Numerical studies of systems depending on parameters

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First example: the shock tube problem

We consider the shock tube problem as a standard example of a hyperbolic system developing shock waves.

1d Euler's system

$$\begin{bmatrix} \rho \\ \rho v \\ \rho e \end{bmatrix}_t + \begin{bmatrix} \rho v \\ \rho v^2 + p \\ (\rho e + p)\, v \end{bmatrix}_x = 0$$

$$e = \frac{p}{\rho(\gamma - 1)} + \frac{1}{2} v^2$$

- $\rho(t, x)$ is the mass density
- $v(t, x)$ is the velocity
- $e(t, x)$ is the energy density
- $p(t, x)$ is the pressure
- $\gamma = c_p / c_v$ is the ratio of specific heats.

Initial and boundary conditions

$$(\rho(0,x), v(0,x), p(0,x)) = \begin{cases} (1,0,1) & \text{if} \quad x \in [0,1/2] \\ (1/8,0,1/10) & \text{if} \quad x \in (1/2,1]\,. \end{cases}$$

The boundary conditions correspond to an open pipe and are enforced by a zeroth order extrapolation.

This setting generates three distinct waves, travelling at speed equal to the eigenvalues of the linearized system:

- A rarefaction wave: $(\lambda_1 = v - c(\gamma))$
- A shock wave: $(\lambda_2 = v)$
- A contact wave: $(\lambda_3 = v + c(\gamma))$

The sound speed is $c(\gamma) = \sqrt{\gamma p/\rho}$.

A variable parameter

The parameter γ is assumed to take values in the interval

$$[\gamma_a, \gamma_b] = [7/5, 5/3]\,;$$

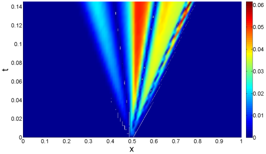
this interval covers both monoatomic and biatomic gases.

We look (at first) for the Taylor expansion with respect to the parameter γ of the solutions of Euler's system, using our version of Taylor Model algorithms.

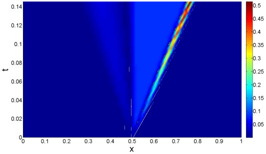
We use Roe's algorithm for integrating the equation, which is the most standard for this kind of hyperbolic problem. The algorithm is implemented as is, except for the fact that we use Taylor objects.

Taylor results

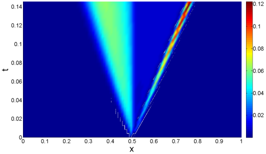
Plot of ρ, v and p with respect to x and t, computed with a Taylor expansion of order 5. We use a grid of 100 points.



$\rho(7/5) - \rho(5/3)$

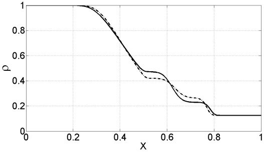


$v(7/5) - v(5/3)$

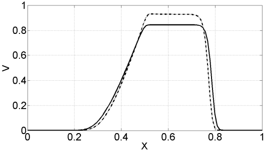


$p(7/5) - p(5/3)$.

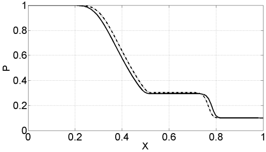
Plot of ρ, v and p with respect to x and $t = 0.15$,



$\rho(7/5)$ and $\rho(5/3)$



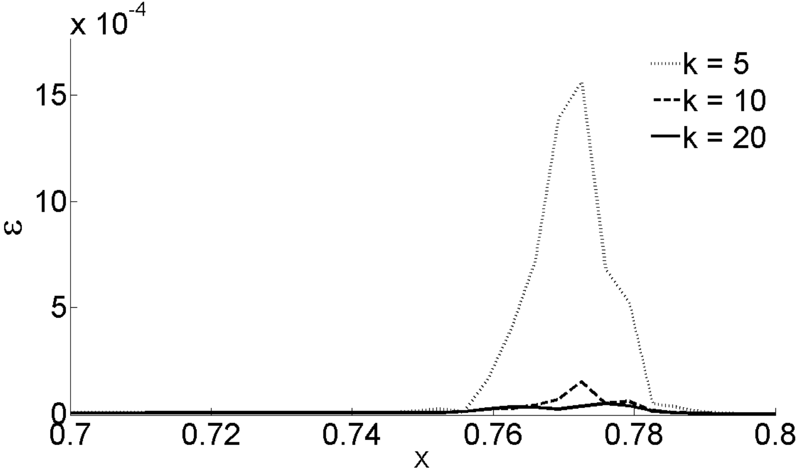
$v(7/5)$ and $v(5/3)$



$p(7/5)$ and $p(5/3)$.

