delta_K &= \quad (K-K_0)/(K_0) \\
r_7 = \quad \delta_m &= \quad (m-m_0)/m_0 \\
r_8 = \quad \delta_z &= \quad (z-z_0)/z_0
\end{aligned}
$$

In which p is the momentum, K is the kinetic energy, v is the velocity, t is the time of flight, γ is the total energy over m_0c^2, m is the mass, and z is the charge. The subscript zero denotes that we are referring to the reference particle.

Helmholtz Theorem

We use Helmholtz's theorem (Fundamental Theorem of Vector Calculus) to obtain: $\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}) + \vec{\nabla} \cdot \phi(\vec{x})$

- $\phi(\vec{x}) = \frac{1}{4\pi} \int_{\partial\Omega} \frac{\vec{n}(\vec{x}_s) \cdot \vec{B}(\vec{x}_s)}{|\vec{x} - \vec{x}_s|} ds - \frac{1}{4\pi} \int_{\Omega} \frac{\nabla \cdot \vec{B}(\vec{x})_v}{|\vec{x} - \vec{x}_v|} dV$
- $\vec{A}(\vec{x}) = -\frac{1}{4\pi} \int_{\partial\Omega} \frac{\vec{n}(\vec{x}_s) \times \vec{B}(\vec{x}_s)}{|\vec{x} - \vec{x}_s|} ds + \frac{1}{4\pi} \int_{\Omega} \frac{\nabla \times \vec{B}(\vec{x})_v}{|\vec{x} - \vec{x}_v|} dV$

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Helmholtz Theorem cont.

Reduction in Free Space: $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = 0$

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Caveats:

- Expansions near the surface diverge
- Requires surface field interpolation
- Endcaps are not measured

"Solutions"

Helmholtz Theorem cont.

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"Solutions"

- Only concerned with Beam Axis
- Interpolation provides some smoothing of Data - Fourier "Interpolation"
- Generally can be considered zero

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RIKEN Data for Q500

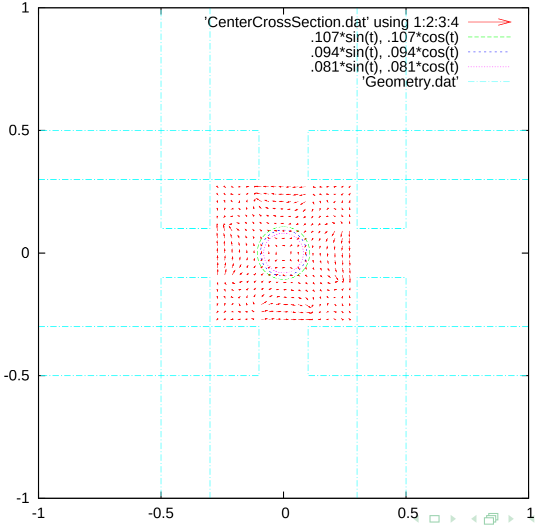
Quadrupole Element - 500mm long

- Given as 9° increments every 10mm
- Data taken well outside of element for field to decay along beam line
- Taken at 3 different radii - approximately at 10cm radius

Test Quadrupole

- Have need for a theoretical model of a finite field Quadrupole with fringe fields.
- Made of 4 superimposed Bar Magnets

Test Quadrupole



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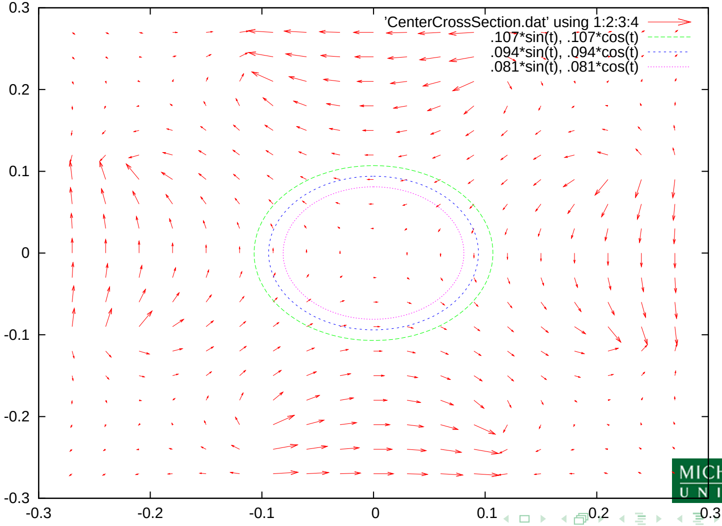
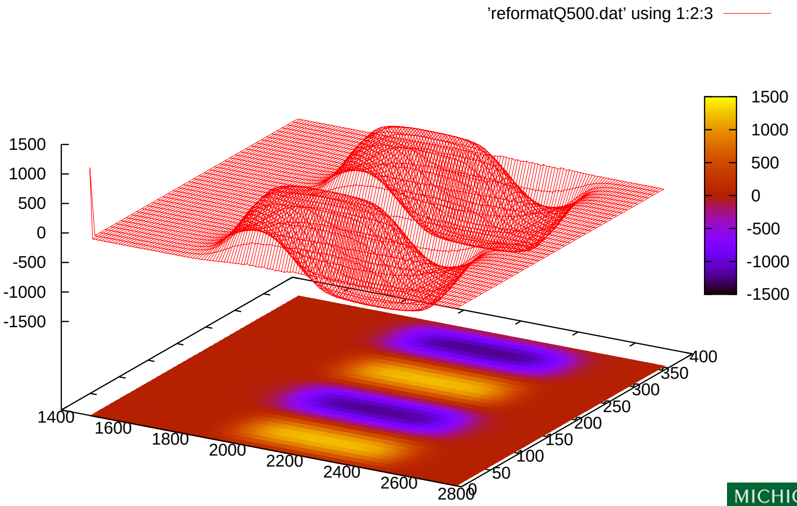
Test Quadrupole Zoomed



