Motivation

- Asteroid survey is a key program for planetary defense
- Accurate orbit determination (OD) is necessary to predict the asteroid orbital parameters and impact probability...
- Accurate OD requires many observations of the same object

- For optical systems (i.e. telescope) a minimum of three observations are necessary to determine an initial guess of the orbit (preliminary OD)
- Due to observation uncertainties the initial orbit can be far from the real one

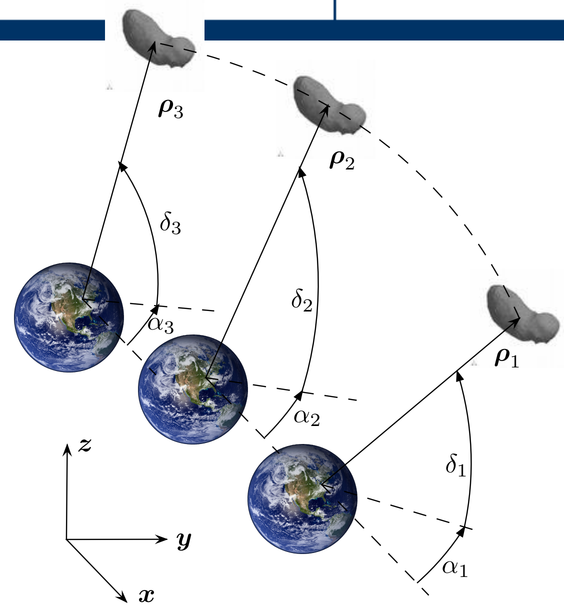
Where should I point telescope to obtain additional observations?

- Differential algebra is used to:
  - Implement of a new high-order iterative procedure for preliminary OD
  - Analytically map observation uncertainties from the observations to the phase space

Outline

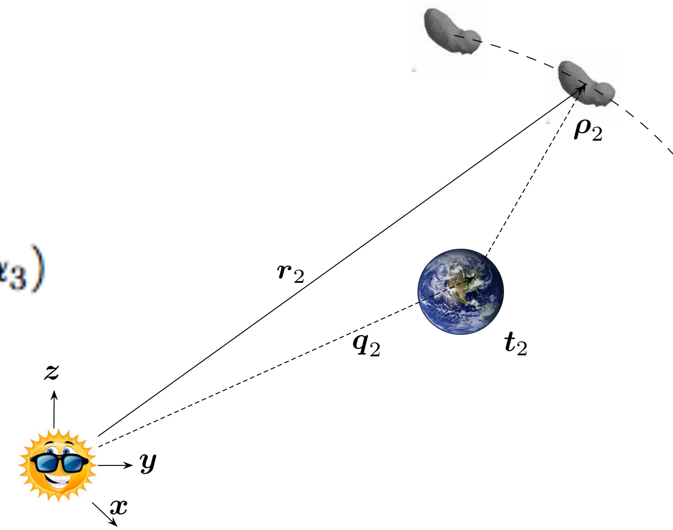
- Preliminary orbit determination problem
  - Gauss Method
- Differential Algebra (DA)
- Solution of parametric implicit equations
- High-order preliminary orbit determination algorithm
- Test cases

Preliminary orbit determination (OD) problem



- Given:
  - three observation epochs t_1, t_2, and t_3
  - three measured right ascensions $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$
  - three measured declinations $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3)$
  - the inertial position of the observer $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$
- Compute the position $\boldsymbol{r}_2$ of the body at t_2 that exactly matches all the observations in a two-body dynamical framework
- Preliminary OD is an old problem (Laplace 1780, Gauss 1809), but it is still the starting point for newly discovered objects
- When more than 3 observations are available a least square problem is set up using the preliminary orbit as first guess
- The measurement uncertainties play a key-role in orbit determination

Preliminary orbit determination (OD) problem



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Preliminary OD: Gauss' method

- Gauss' method was derived for the recovery of the dwarf planet Ceres (1801)
- Preliminary OD is reduced to a two-step procedure:
  1. It exploits truncated Lagrange coefficients expansions to determine a first guess solution at epoch t_2 (solving a 8-th order polynomial)
  2. It iterates using the exact Lagrange coefficients until the three sets of observations are exactly satisfied in the two-body dynamical framework (when it converges)
- Measurement accuracies are considered only when additional observations are available, i.e. propagation of the covariance matrix

Differential Algebra is exploited to:

- Implement a new high-order iterative procedure for preliminary OD
- Deal with measurement uncertainties analytically from preliminary OD

Differential Algebra (DA)

- Differential Algebra (DA) is an automatic differentiation technique

Algebra of real numbers ⇨ Algebra of Taylor polynomials

- Unlike standard automatic differentiation tools, the analytic operations of differentiation and antiderivation are introduced

- DA can be easily implemented in a computer environment (COSY-Infinity, Berz and Makino, 1998)

- Given any sufficiently regular function f of v, DA enables the computation of its Taylor expansion up to an arbitrary order n with respect to all v variables and any additional parameter

Solution of parametric implicit equations

- We want to find the solution $x(p)$ of the parametric implicit equation (PIE)

$$f(x,p) = 0 \tag{1}$$

- Compute the solution x for the nominal value of the parameter p by a pointwise method (e.g. Newton's method)
- Initialize the state and the parameter as DA variables

$$\begin{aligned} [x] &= x + \delta x \\ [p] &= p + \delta p. \end{aligned} \tag{2}$$

- Evaluate (1) in the DA-framework to obtain

$$\delta f = \mathcal{M}_f(\delta x, \delta p)$$

- Map (2) has no constant part as x is solution of (1) for p
- Build the map

$$\begin{bmatrix} \delta f \\ \delta p \end{bmatrix} = \begin{bmatrix} \mathcal{M}_f \\ \mathcal{I}_p \end{bmatrix} \begin{bmatrix} \delta x \\ \delta p \end{bmatrix}$$

Solution of parametric implicit equations

- Invert the Taylor map (ad hoc inversion algorithm available)

$$\begin{bmatrix} \delta x \\ \delta p \end{bmatrix} = \begin{bmatrix} \mathcal{M}_f \\ \mathcal{I}_p \end{bmatrix}^{-1} \begin{bmatrix} \delta f \\ \delta p \end{bmatrix}$$

- Evaluate the map in $\delta f = 0$ to satisfy equation

$$\begin{bmatrix} \delta x \\ \delta p \end{bmatrix} = \begin{bmatrix} \mathcal{M}_f \\ \mathcal{I}_p \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \delta p \end{bmatrix} \quad (3)$$

