RMS-Parameters And Their Optimization

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ABSTRACT

This work is devoted to problems of optimization of RMS characteristics of beam of charged particles in accelerators. Optimization problems of minimization of the effective emittance growth are considered. Special minimax functionals are introduced. Linear and nonlinear beam dynamics models can be used. Numerical results on optimization problem of radial matching section in RFQ accelerator are presented.

INTRODUCTION

Problem of transverse motion parameters optimization could be formulated in various ways and those problems are usually complicated multi parameters problems. In case of linear equations of transverse motion it is suitable to consider beam envelops as beam characteristics. In nonlinear case RMS parameters of beam of trajectories of charged particles could be used for motion parameters estimation and for setting functionals of quality.

Problems of optimization of transverse motion one should consider step by step to achieve desired results. Among such various problems the main are the problems of beam focusing, minimization of effective emittance growth and also a problem of radial matching of beam parameters at RFQ accelerator structure input.

INTRODUCTION (2)

Suggested approach to mentioned problems based on solution of mathematical formal optimization problem is considered. Capabilities of the method are shown on example of solution of a particular problem – problem of definition of radial matching section parameters. The use of such section allows providing matching of time independent constant beam input with time dependent acceptance at the entrance of the RFQ accelerator which rotates with the frequency of the RF field. The successful resolution of the problem is important because the quality of radial matching directly connected with beam losses in the channel. Usually geometry of radial matching section was described by certain function of few variable parameters (for example, polynomial with uncertain coefficients).

INTRODUCTION (3)

It is shown that with use of chosen form of quality functional one can get variation of the functional in analytic representation. It gives us opportunity significantly speed up calculations and increase the number of controllable parameters. In given example controllable parameters were the values of aperture of the channel at the beginning and in some certain points of the cells in the radial matching section (in between the geometry is described by splines). In that case the class of possible solutions became wider.

PROBLEM STATEMENT

Let us consider charged particle beam dynamics in RFQ structure:

$$\frac{d^2 z}{d\tau^2} = 4\frac{eUT}{W_0 L}\cos(Kz)\cos(\theta\tau + \varphi_0) = F_z,$$

$$\frac{d^2 x}{d\tau^2} = \left(\frac{eU\kappa}{W_0 a^2} + 8\pi\frac{eUT}{W_0 L^2}\sin(Kz)\right)\cos(\theta\tau + \varphi_0)x$$

$$- \frac{eI}{2W_0\pi\varepsilon_0 v r_x r_y}\left(1 - \frac{r_x - r_y}{r_x + r_y}\right)x = Q_x(\tau, \varphi_0, r_x, r_y)x$$

$$\frac{d^2 y}{d\tau^2} = \left(-\frac{eU\kappa}{W_0 a^2} + 8\pi\frac{eUT}{W_0 L^2}\sin(Kz)\right)\cos(\theta\tau + \varphi_0)x$$

$$- \frac{eI}{2W_0\pi\varepsilon_0 v r_x r_y}\left(1 + \frac{r_x - r_y}{r_x + r_y}\right)y = Q_y(\tau, \varphi_0, r_x, r_y)y$$

Here $a = a(z)$ is a control function to be defined from an optimization problem.

The equations can be transformed to the linear system of ODEs:

$$\frac{d\tilde{x}}{d\tau} = A(\tau)\tilde{x},$$

$\tilde{x} \in R^2$; $\tilde{x}_1 = x$, $\tilde{x}_2 = dx/d\tau$ and $\tilde{x}_1 = y$, $\tilde{x}_2 = dy/d\tau$.

And matrix $A(\tau)$ has the following form:

$$A(\tau) = \begin{pmatrix} 0 & 1 \\ Q_{x,y} & 0 \end{pmatrix}.$$

Let the set of conditions for the linear system at initial τ fills the ellipses in the planes $\left(x, \frac{dx}{d\tau}\right)$ and $\left(y, \frac{dy}{d\tau}\right)$ correspondingly

$$\tilde{x}_0^{\,*} B_0^{\,x,y}\, \tilde{x}_0 \le 1.$$

Then to describe the system evolution the following matrix equation for beam envelopes can be written