Field Profile - RIKEN Q500



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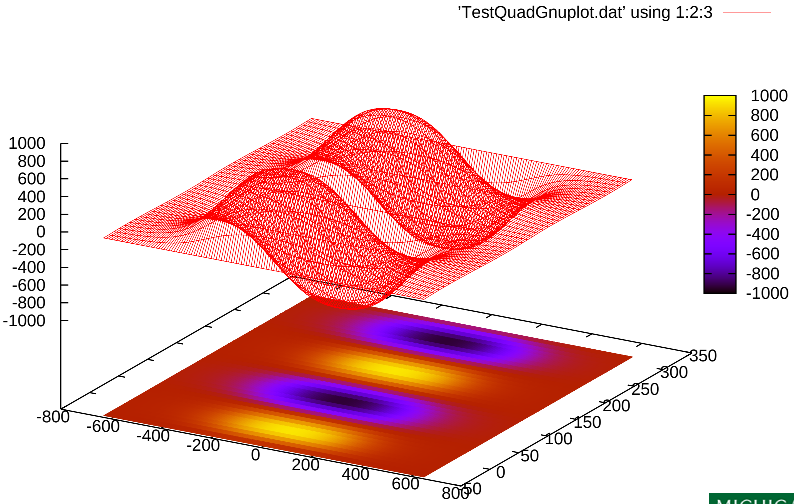
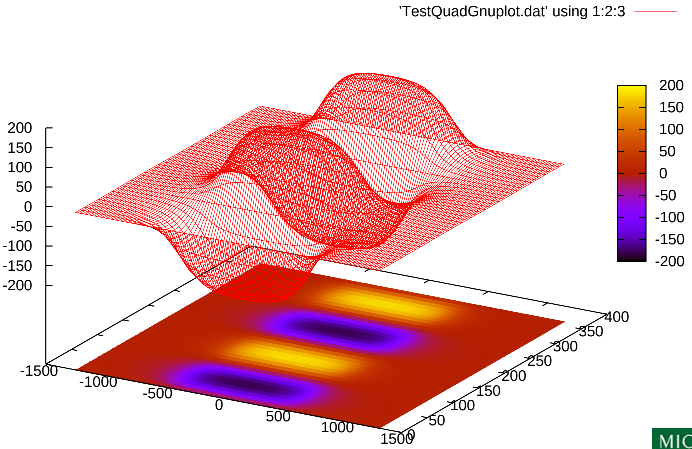


Field Profile - Test Quadrupole Longer



Fourier Analysis Method

- Solving Laplace's equation $\nabla^2 V = 0$ in Free Space
- Ansatz: $V = \sum_k \sum_l M_{k,l}(s) cos(l\phi + \theta_{k,l}) r^k$
- Discrete Fourier Transform: $X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi nk/N}$
- $V_{r_0} = \sum_n a_n cos(n\phi + \phi_n)$
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- $\begin{pmatrix} a_{2,r_1} \\ a_{2,r_2} \end{pmatrix} = \begin{pmatrix} r_1^2 & r_1^4 \\ r_2^2 & r_2^4 \end{pmatrix} \times \begin{pmatrix} M_{2,2} \\ M_{4,2} \end{pmatrix}$

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- For Magnetic Field: $\begin{pmatrix} a_{2,r_1} \\ a_{2,r_2} \\ a_{2,r_3} \end{pmatrix} = \begin{pmatrix} 2r_1 & 4r_1^3 & 6r_1^5 \\ 2r_2 & 4r_2^3 & 6r_2^5 \\ 2r_3 & 4r_3^3 & 6r_3^5 \end{pmatrix} \times \begin{pmatrix} B_{2,2} \\ B_{4,2} \\ B_{6,2} \end{pmatrix}$

