Computer Assisted Proof of the Existence of High Period Fixed Points in Hénon Maps

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Outline

The fixed point problem

- We numerically want to find $x \in \mathbb{R}^n$ so that for $f : \mathbb{R}^n \to \mathbb{R}^n$ we have
$$f(x) = x$$
- Equivalently we can find the zeros of
$$f(x) - x = 0$$
- Those points can tell us a lot about the dynamics of the system

Numerical limitations

Mathematically it is obvious how to check if a given point x is a fixed point by just checking if $f(x) = x$. However numerically there are round off errors that make this check invalid.

Example: Fake fixed point

$$f : x \mapsto x + 10^{-16}$$

In double precision floating point arithmetic roughly all points $|x| > 1$ will look like "fixed points".

Another Example

A more realistic example for numerical problems is the determination of the order of a fixed point.

Example: Order of a fixed point

f : Hénon map with $A = 1.422$ and $B = 0.3$

$$
\begin{array}{lll}
f^{(00)}: & x = -0.0869282203452\underline{939} & y = 0.23915367507167\underline{47} \\
f^{(15)}: & x = -0.0869282203454\underline{442} & y = 0.23915367507169\underline{64} \\
f^{(30)}: & x = -0.0869282203452\underline{939} & y = 0.23915367507167\underline{47} \\
f^{(45)}: & x = -0.0869282203454\underline{442} & y = 0.23915367507169\underline{64} \\
f^{(60)}: & x = -0.0869282203452\underline{939} & y = 0.23915367507167\underline{47} \\
\vdots & \vdots & \vdots
\end{array}
$$

How to prove the existence of fixed points

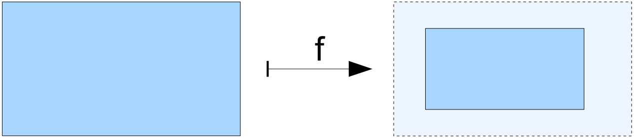
- We need to use rigorous numerical methods for our calculations to contain round off errors (Intervals, Taylor Models, . . .)
- We have to use certain purely mathematical results that imply the existence of fixed points.
- Those mathematical theorems have to be suited to be checked by numerical methods

Schauder fixed point theorem

Theorem

Let K be a non-empty, compact, and convex set. Then any continuous map $f : K \to K$ has a fixed point in K.

Note that the requirements on f are very weak.

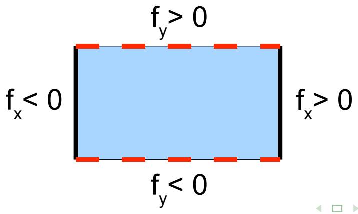


Miranda theorem

Theorem

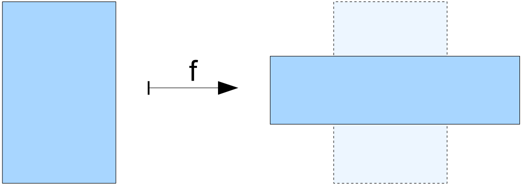
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, and $K = (x_1, x_2, \ldots, x_n)$ with $x_i = [x_i^-, x_i^+]$ be a box in $\mathbb{R}^n$.
If $f_i < 0$ on $K_i^- = \{x \in K | x_i = x_i^-\}$ and $f_i > 0$ on $K_i^+ = \{x \in K | x_i = x_i^+\}$ for all i then f has a zero in K.

This is essentially the intermediate value theorem for higher dimensions.



Corollary to the Miranda theorem

Applying this theorem to the function $g(x) = f(x) - x$ we find that $f(x)$ has a fixed point in K if $f_i(x) < x_i^-$ on K_i^- and $f_i(x) > x_i^+$ on K_i^-.



The Hénon map

We use the Hénon map defined as

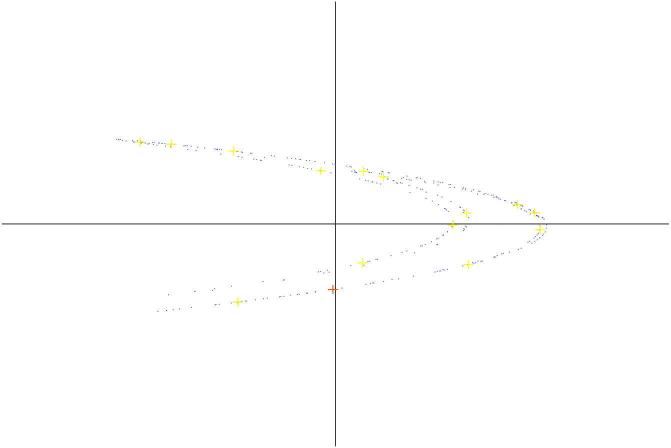
$$\mathcal{H} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 + y - Ax^2 \\ Bx \end{pmatrix}$$

Where A and B are parameters. In the standard Hénon map $A = 1.4$ and $B = 0.3$.

A Hénon map near standard parameters

- For our first example we use a map very close to the standard parameters with $A = 1.422$ and $B = 0.3$.
- While close to the standard map this map seems to have an attractive period 15 fixed point near $x = -0.01465994336066556$ and $y = -0.2948278571848495$.
- That point has a large basin of attraction, lots of starting conditions eventually seem to fall into that point.

Attractive period fifteen fixed point



Verification process

- Since the point is attractive we simply start somewhere nearby and iterate until we have an approximation up to floating point accuracy.
- We want to use the Schauder fixed point theorem so we are looking for a box around the alleged fixed point that after 15 iterations is mapped back into itself.
- To achieve this we have to chose our box along the eigen-vectors of the linearisation by switching into eigen-coordinates. So we consider the topologically conjugated map $M^{-1} \circ \mathcal{H}^{(15)} \circ M$ where M is the eigen-vector basis.