$$\frac{dS}{d\tau} = A(\tau)S + SA^{*}(\tau),$$

with initial condition $$S(0) = B_0^{-1}.$$

Or one can rewrite the equation in the form

$$\frac{dS_{11}^{x,y}}{d\tau} = 2S_{12}^{x,y},$$

$$\frac{dS_{12}^{x,y}}{d\tau} = Q_{x,y} \cdot S_{11}^{x,y} + S_{22}^{x,y},$$

$$\frac{dS_{22}^{x,y}}{d\tau} = 2Q_{x,y} \cdot S_{12}^{x,y}.$$

Here $r_x = \sqrt{S_{11}{}^{x}}$, $r_y = \sqrt{S_{11}{}^{y}}$.

For optimization problem it is often suitable to consider longitudinal coordinate z as independent variable. In this case equations of longitudinal motion can be transformed to

$$\frac{d\varphi}{dz} = \frac{\tilde{\omega}}{\beta}, \qquad \frac{d\beta}{dz} = \frac{F_z}{\beta},$$

where $\varphi = \tilde{\omega}\tau + \varphi_0$ - phase of particle, $\beta = \frac{\dot{z}}{c}$.

And then transverse motion equations have the following form

$$\frac{d^2x}{dz^2} = \frac{1}{\beta^2}\left(Q_x x - \frac{dx}{dz}F_z\right),$$

$$\frac{d^2y}{dz^2} = \frac{1}{\beta^2}\left(Q_y y - \frac{dy}{dz}F_z\right).$$

With use of new shape of charged particle beam dynamics one can get the following equations for beam envelopes:

$$\frac{dS_{11}^{x,y}}{dz} = 2S_{12}^{x,y},$$

$$\frac{dS_{12}^{x,y}}{dz} = \frac{Q_{x,y}}{\beta} \cdot S_{11}^{x,y} - \frac{F_z}{\beta^2} S_{12}^{x,y} + S_{22}^{x,y},$$

$$\frac{dS_{22}^{x,y}}{dz} = 2\frac{Q_{x,y}}{\beta} \cdot S_{12}^{x,y} - 2\frac{F_z}{\beta^2} S_{22}^{x,y}.$$

PROBLEM STATEMENT (2)

Let the system describes transverse motion can be written as linear differential equation systems

$$\frac{d\xi}{dz} = A_x\xi \ , \qquad \frac{d\eta}{dz} = A_y\eta \ ,$$

where $\xi = (\xi_1, \xi_2)$, $\xi_1 = x$, $\xi_2 = \dfrac{dx}{dz}$, $\eta = (\eta_1, \eta_2)$, $\eta_1 = y$, $\eta_2 = \dfrac{dy}{dz}$, a and matrices A_x and A_y have

the form

$$A_x = \frac{1}{\beta^2}\begin{pmatrix} 0 & \beta^2 \\ Q_x & -F_z \end{pmatrix}; \qquad A_y = \frac{1}{\beta^2}\begin{pmatrix} 0 & \beta^2 \\ Q_y & -F_z \end{pmatrix}.$$

Let the set of conditions for the system at some instant τ fill the ellipses

$$\xi^* G_x \xi \le 1, \quad \eta^* G_y \eta \le 1,$$

in the planes $\left(x, \frac{dx}{dz}\right)$ and $\left(y, \frac{dy}{dz}\right)$ correspondingly. Then, the matrices G_x and G_y satisfy the following system of matrix equations

$$\frac{d}{d\tau} G_x = -A_x^* G_x - G_x A_x, \qquad \frac{d}{d\tau} G_y = -A_y^* G_y - G_y A_y.$$

The system of equations should be solved on the interval from the entrance into the regular part of the accelerator to the end of structure, i.e. from $z = 0$ to $z = Z$. Initial conditions for the system are the matrices of ellipses defining acceptances of the regular part of the accelerator, depending on an initial phase φ_0 *($0<\varphi_0<2\pi$)* :

$$G_x(0, \varphi_0) = G_{x,0}(\varphi_0), \quad G_y(0, \varphi_0) = G_{y,0}(\varphi_0).$$

The optimization problem for the effective emittence growth minimization problem is to find a function $a(z)$*, i.e. law of the radius change along the accelerator structure, providing under the initial conditions the maximum possible overlapping of families of ellipses at the end of the accelerating structure.*