Fourier Analysis from Recurrence

Another possible method from Laplace equation recurrence relation:

$$M_{l+2,l}(s) = \frac{M_{l,l}^{(2n)(s)}}{\prod_{v=1}^{n}((l)^2-(l+2v)^2))}$$

For the Quadrupole $M''_{2,2}(s) = -12M_{4,2}(s)$

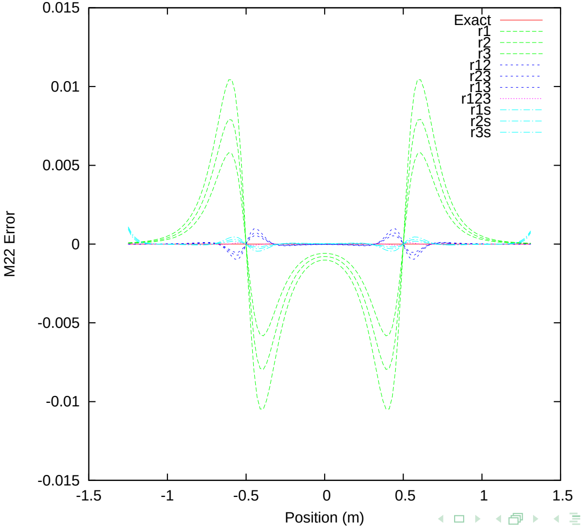
Like a finite difference method for a PDE, we can discretize along the beam axis to obtain:

$$a_{2,r_1}(s) = M_{2,2}(s)r_1^2 + (-\frac{1}{12}\frac{M_{2,2}(s-h)-2M_{2,2}(s)+M_{2,2}(s+h)}{h^2})r_1^4$$

Giving the system of equations:

$$\begin{pmatrix} r_1^2-\frac{r_1^4}{6h^2} & \frac{r_1^4}{12h^2} & 0 & 0 & \cdots \\ \frac{r_1^4}{12h^2} & r_1^2-\frac{r_1^4}{6h^2} & \frac{r_1^4}{12h^2} & 0 & \cdots \\ 0 & \frac{r_1^4}{12h^2} & r_1^2-\frac{r_1^4}{6h^2} & \frac{r_1^4}{12h^2} & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & & & \end{pmatrix} \begin{pmatrix} M_{2,2}(s_1) \\ M_{2,2}(s_2) \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{2,r_1}(s_1) \\ a_{2,r_1}(s_2) \end{pmatrix}$$

Test Quadrupole Results



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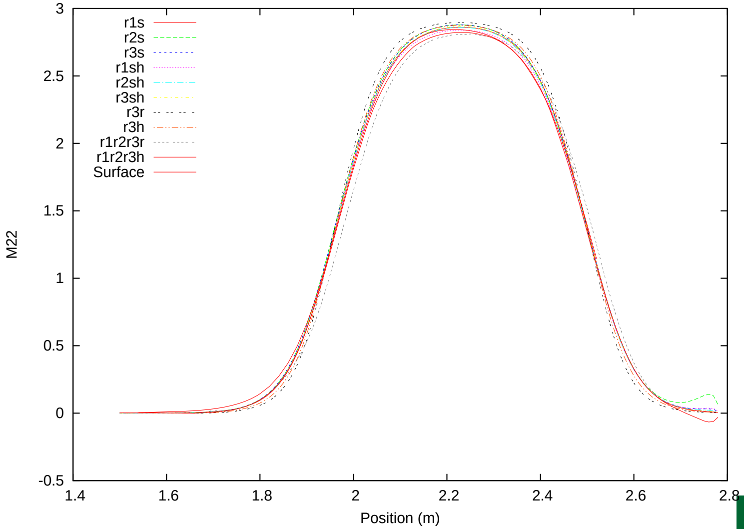
RIKEN Q500 Results



Summary of Method

- Obtain Smooth Interpolated Data using Fourier Expansion
- Generate Local Field Expansions via Helmholtz Decomposition around any point we desire to perform a Flow Integration
- Perform a flow integrations until we straddle the proposed Poincaré Section
- Obtain Crossing time of flow with Section
- Evaluate Flow at the Crossing time for the Poincaré Section
- Convert to Particle Optical Coordinates

Verification

- Taylor Models instead of Differential Algebraic objects
- Requires Verified Inversion to obtain Verified Crossing Time
  - Alternatively, Can take Heuristic bound on non-verified crossing time
  - Range bound constraint over the the complement of the space away from zero
- COSY-VI for integration
- Helmholtz decomposition integrals can be done with Taylor Models (Alternate implementation of normal vectors may need to be used.)

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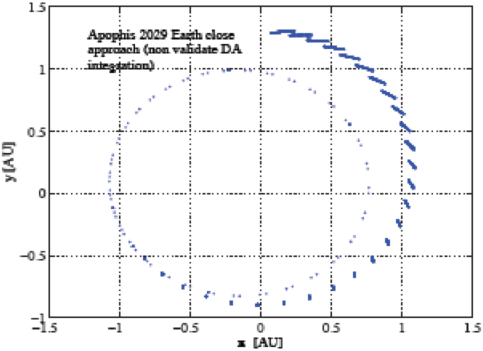
Possible Endeavor - Poincaré Integrator

Apophis



Integration and Poincaré section for reducing size of phase space.
Long Term integrations.

End

- Thanks for your time!