- Extract the first line of (3)

$$\delta x = \mathcal{M}_x(\delta p)$$

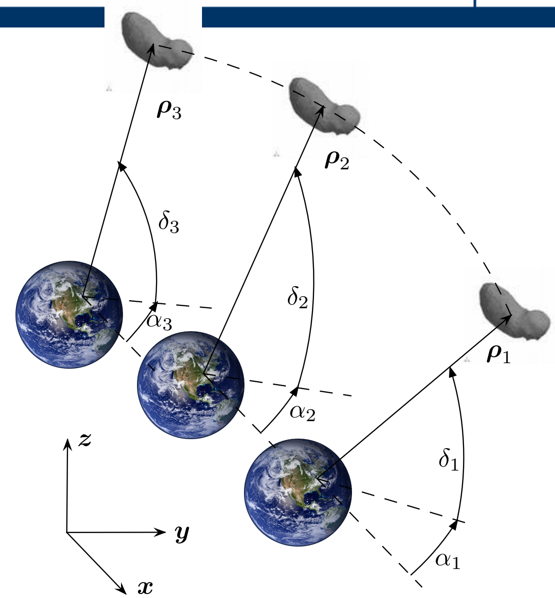
- Obtain

$$[x] = x + \mathcal{M}_x(\delta p) \quad (4)$$

- The solution $x(p)$ is approximated by a Taylor polynomial of arbitrary order: given any δp the evaluation of (4) delivers the new solution of the parametric implicit equation

Preliminary OD: DA-based method

- The solution of Gauss' 8-th order polynomial is the starting point
- Gauss' iterations are replaced by an high-order iterative procedure solving:
  - Lambert's problem from t_2 and t_3
  - Kepler's problem from t_2 and t_1



- Lambert's and Kepler's problems are solved via the high-order DA algorithm for the solution of parametric implicit equations
- The result is the state $\boldsymbol{r}_2$ that exactly matches the observations + its dependence from the observation accuracies in terms of a Taylor map:

$$\boldsymbol{r}_2 = \boldsymbol{r}_2(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta})$$

- The Taylor polynomial of $\boldsymbol{r}_2$ is mapped forward in time to analytically describe the regions of the sky to be scanned for successive observations

High-order iterative OD

- Initialize the observation sets as DA variables

$$[\boldsymbol{\alpha}] = \boldsymbol{\alpha} + \delta\boldsymbol{\alpha}$$
$$[\boldsymbol{\delta}] = \boldsymbol{\delta} + \delta\boldsymbol{\delta}.$$

- Thus, the lines of sight at t_2 and t_3 are

$$[\hat{\boldsymbol{\rho}}_2] = \hat{\boldsymbol{\rho}}_2 + \mathcal{M}_{\hat{\boldsymbol{\rho}}_2}(\delta\alpha_2, \delta\delta_2)$$
$$[\hat{\boldsymbol{\rho}}_3] = \hat{\boldsymbol{\rho}}_3 + \mathcal{M}_{\hat{\boldsymbol{\rho}}_3}(\delta\alpha_3, \delta\delta_3).$$

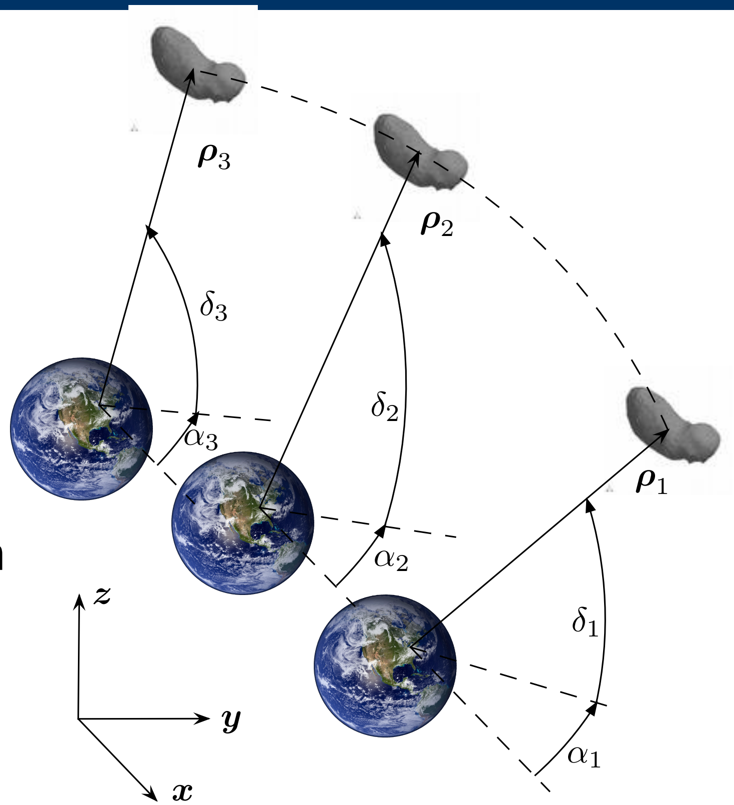
- Compute Gauss' 8-th order polynomial solution
- Initialize the slant ranges at t_2 and t_3 as DA variables

$$[\rho_2]^{1^-} = \rho_2^{1^-} + \delta\rho_2$$
$$[\rho_3]^{1^-} = \rho_3^{1^-} + \delta\rho_3,$$

where the constant parts are from the solution of Gauss' 8th order polynomial

- Thus, compute the Taylor polynomial of the asteroid position at t_2 and t_3 :

$$[\boldsymbol{r}_2] = \boldsymbol{r}_2 + \mathcal{M}_{\boldsymbol{r}_2}(\delta\alpha_2, \delta\delta_2, \delta\rho_2)$$
$$[\boldsymbol{r}_3] = \boldsymbol{r}_3 + \mathcal{M}_{\boldsymbol{r}_3}(\delta\alpha_3, \delta\delta_3, \delta\rho_3).$$