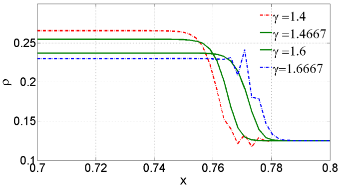
Estimate of the error

Error on density reconstruction with a grid $N = 300$ for different orders. Only the region close to the shock wave is represented.

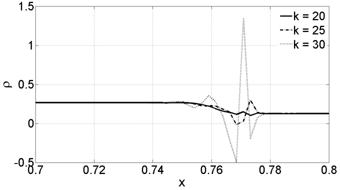


Finer grid

In order to test the effectiveness of the method to work with discontinuities, we refine the grid. The pictures represent the density computed with $N = 400$ and the corresponding error, in the region close to the shock wave. It is quite clear that the method fails due to the fact that the radius of convergence of the Taylor expansion drops, when the grid is refined.



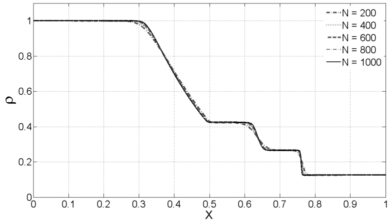
Density



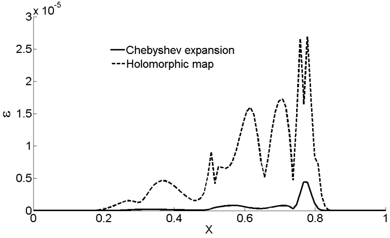
Error at $\gamma = 7/5$.

Chebyshev expansion

We need to change the representation. We chose Chebyshev series instead of Taylor's. Note that it is very simple to adapt the algoritms for the automatic computation of the expansion. Provided the order of the expansion is high enough, the errors can be kept very low, even with a very fine grid ($N = 1000$).



Density



Errors

Chebyshev vs. Taylor (or....)

The numerical complexity of the Taylor and Chebyshev approaches is very similar, with a slight advantage for the Taylor expansion, due to a simpler formula for the multiplication of a Taylor series. Also, the Taylor series has the advantage of providing directly the values of the derivatives of the functions under consideration, and for this reason it looks more useful e.g. for sensitivity analyses. On the other hand, the Chebyshev expansion is much more powerful when one wishes to be able to represent sharp discontinuities (which will nonetheless appear smoothed by the numerical discretization), or when one need to compose the expansion with a less than smooth function (such as the spectral projection used in Roe's algorithm). In these cases, by choosing a sufficiently high order, it is possible to obtain an approximation as good as required.

Second example: an airfoil of variable geometry

Now we consider a standard problem with an elliptic equation in a 2d domain, solved with a standard finite element algorithm. The main novelty is due to the fact that the domain is parametrized.

The Joukowski map

$$J(z) = z + \frac{1^2}{z}$$

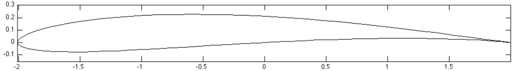
is conformal in $\mathbb{C} \setminus \{0, \pm 1\}$, while at ± 1 is doubles the angles.

When J is applied to a circle C_{z_0} centered in $z_0 = a + \mathrm{i}\, b$ and passing through $z = 1$, it gives the typical shape of an airfoil.

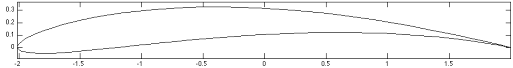
An airfoil of variable geometry

a is related to the witdh of the airfoil

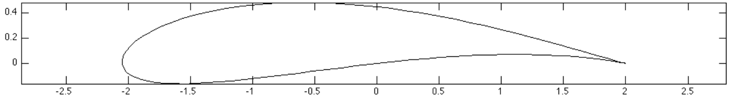
b is related to the curvature of the airfoil



$a = 0.05,\ b = 0.05$



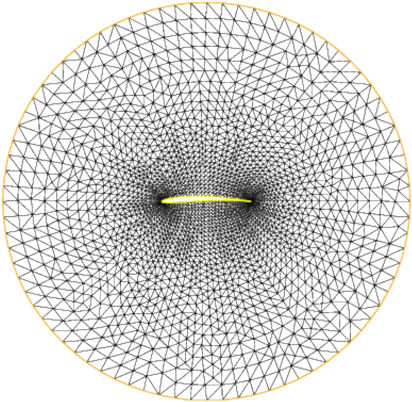
$a = 0.05,\ b = 0.1$

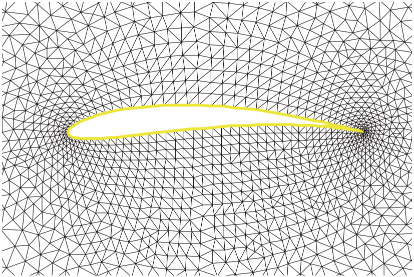


$a = 0.1,\ b = 0.1$

The mesh

We build a mesh in the annulus $r_0 \leq \rho \leq 1$ and we transform it by the Joukowski map:





A potential flow

We consider a flow which is:

- irrotational: $\nabla \times \mathbf{v} = 0$
- incompressible: $\nabla \cdot \mathbf{v} = 0$
- non viscous.