METHOD OF SOLUTION

Let's represent elements of the matrices $G_x(z,\varphi_0)$ and $G_y(z,\varphi_0)$ as follows

$$G_x = \begin{pmatrix} \dfrac{\Delta_x + \alpha_x^2}{\nu_x} & \alpha_x \\ \alpha_x & \nu_x \end{pmatrix}, \qquad G_y = \begin{pmatrix} \dfrac{\Delta_y + \alpha_y^2}{\nu_y} & \alpha_y \\ \alpha_y & \nu_y \end{pmatrix},$$

where $\nu_x/\Delta_x = r_x^2, \nu_y/\Delta_y = r_y^2,\ \Delta_x = \det G_x,\ \Delta_y = \det G_y.$

Notice that values $\dfrac{\alpha_x}{\sqrt{\Delta_x}}, \dfrac{\nu_x}{\sqrt{\Delta_x}}$ and $\dfrac{\alpha_y}{\sqrt{\Delta_y}}, \dfrac{\nu_y}{\sqrt{\Delta_y}}$ are the so-called Courant-Snyder or Twiss parameters.

Now the equations can be written in the following form

$$\frac{d\nu_x}{d\tau} = -2\alpha_x, \quad \frac{d\alpha_x}{d\tau} = -Q_x\nu_x - \frac{\Delta_x + \alpha_x^2}{\nu_x}$$

$$\frac{d\nu_y}{d\tau} = -2\alpha_y, \quad \frac{d\alpha_y}{d\tau} = -Q_y\nu_y - \frac{\Delta_y + \alpha_y^2}{\nu_y}$$

Let's also consider the following functional which characterizes the quality of transverse motion by mismatch of ellipses $G_x(Z,\varphi_0)$ and $G_y(Z,\varphi_0)$ with given ellipses B_x and B_y:

$$J(a) = \max_{\varphi_0} \lambda_x^{-1}(\varphi_0) + \max_{\varphi_0} \lambda_y^{-1}(\varphi_0),$$

Where

$$\lambda_x^{-1}(\varphi_0) = \lambda^{-1}(G_x(Z,\varphi_0), B_x),$$

$$\lambda_y^{-1}(\varphi_0) = \lambda^{-1}(G_y(Z,\varphi_0), B_y).$$

Here $\lambda = \min(\lambda_1, \lambda_2)$ is a minimum eigenvalue of a cluster of quadratic forms generated by a pair of ellipses with the matrices G and B: $\chi(\lambda) = \det(G - \lambda B) = 0$, $\chi(\lambda_1) = \chi(\lambda_2) = 0$.

The value of the inverse minimum eigenvalue characterizes the degree of mismatch pairs of ellipses. In the case of fully identical ellipses, this value is equal to unity. So always $\lambda^{-1} \geq 1$.

Matrices B_x and B_y describing the desired phase portrait of the beam at the end of the accelerator structure.

The problem of minimizing this functional is the minimax optimization problem and can be decided by directed methods of optimization.

Rewrite the last system as a system of the equations of the form

$$\frac{d\sigma}{dz} = f(z, \sigma, \varphi, u),$$

with the corresponding initial conditions

$$\sigma(0) = \sigma(0, \varphi), \quad \varphi \in [\varphi_1, \varphi_2].$$

Here $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, $\sigma_1 = \alpha_x$, $\sigma_2 = v_x$, $\sigma_3 = \alpha_y$, $\sigma_4 = v_y$; $u = a(z)$ is a control function; initial phase value $\varphi = \varphi_0$ belongs to some interval.

The functional can be written in the form

$$J(u) = \max_{\varphi_0} g_1(\sigma(Z, \varphi_0), \varphi_0) + \max_{\varphi_0} g_2(\sigma(Z, \varphi_0), \varphi_0).$$

It could be shown that in this case one have the following representation of variation of the functional:

$$\delta J = -\int_0^Z p^*(z)\Delta u(z)dz,$$

Where vector-function $p(\tau)$ should be defined by formula

$$p(z) = \frac{\partial f(z,\sigma(z,\varphi_x),\varphi_x,u(z))^*}{\partial u}\psi_1(z,\varphi_x) + \frac{\partial f(z,\sigma(z,\varphi_y),\varphi_y,u(z))^*}{\partial u}\psi_2(z,\varphi_y).$$