High-order iterative OD

- Solve Lambert's problem from t_2 and t_3 using the DA-solution solution of PIE

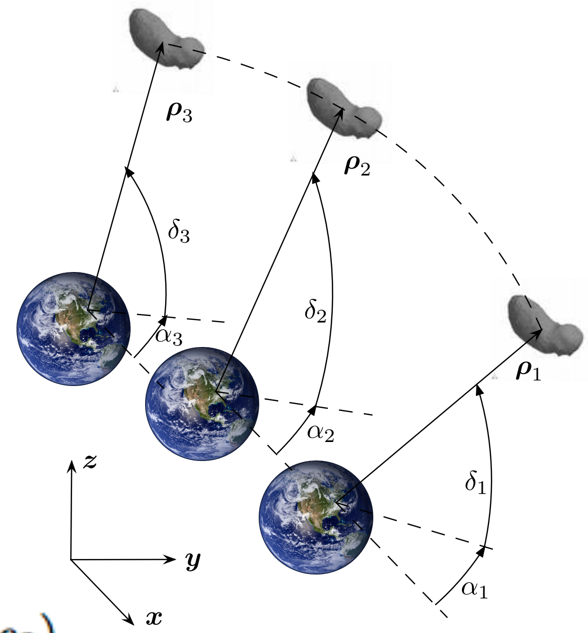
$$[\boldsymbol{v}_2] = \boldsymbol{v}_2 + \mathcal{M}_{\boldsymbol{v}_2}(\delta\alpha_2, \delta\delta_2, \delta\alpha_3, \delta\delta_3, \delta\rho_2, \delta\rho_3)$$

- Solve Kepler's problem from t_2 and t_1

$$[\boldsymbol{r}_1] = \boldsymbol{r}_1 + \mathcal{M}_{\boldsymbol{r}_1}(\delta\alpha_2, \delta\delta_2, \delta\alpha_3, \delta\delta_3, \delta\rho_2, \delta\rho_3)$$

- Compute the associated values of the observables

$$[\tilde{\alpha}_1] = \tilde{\alpha}_1 + \mathcal{M}_{\tilde{\alpha}_1}(\delta\alpha_2, \delta\delta_2, \delta\alpha_3, \delta\delta_3, \delta\rho_2, \delta\rho_3)$$
$$[\tilde{\delta}_1] = \tilde{\delta}_1 + \mathcal{M}_{\tilde{\delta}_1}(\delta\alpha_2, \delta\delta_2, \delta\alpha_3, \delta\delta_3, \delta\rho_2, \delta\rho_3).$$



- Build the map of defects

$$[\Delta\tilde{\alpha}_1] = [\tilde{\alpha}_1] - [\alpha_1] = \Delta\tilde{\alpha}_1 + \mathcal{M}_{\Delta\tilde{\alpha}_1}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta}, \delta\rho_2, \delta\rho_3)$$
$$[\Delta\tilde{\delta}_1] = [\tilde{\delta}_1] - [\delta_1] = \Delta\tilde{\delta}_1 + \mathcal{M}_{\Delta\tilde{\delta}_1}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta}, \delta\rho_2, \delta\rho_3),$$

where the constant part is due to initial guess (it is not the exact solution)

- Build an origin preserving map

$$[\Delta\alpha_1] = [\Delta\tilde{\alpha}_1] - \Delta\tilde{\alpha}_1 = \mathcal{M}_{\Delta\alpha_1}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta}, \delta\rho_2, \delta\rho_3)$$
$$[\Delta\delta_1] = [\Delta\tilde{\delta}_1] - \Delta\tilde{\delta}_1 = \mathcal{M}_{\Delta\delta_1}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta}, \delta\rho_2, \delta\rho_3)$$

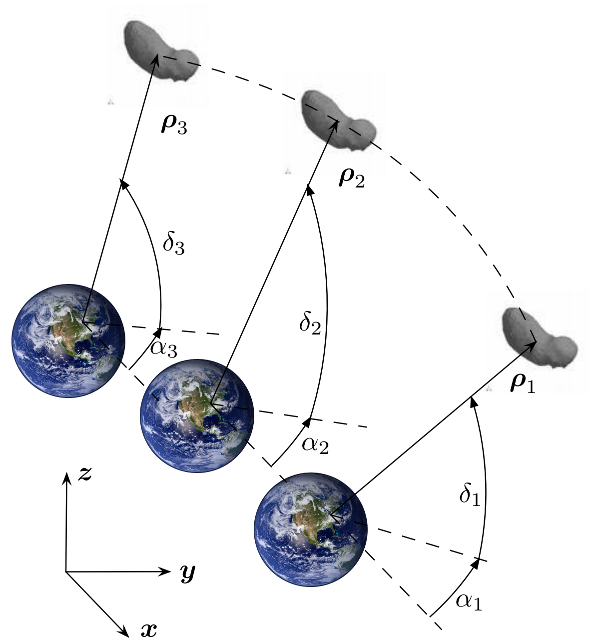
High-order iterative OD

- Build an augmented map

$$\begin{bmatrix} \Delta\alpha_1 \\ \Delta\delta_1 \\ \delta\boldsymbol{\alpha} \\ \delta\boldsymbol{\delta} \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{\Delta\alpha_1} \\ \mathcal{M}_{\Delta\delta_1} \\ \mathcal{I}_{\boldsymbol{\alpha}} \\ \mathcal{I}_{\boldsymbol{\delta}} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\alpha} \\ \delta\boldsymbol{\delta} \\ \delta\rho_2 \\ \delta\rho_3 \end{bmatrix}$$

- Invert it with Taylor polynomial algorithm

$$\begin{bmatrix} \delta\boldsymbol{\alpha} \\ \delta\boldsymbol{\delta} \\ \delta\rho_2 \\ \delta\rho_3 \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{\Delta\alpha_1} \\ \mathcal{M}_{\Delta\delta_1} \\ \mathcal{I}_{\boldsymbol{\alpha}} \\ \mathcal{I}_{\boldsymbol{\delta}} \end{bmatrix}^{-1} \begin{bmatrix} \Delta\alpha_1 \\ \Delta\delta_1 \\ \delta\boldsymbol{\alpha} \\ \delta\boldsymbol{\delta} \end{bmatrix}$$



- Extract the map

$$\begin{bmatrix} \delta\rho_2 \\ \delta\rho_3 \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{\rho_2} \\ \mathcal{M}_{\rho_3} \end{bmatrix} \begin{bmatrix} \Delta\alpha_1 \\ \Delta\delta_1 \\ \delta\boldsymbol{\alpha} \\ \delta\boldsymbol{\delta} \end{bmatrix}$$

High-order correction map!