We can define the *stream function* $\psi(x, y)$:

$$\mathbf{v}_1 = \frac{\partial \psi}{\partial y} \quad \mathbf{v}_2 = -\frac{\partial \psi}{\partial x}$$

and then we have

$$\Delta \psi = 0\,.$$

Boundary conditions

- at infinity the flow is constant.

$$u = V_{\infty,x} \; v = V_{\infty,y} \implies \psi = (V_{\infty,x})\, y - (V_{\infty,y})\, x = y - \tan(\alpha)\, x$$

- at the airfoil Γ_2 the flow is tangent.

$$\langle \mathbf{v}, \mathbf{n} \rangle = 0 \implies \frac{\partial \psi}{\partial x} n_x - \frac{\partial \psi}{\partial y} n_y = 0$$
$$\implies \frac{\partial \psi}{\partial s} = 0 \text{ su } \Gamma_2 \implies \psi = c$$

We have a bounded, regular domain Ω_t depending on parameters. The system is:

$$\begin{cases} \Delta \psi_t = 0 & \text{in } \Omega_t \\ \psi_t = y - \tan(\alpha)\, x & \text{on } \Gamma_1^t \\ \psi_t = c_t & \text{on } \Gamma_2^t \end{cases}$$

Where c_t must be computed in order to satisfy the Kutta-Joukowski condition.

Kutta-Joukowski condition

The Kutta-Joukowski condition can be stated as follows:
The streamlines leave smoothly the trailing edge.

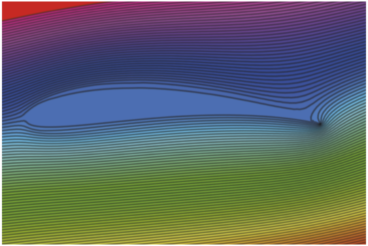
- it comes from experimental considerations
- it is related to the circulation of th flow around the airfoil
- it allows the assumption of non viscous fluid.

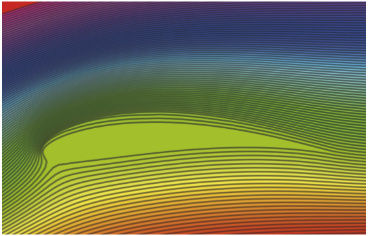
$$\psi_t = \psi_{0,\,t} + c_t\,\psi_{1,\,t}$$

$$\Longrightarrow\ c_t \in \mathcal{T}\ \text{ t.c. } \left[\frac{\partial\psi_{0,\,t}}{\partial\mathbf{n}} + c_t\,\frac{\partial\psi_{1,\,t}}{\partial\mathbf{n}}\right]_{TE^-}^{TE^+} = 0$$

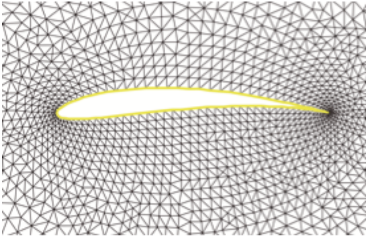
The normal derivaties are discretized as finite differences.

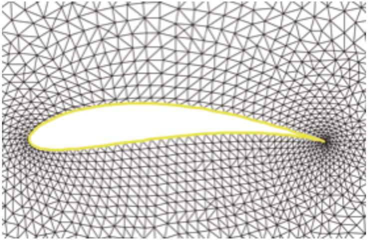
Effect of the Kutta-Joukowski condition:



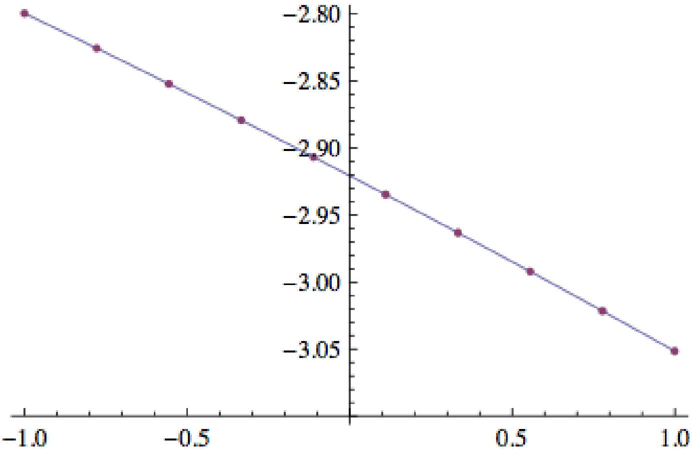


Airfoils:





Lift as a function of the parameter, with Taylor of order 6.



The relative error is less than 10^{-6}.