Here φ_x and φ_y correspond to the "worst" phases in corresponding planes, i.e. $\max\limits_{\varphi} g_1(\sigma(Z,\varphi),\varphi) = g_1(\sigma(Z,\varphi_x),\varphi_x)$ and $\max\limits_{\varphi} g_2(\sigma(Z,\varphi),\varphi) = g_2(\sigma(Z,\varphi_y),\varphi_y)$; vector-functions ψ_1 and ψ_2 are solutions of the following systems with special initial conditions:

$$\frac{d\psi_i}{dz} = -\left(\frac{\partial f(z,\sigma(z,\varphi),\varphi,u(z))}{\partial \sigma}\right)^*\psi_i, \quad \psi_i(Z,\varphi) = \frac{\partial g_i(\sigma(Z,\varphi),\varphi)}{\partial \sigma}, \quad i = 1,2.$$

The vector-function $p(z)$ defines the direction of minimization:

$$u_\varepsilon(\tau) = u(\tau) + p(\tau)\varepsilon, \qquad \text{where } \varepsilon > 0, \text{ and } u_\varepsilon(\tau) \text{ - possible new control}.$$

Radial matching section optimization

For the radial matching section of the accelerator, charged particle dynamics in the (x, y) plane which is perpendicular to the longitudinal axis in the case of micro canonical charge distribution can be described by the following system of equations:

$$\frac{d\xi}{d\tau} = A_x \xi \,, \qquad \frac{d\eta}{d\tau} = A_y \eta \,, \tag{1}$$

where $\xi = (\xi_1, \xi_2)$, $\xi_1 = x$, $\xi_2 = \frac{dx}{d\tau}$, $\eta = (\eta_1, \eta_2)$, $\eta_1 = y$, $\eta_2 = \frac{dy}{d\tau}$, and the matrices A_x and A_y have the form

$$A_x = \begin{pmatrix} 0 & 1 \\ Q_x & 0 \end{pmatrix}; \qquad A_y = \begin{pmatrix} 0 & 1 \\ Q_y & 0 \end{pmatrix}. \tag{2}$$

Here

$$Q_x(\tau, z, \varphi_0, r_x, r_y) = \frac{eU}{W_0 a^2}\cos(\theta\tau + \varphi_0) - \frac{4I/I_0}{\beta_z r_x (r_x + r_y)}, \tag{1}$$

$$Q_y(\tau, z, \varphi_0, r_x, r_y) = -\frac{eU}{W_0 a^2}\cos(\theta\tau + \varphi_0) - \frac{4I/I_0}{\beta_z r_y (r_x + r_y)}, \tag{2}$$

where $\tau = ct$, $\theta = 2\pi\omega/c$, U is the intervane voltage, W_0 is the charged particle rest energy, ω is the accelerating field frequency, φ_0 is the initial phase, c is the velocity of light, a is the radius of the channel, $v = \dot{z}$ is the longitudinal velocity of a particle which is constant along the matching section, r_x and r_y are the beam envelopes, I is the beam current.

Let the set of conditions for system (1) at some instant τ fill the ellipses

$$\xi^* G_x \xi \leq 1, \quad \eta^* G_y \eta \leq 1, \tag{5}$$

in the planes $\left(x, \frac{dx}{d\tau}\right)$ and $\left(y, \frac{dy}{d\tau}\right)$ correspondingly.

Then, the matrices G_x and G_y satisfy the following system of matrix equations

$$\frac{d}{d\tau} G_x = -A_x^* G_x - G_x A_x, \quad \frac{d}{d\tau} G_y = -A_y^* G_y - G_y A_y. \tag{6}$$

The system of equations (6) should be solved on the interval from the entrance into the regular part of the accelerator to the entrance into the radial matching section, i.e. from $\tau = \mathrm{T}$ to $\tau = 0$. Initial conditions for the system (6) are the matrices of ellipses defining acceptances of the regular part of the accelerator, depending on an initial phase φ_0:

$$G_x(\mathrm{T}, \varphi_0) = G_{x,\mathrm{T}}(\varphi_0), \quad G_y(\mathrm{T}, \varphi_0) = G_{y,\mathrm{T}}(\varphi_0). \tag{7}$$

The optimization problem for the radial matching section is to find a function $a(\tau)$, i.e. law of the radius change along the matching sections, providing under the conditions (7) the maximum possible overlapping of families of ellipses at the entrance of the radial matching section.