- Evaluate in $[\Delta\alpha_1] = -\Delta\tilde{\alpha}_1$ and $[\Delta\delta_1] = -\Delta\tilde{\delta}_1$
- Iterate until $\Delta\tilde{\alpha}_1 = 0$ and $\Delta\tilde{\delta}_1 = 0$

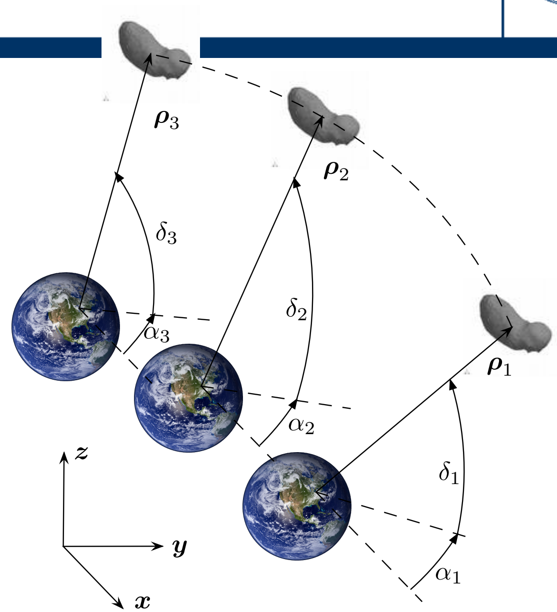
High-order iterative OD

- The high-order iterative method delivers

$$\begin{aligned}[\boldsymbol{r}_2] &= \boldsymbol{r}_2 + \mathcal{M}_{\boldsymbol{r}_2}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta}) \\ [\boldsymbol{v}_2] &= \boldsymbol{v}_2 + \mathcal{M}_{\boldsymbol{v}_2}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta})\end{aligned} \quad (4)$$

  where

  - $\boldsymbol{r}_2$ and $\boldsymbol{v}_2$ exactly satisfy the nominal observations
  - the polynomial part analytically maps the observations uncertainties in the phase space



- A DA-Kepler problem can be used to map the state to any arbitrary epoch t_4

$$\begin{aligned}[\boldsymbol{r}_4] &= \boldsymbol{r}_4 + \mathcal{M}_{\boldsymbol{r}_4}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta}) \\ [\boldsymbol{v}_4] &= \boldsymbol{v}_4 + \mathcal{M}_{\boldsymbol{v}_4}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta})\end{aligned} \quad (5)$$

Easily extended to n-body dynamics (Armellin et al 2010)

- This map can be used to analytically map the observation uncertainties in the sky at epoch t_4

$$\begin{aligned}[\alpha_4] &= \alpha_4 + \mathcal{M}_{\alpha_4}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta}) \\ [\delta_4] &= \delta_4 + \mathcal{M}_{\delta_4}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta})\end{aligned} \quad (6)$$

it tells where to look for successive observations

The Use of the high-order maps

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- The measurements performed by telescope (in general from any device) are affected by errors modeled by gaussian distributions
- Given any $\delta\boldsymbol{\alpha}$ and $\delta\boldsymbol{\delta}$, the Taylor polynomial maps:

$$[\alpha_4] = \alpha_4 + \mathcal{M}_{\alpha_4}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta})$$
$$[\delta_4] = \delta_4 + \mathcal{M}_{\delta_4}(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\delta})$$

  analytically describe how deviations from the nominal observation affect α_4, and δ_4
- For given values of measurements standard deviations σ, the Taylor maps can be used to
  - Perform fast Monte Carlo simulations based on the evaluation of the polynomial maps (running only one DA-based OD solver)
  - Analytically map the covariance matrix of the measurements in the phase space (using Park and Scheeres 2006)
  - Draw the propagated uncertainty sets by sampling each uncertain variable in -3σ, 0, 3σ

Test cases: simulated observations

- Simulated observations of two known asteroids to assess the method performances

Body	Epoch	a	e	i	Ω	ω	M
	JD	AU	–	deg	deg	deg	deg
J95E01K	2455200.5	2.2653859	0.7756466	8.85709	355.44679	296.83636	103.43357
K09S19T	2455200.5	2.3644574	0.5917237	6.68904	0.49354	336.40458	34.53061
Earth	2455200.5	0.9999880	0.0167168	8.854e-4	175.40648	287.61578	2.57607

- Observatory of discovery and *epoch* of discovery

Observatory	Asteroid	year	month	day	UT [hr]
Oizumi	J95E01K	1995	3	7	18.82256
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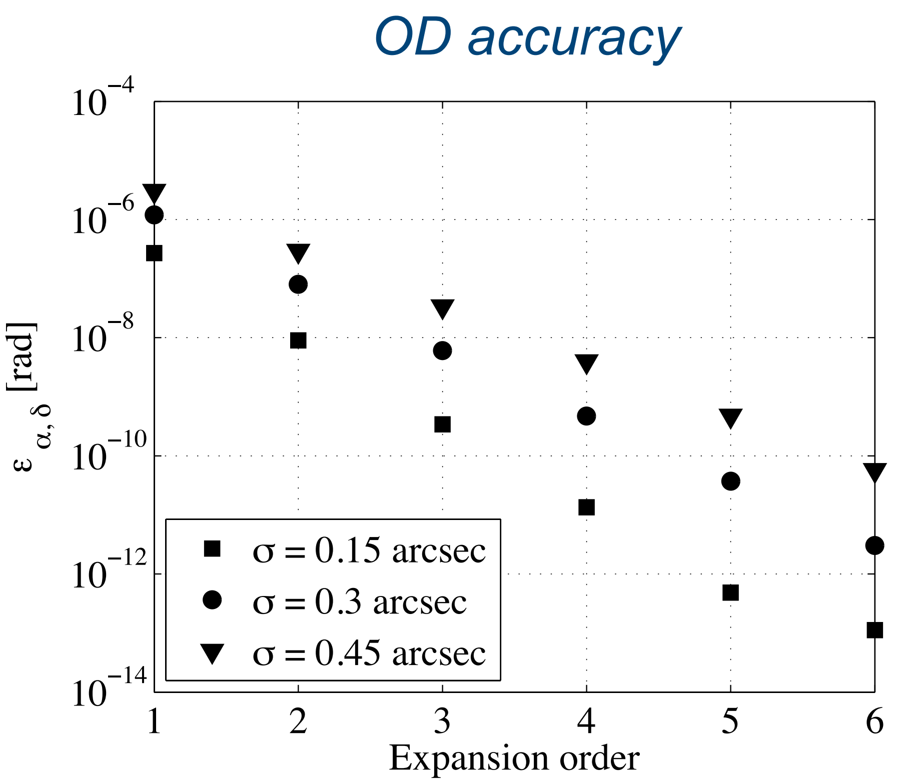
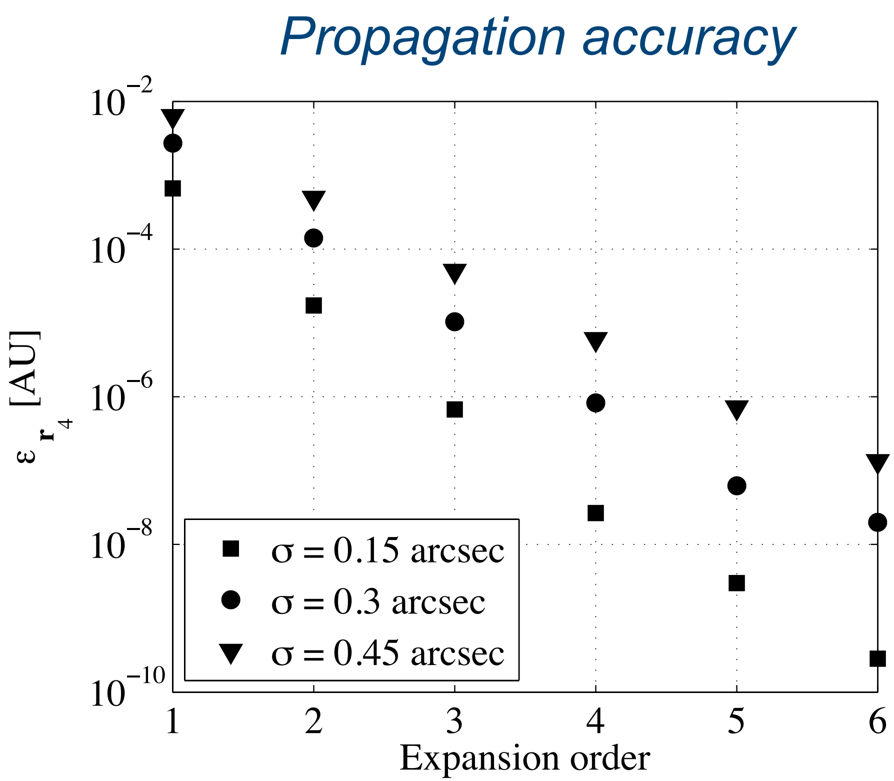
$\Rightarrow t_2$

