A Rigorous Implementation of Taylor Model-Based Integrator in Chebyshev Basis Representation [1]

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Motivation

Hybrid System: A dynamic system with both discrete and continuous behavior

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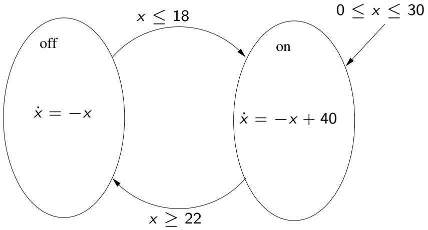
Example: Thermostat



Safety Verification of Hybrid Systems

Continuous variables evolve according to differential equations

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Computing of rigorous enclosures of continuous behavior required

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Implementation in tool HSOLVER [Ratschan and She, 2007]
`http://hsolver.sourceforge.net`
Uses degree one Taylor expansions of continuous evolution

Evolution Enclusure in Hybrid Systems Verification

We want to use better than degree one Taylor expansions

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⇒ Use of rigorous Taylor model-based integrator

We want to handle highly non-linear input models

⇒ We propose new method for long-term wrapping effect suppresion

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$\Rightarrow$ We propose new method for long-term wrapping effect suppresion

We want to handle high dimensional examples with small models

$\Rightarrow$ Use of close to optimal Chebyshev basis approximations

Verified Taylor Model-Based Integrator

Given:

- ODE: $\dot{x} = f(x) + I_1$
- Initial condition $g(a) + I_2$
- Time bound t_{max}

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- I_3 is always wider than I_2
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Solution: Make interval I_2 empty or very small

Long-Term Stabilization of Integrator

Known methods:

- Shrink wrapping - Include error interval into the polynomial part of Taylor model
- Preconditioning - Use composition of two Taylor models where the outer Taylor model is error free

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Problem: Non-linear initial condition

New Long-Term Stabilization Method

We propose a new method based on affine arithmetic

In each integration step:

- Use affine combination of new variables b to create Taylor model $g'(b)$ of interval I_2
- Use $g(a) + g'(b)$ as a new initial condition
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Problem: In each step we add new set of additional variables b

Multi-Step Integration

In $i - th$ integration step:

- Split initial condition $g_i(a, b_{i-1}) + I_2$ into $g_i^*(a)$ and $g_i^+(a, b_{i-1}) + I_2$

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We can estimate the error through back-substitution, since the dependency between b_{i-1} and b_i is known

New Method Properties

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Additional variables that represent the unknown error required

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Expanding function in Chebyshev polynomials:

$$f(x) = \sum_{i=0}^{\infty} a_i T_i(x)$$

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Approximation is orders of accuracy more accurate than expansions in Taylor series [Kaucher and Miranker, 1988]

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Integration:

- $\int_0^y T_0(x)dx = T_1(y)$
- $\int_0^y T_1(x)dx = (T_0(y) + T_2(y))/4$
- for even $i > 1$:
  $\int_0^y T_i(x)dx = (-T_{i-1}(y)/(i-1) + T_{i+1}(y)/(i+1))/2$
- analogous for odd $i > 1$

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Possibility to extend all operations with rigorous error estimation

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$x_0 \in [0.95, 1.05] \quad y_0 \in [2.95, 3.05]$
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COSY computes resulting Taylor model in time t_{max} with error interval of width 3×10^{-9} (degree 12 Taylor model)
[Makino and Berz, 2006]

Computational Experiments

Table of error interval width:

	Our Implementation		COSY
Order	Taylor series	Chebyshev poly.	
4	4.4E-2	7.4E-3	
6	1.1E-3	6.6E-5	
8	3.4E-5	6.2E-7	
10	1.1E-6	5.7E-9	
12	3.4E-8	5.2E-11	3E-9
14	1.1E-9	9.8E-13	

Computation of our tool used fixed time step

Conclusion

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Computational experiments demonstrate the usefulness of the method

Ongoing, Future Work

- Automatic selection of the time step length

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I want to thank Stefan Ratschan for his comments that helped me to prepare this presentation