Let's consider the functions

$$\Phi_x(\varphi_0) = \mathrm{Sp}(G_x(0,\varphi_0) - B_x)^2, \tag{8}$$

$$\Phi_y(\varphi_0) = \mathrm{Sp}(G_y(0,\varphi_0) - B_y)^2, \tag{9}$$

where B_x, B_y are given matrices, Sp is the trace of the corresponding matrix. Functions $\Phi_x(\varphi_0)$ and $\Phi_y(\varphi_0)$ characterize deviations of ellipses G_x and G_y at $\tau = 0$ from the given ellipses B_x and B_y, accordingly.

Here two approaches based on introduction of two different functionals are considered.

Introduce the functional

$$J(a) = c_1 \int_{\varphi_1}^{\varphi_2} \Phi_x(\varphi_0)\, d\varphi_0 + c_2 \int_{\varphi_1}^{\varphi_2} \Phi_y(\varphi_0)\, d\varphi_0, \qquad (10)$$

estimating the degree of mutual overlapping of ellipses corresponding to various initial phases at the entrance of the matching section. Here φ_1 and φ_2 are limits of variation of initial phase φ_0; c_1, c_2 are some positive constants.

Note that the optimization of the functional (10) over the control function $a(\tau)$ can be viewed as a nonstandard problem of the optimal control theory.

Let's also consider the following functional which characterizes the quality of the matching section by mismatch of ellipses $G_x(0,\varphi_0)$ and $G_y(0,\varphi_0)$ with given ellipses B_x and B_y:

$$J(a) = \max_{\varphi_0} \lambda_x^{-1}(\varphi_0) + \max_{\varphi_0} \lambda_y^{-1}(\varphi_0), \qquad (11)$$

where

$$\lambda_x^{-1}(\varphi_0) = \lambda^{-1}\left(G_x(0,\varphi_0), B_x\right), \qquad (12)$$

$$\lambda_y^{-1}(\varphi_0) = \lambda^{-1}\left(G_y(0,\varphi_0), B_y\right). \qquad (13)$$

Here $\lambda = \min(\lambda_1, \lambda_2)$ is a minimum eigenvalue of a cluster of quadratic forms generated by a pair of ellipses with the matrices G and B:

$$\chi(\lambda) = \det(G - \lambda B) = 0, \ \chi(\lambda_1) = \chi(\lambda_2) = 0. \qquad (14)$$

The value of the inverse minimum eigenvalue characterizes the degree of mismatch pairs of ellipses.

In the case of fully identical ellipses, this value is equal to unity. So always $\lambda^{-1} \geq 1$.

Matrices B_x and B_y describing the desired phase portrait of the beam at the beginning of the matching section.

The problem of minimizing the functional (11) is the minimax optimization problem.

RESULTS OF NUMERICAL OPTIMIZATION

The analytic representations of the variations of the functionals (10),(11) were used to find geometric parameters of radial matching section of the RFQ accelerator of protons (initial energy 45keV, intervane voltage 100kV, RF field frequency 352 MHz, current 75mA, emittance $4 \cdot 10^{-7}$ m·rad). One of the possible choices of the law of variation of the channel radius along the radial matching section is presented in Figure 3. In Figure 1 the RFQ acceptances without radial matching section are shown. The illustration of the radial matching section effect is shown in Figure 2.