- Observations separation:
  - $t_1 \longrightarrow$ one day $\longrightarrow t_2 \longrightarrow$ one day $\longrightarrow t_3$
  - $t_1 \longrightarrow$ two hours $\longrightarrow t_2 \longrightarrow$ one day $\longrightarrow t_3$
- Standard deviations $\sigma = 0.15$, 0.3 and 0.45 arcsec

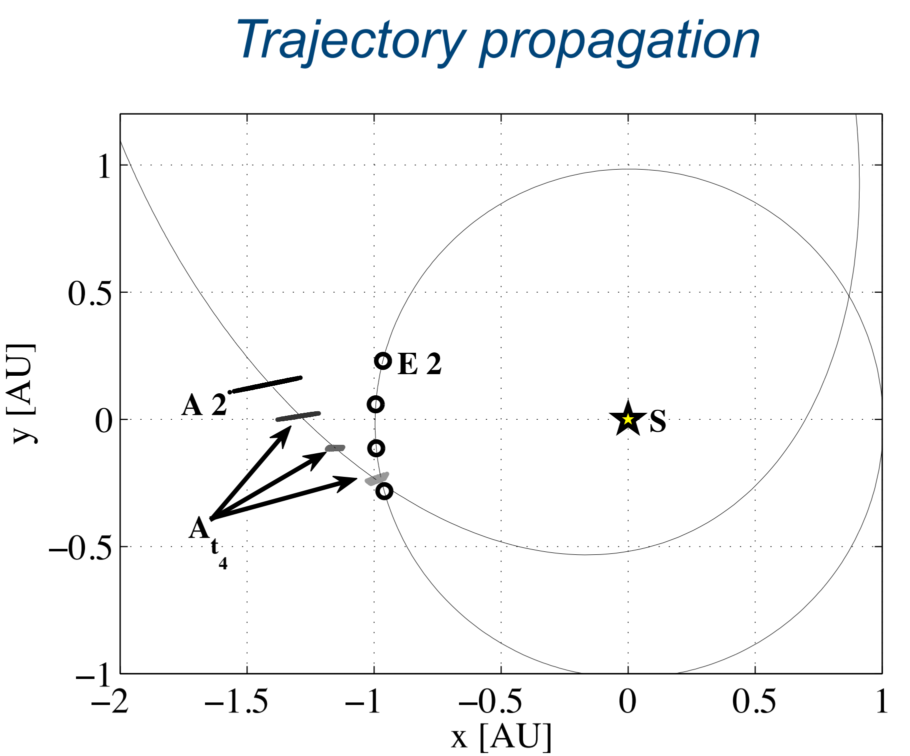
J95E01K: 1 day separated observations

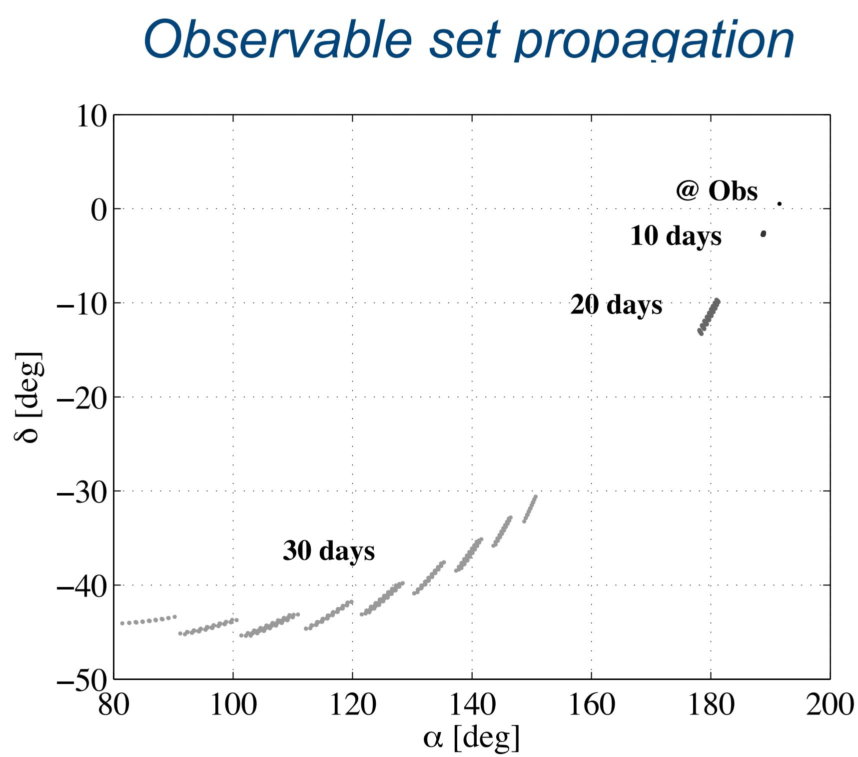




- Convergence in two iterations (vs. ~50 Gauss' iterations)
- Large observation uncertainties can be managed by high order computation
- Computational time for the solution, expansion, and evaluations of 1000 samples
  - 1.25 s for 5th order, 6.23 s for 6th order (Mac OS X, 2 GHz Intel Core Duo)

J95E01K: 1 day separated observations

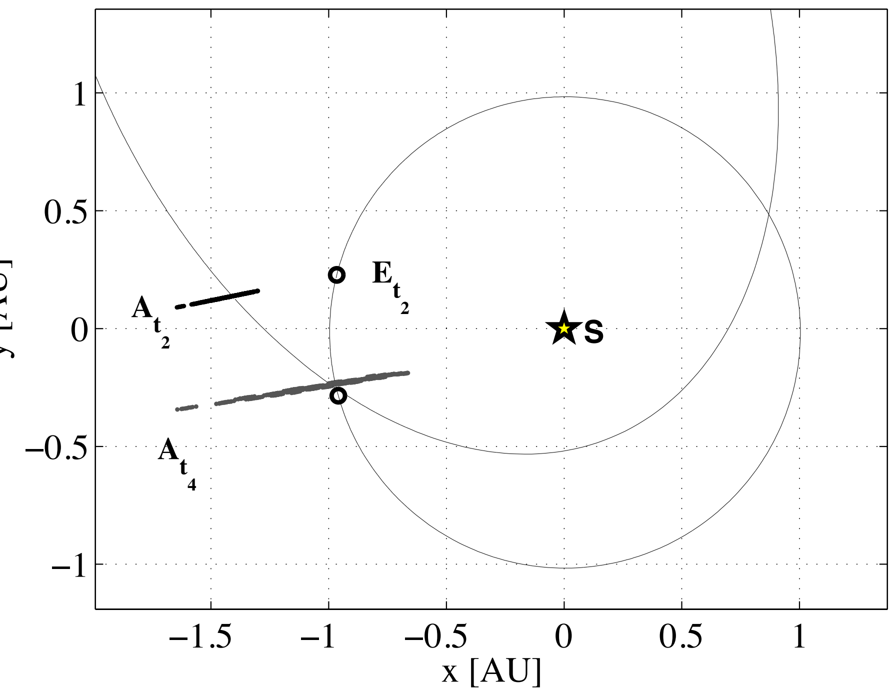




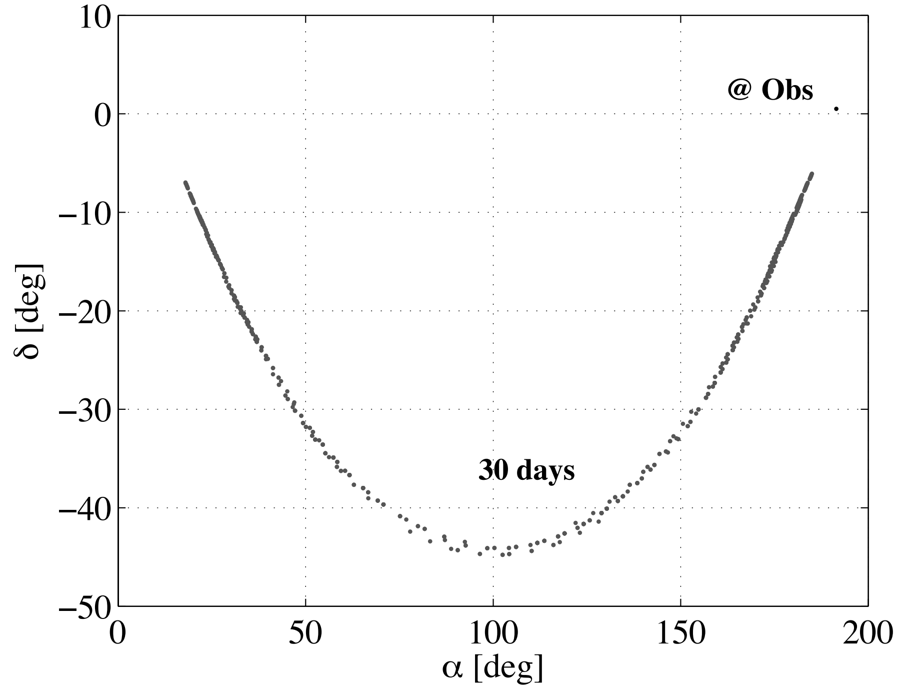
- Observation close to opposition with $\sigma = 0.3$ arcsec
- Set propagated up to 30 days
- Plot obtained by sampling each variable in -3σ, 0, 3σ