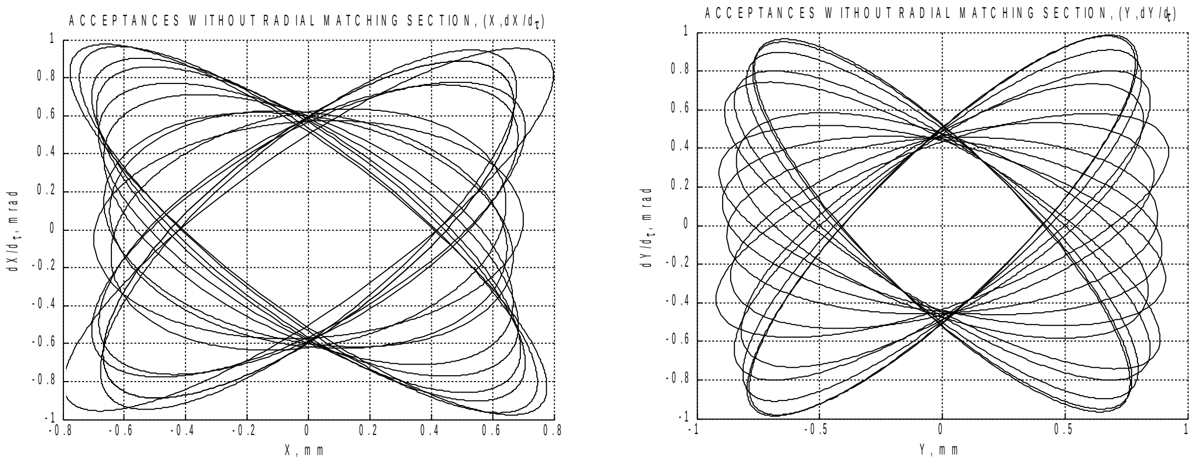


Figure 1.

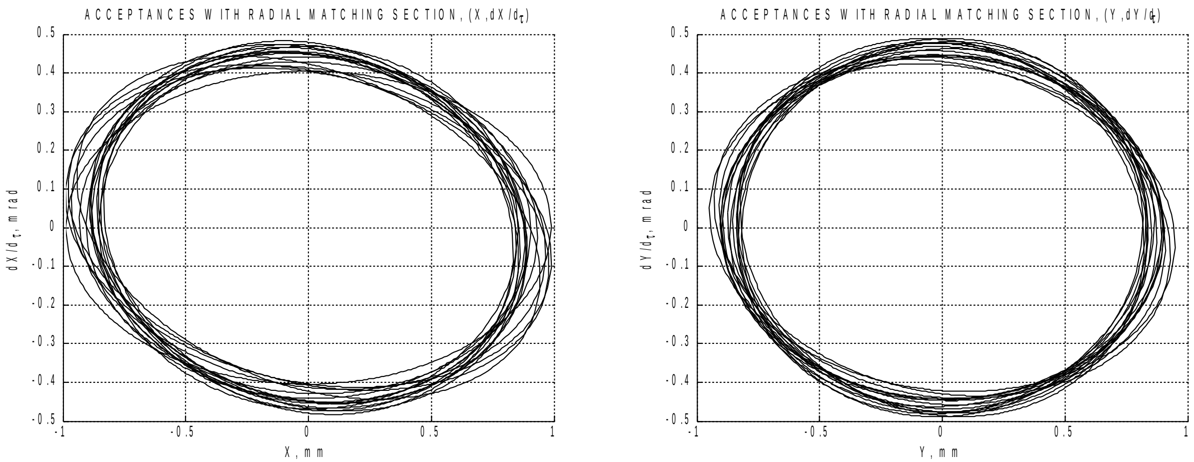


Figure 2.