J95E01K: 2 hour - 1 day separation

Trajectory propagation

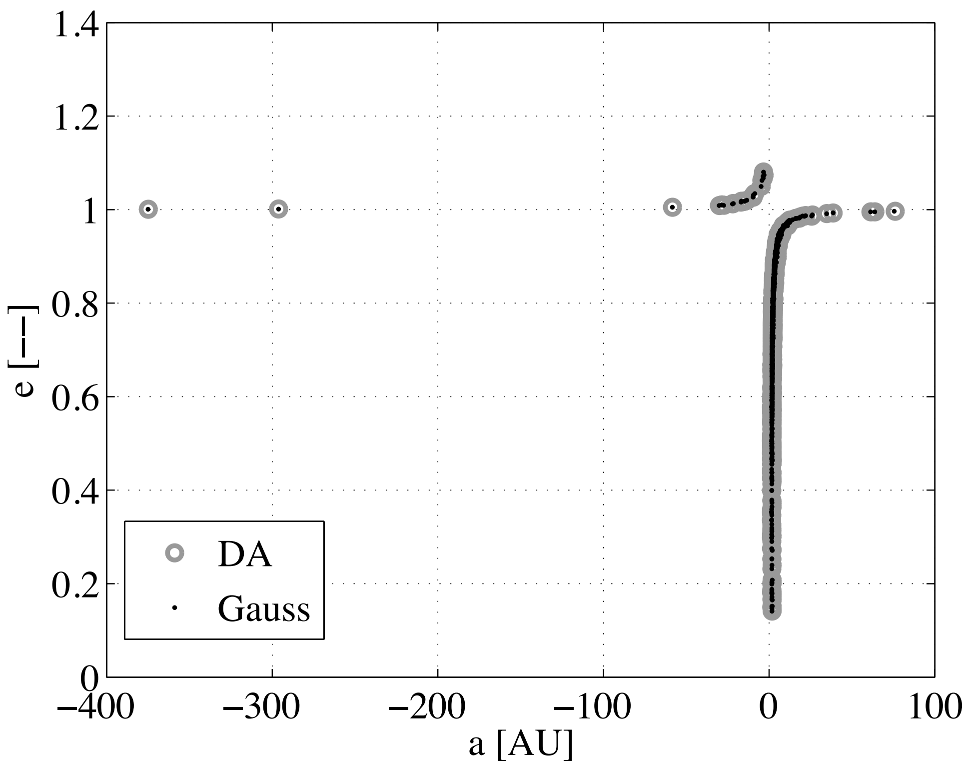


Observable set propagation



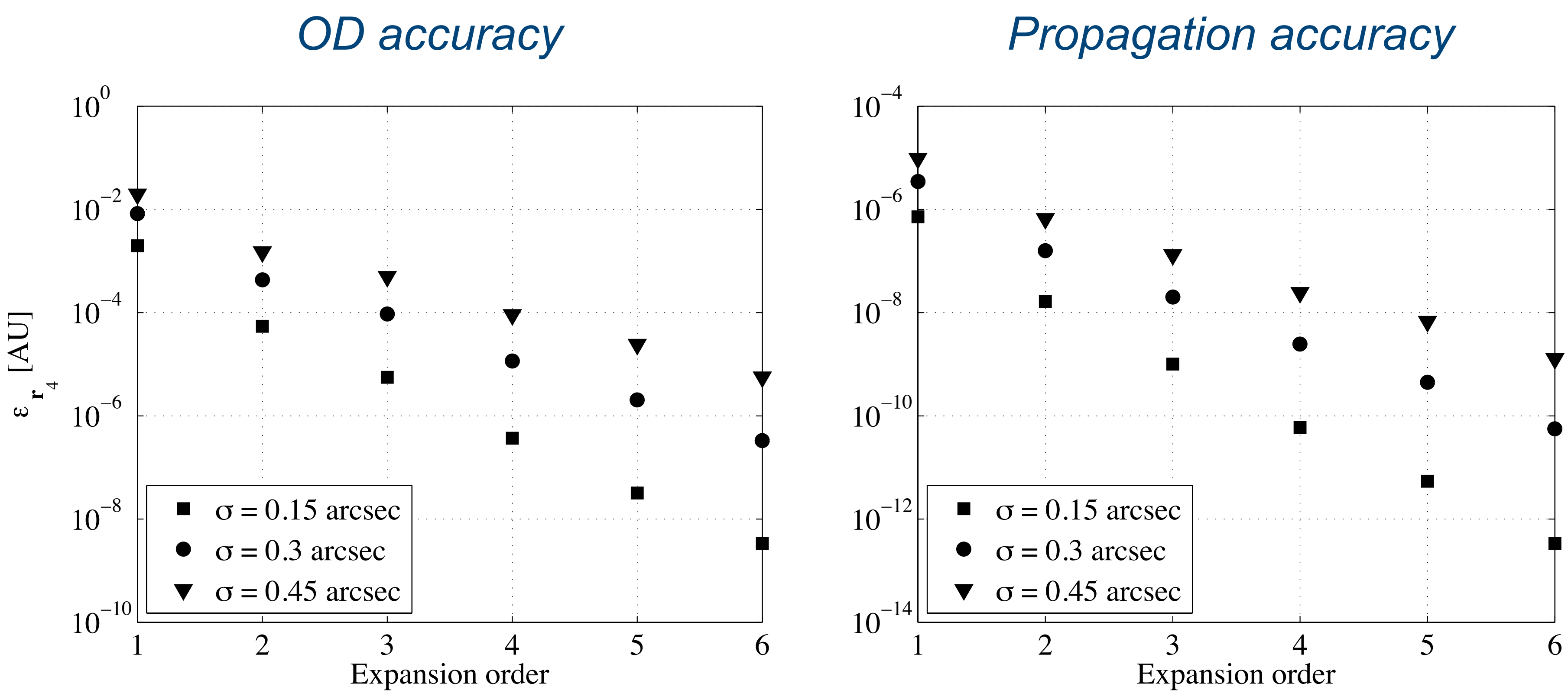
- Observation close to opposition with $\sigma = 0.15$ arcsec
- Closer observations yield larger uncertainty sets
- Plot obtained by sampling each variable in -3σ, 0, 3σ

J95E01K: 2 hour - 1 day separation



- Observation close to opposition with $\sigma = 0.3$ arcsec
- Within this uncertainty set both elliptic and hyperbolic solutions appear
- A single 7th order expansion capture both elliptic and hyperbolic solutions

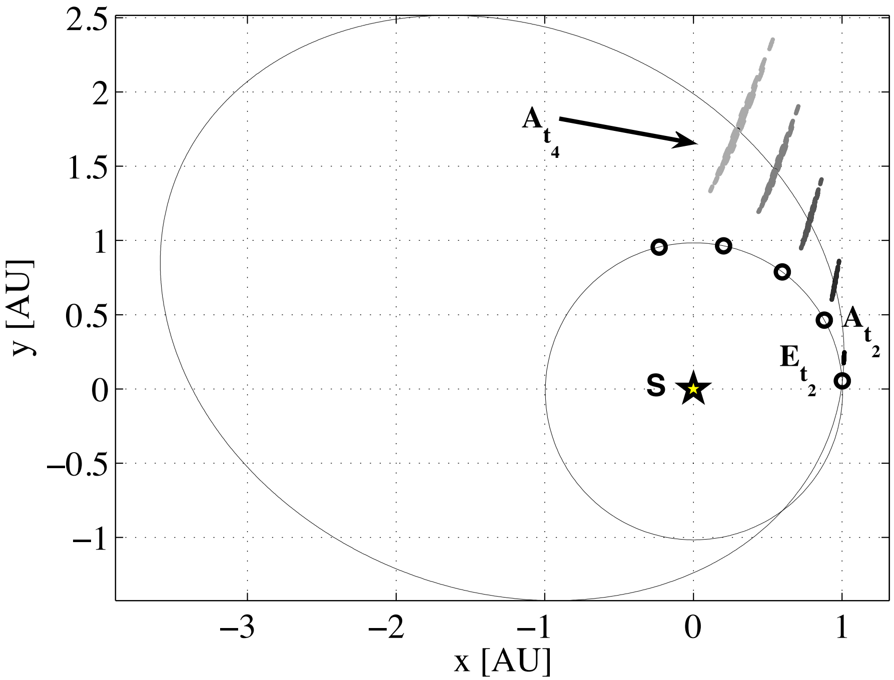
K095197: 1 day separated observations



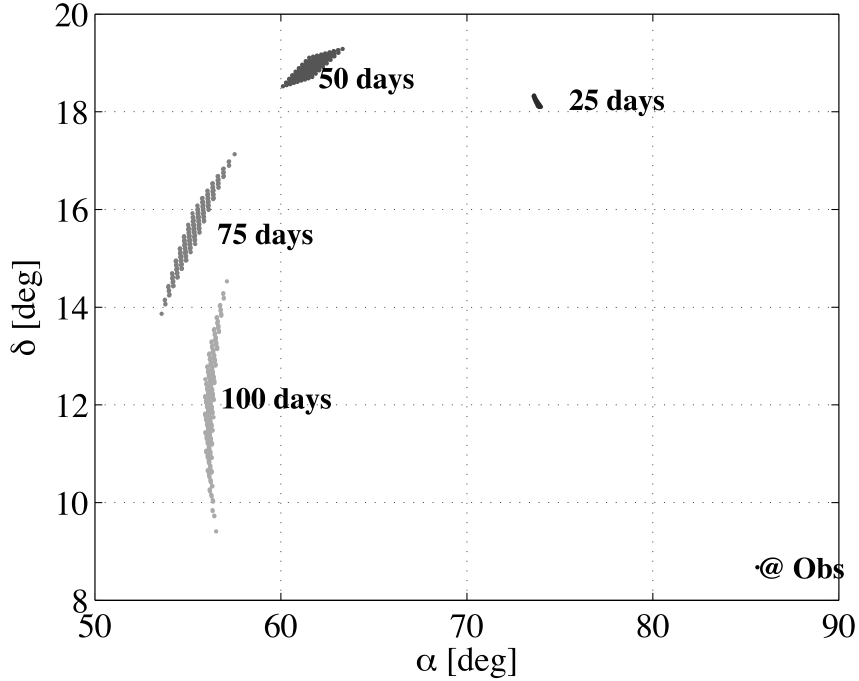
- Convergence in two iterations
- Gauss does not converge to the true solution

K095197: 1 day separated observations

Trajectory propagation



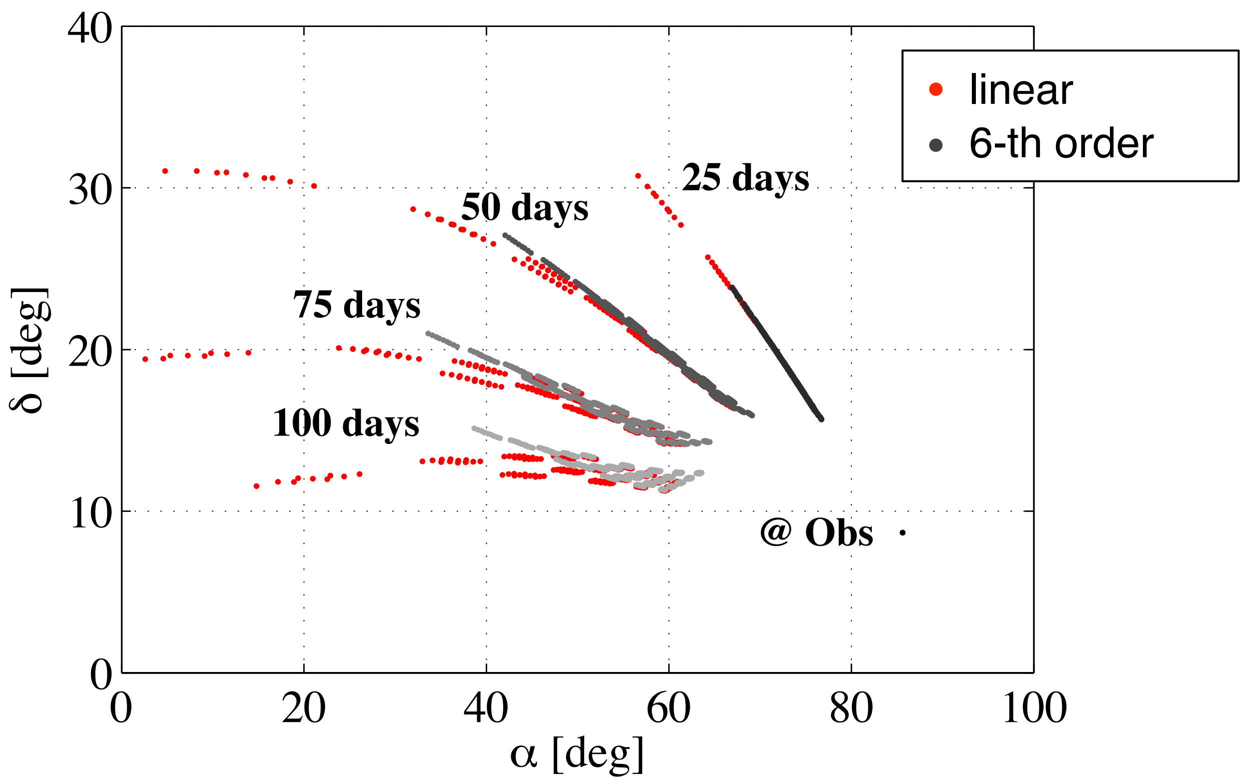
Observable set propagation



- Observation close to quadrature with $\sigma = 0.3$ arcsec
- Set propagated up to 100 days
- Plot obtained by sampling each variable in -3σ, 0, 3σ

K095197: 2 hour - 1 day separation

Linear vs. 6-th order



- Observation close to quadrature with $\sigma = 0.45$ arcsec
- A single 6th order expansion greatly **reduces** the region of the sky to be scanned for future observations

Conclusion and Future Work

Conclusions

- A new high-order iterative procedure for preliminary OD is presented
- The method is based on Taylor differential algebra implemented in COSY-Infinity
- The method converges with few iterations (maximum 3)
- The method can converge also when classical methods do not
- Observation uncertainties are analytically mapped in the phase space
- Regions of the sky to be scanned for successive observations are described by Taylor polynomial

Future Work

- A detailed convergence analysis is missing
- Application of the method from simulated observation to real observations

7th International Workshop on Taylor Models Methods

Key West - Florida

December 14-17, 2011

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High-order Uncertainty Management in Preliminary Orbit Determination

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