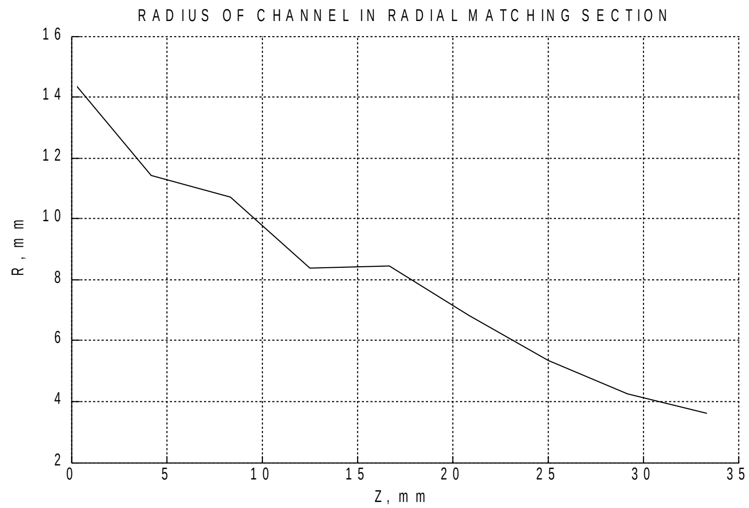
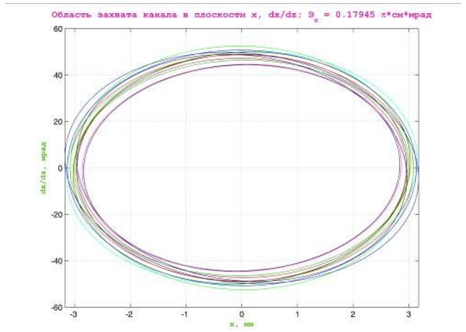
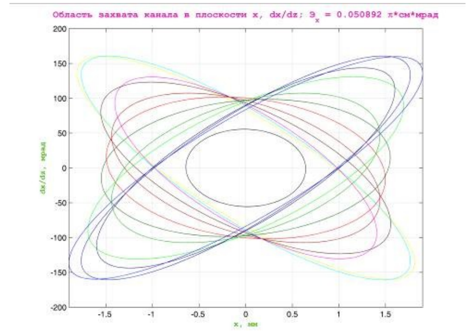


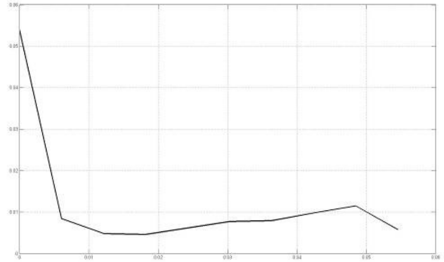
Figure 3.

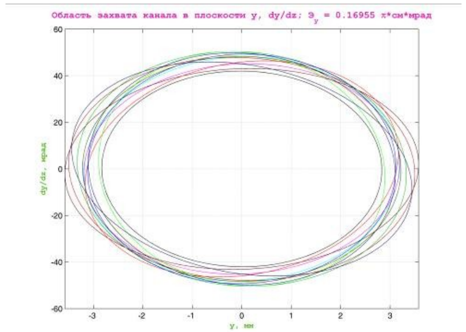
Also computations were used to find geometric parameters of radial matching section of the RFQ accelerator of protons (initial energy 95keV, output energy 5 MeV, intervane voltage 100kV, RF field frequency 352 MHz, initial cell length 6.06 mm).

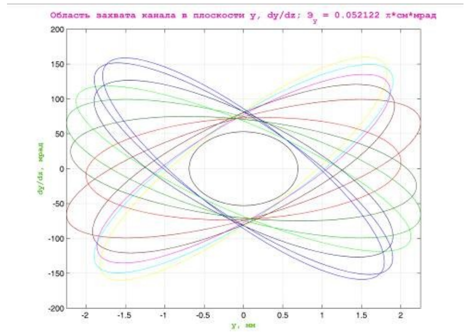
OPTIMIZATION RESULTS



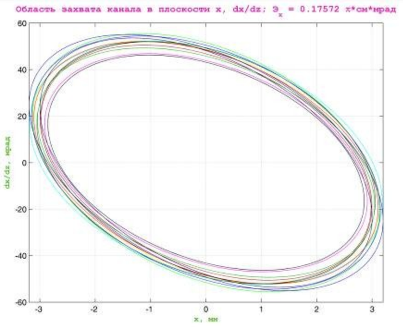


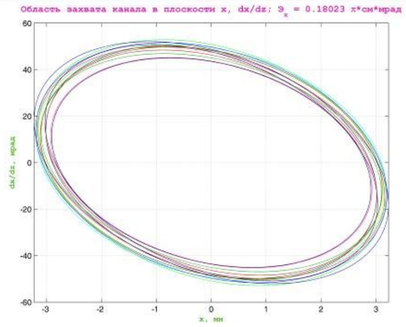


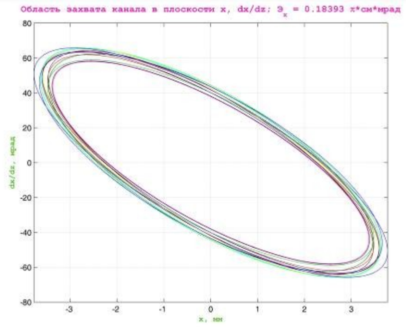


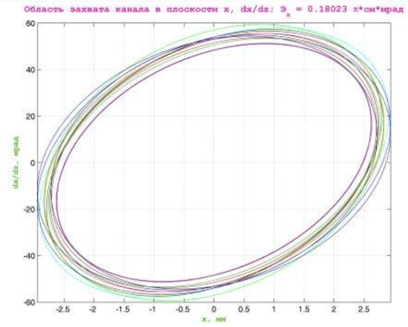


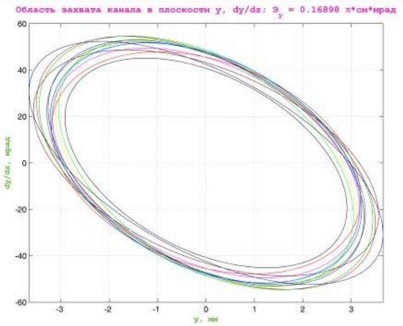
OPTIMIZATION RESULTS

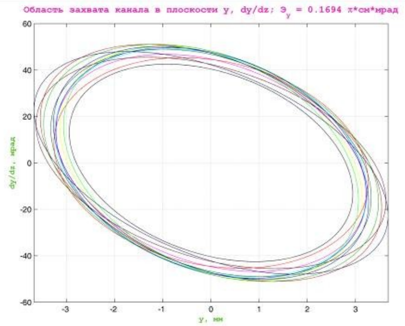


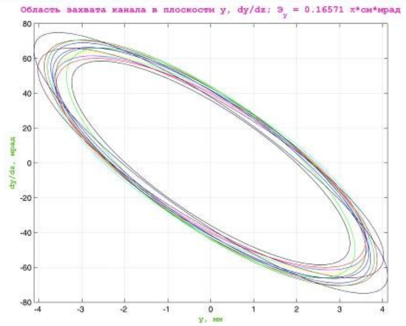


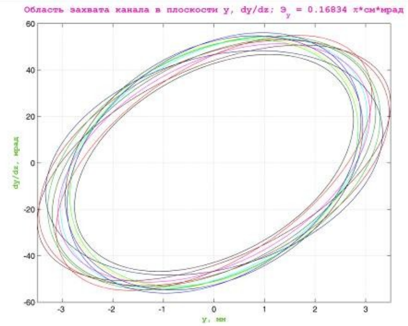


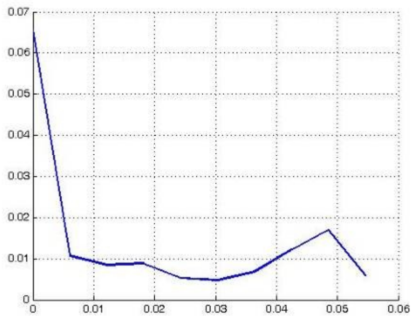


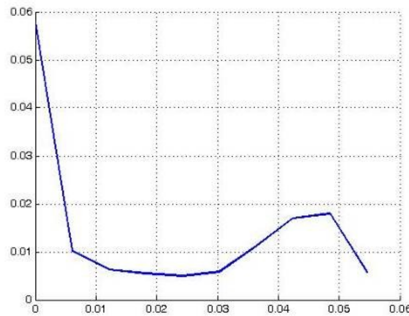


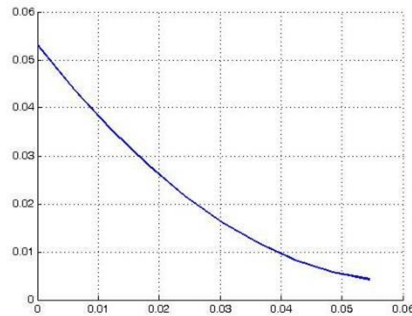












CONCLUSION

New mathematical models and methods of the RFQ structure optimization were suggested. In this work the optimization approach with use of two different functionals to find geometric parameters of radial matching section is considered. It should be noted, that the proposed approach can be utilized to optimize the transverse dynamics in accelerators if the dynamics is adequately described by linear equations or with use RMS parameters. In particular, this method can be used to minimize the growth of the effective emittance in the RFQ channel.

THE END