Transformation to eigen-coordinates

- We can use a DA expansion of $\mathcal{H}^{(15)}$ to find an approximate linearisation M. Since we are free to chose any coordinate system we do not need to be rigorous in this step.
- What has to be done rigorously is the inversion of M. We use simple interval arithmetic to achieve that.
- We observe that the eigenvalues of the linearisation at the fixed point are -0.944 and $0.152 \cdot 10^{-7}$, so there is one direction with very little contraction making the proof harder.
- We can chose whichever of the 15 fixed points we want, while the eigenvalues are the same for each we chose one that has near orthogonal eigen-vectors.

Verification with interval methods

- Sending an interval through the map shows a blowup of about a factor 3000 in each coordinate. This illustrated how big the overestimation is due to the dependency problem.
- Based on that we can already estimate the number of interval boxes needed to be at least around 10 Million boxes.
- For the actual verification we use a simple stack-based approach:
  1. Take a box of the stack and map it
  2. If the box is mapped into the starting box we discard it; otherwise split the box into four and push them on the stack

Disadvantages of interval methods

This method works and eventually verifies that the original box is mapped into itself. However it suffers from several problems:

- It turns out that we need about 300 million interval boxes to complete the verification.
- The initial box had to be split about 12-15 times to actually map into the original box.
- The whole computation has to be done in high precision because the initial box already has to be very close to the fixed point (we used a width of 10^{-20} with 45 digits precision)
- The whole procedure takes anywhere up to two hours.

Verification with Taylor Models

Doing the same calculation with Taylor Models yields a much faster result.

- We use one single Taylor Model of width 10^{-5} in x and y.
- That single Taylor Model already maps into itself, the calculation taking less than a second.
- This is due to the significant reduction of the dependency problem when using Taylor Models.

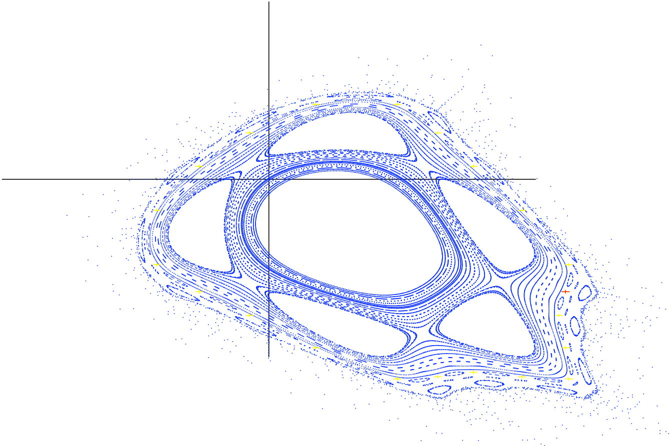
An Area preserving Hénon map

Another interesting Hénon map:

- $A = -0.4375$, $B = -1$
- area preserving, i.e. $\det D\mathcal{H} = 1$
- has several high period hyperbolic fixed points

A period 21 fixed point

For further studies we pick a fixed point of period 21:

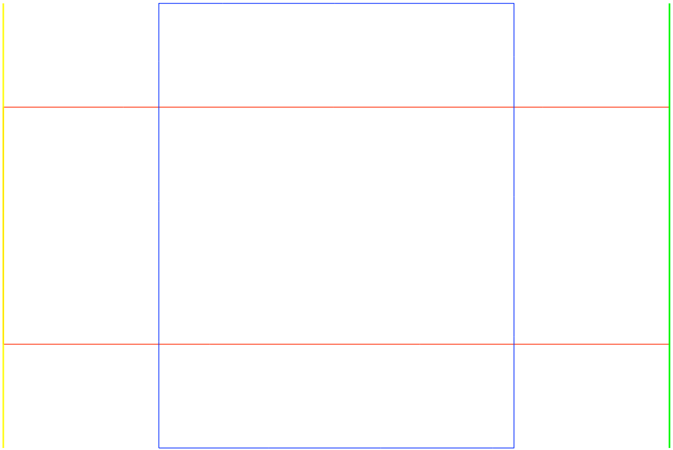


Verification using Taylor Models

To verify those fixed points we use the Miranda theorem

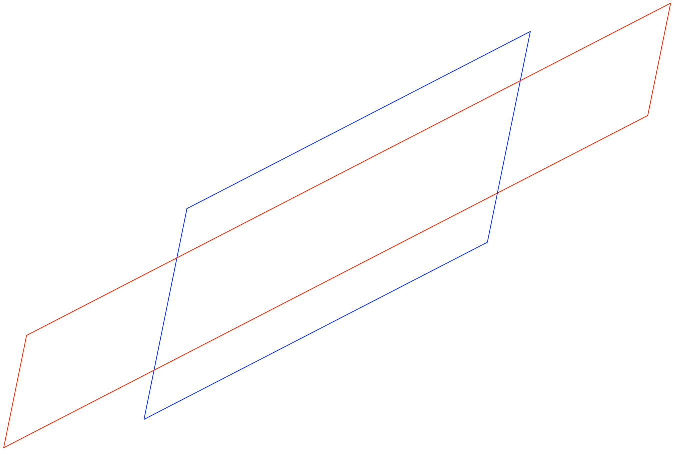
1. Take approximate fixed point and perform several Newton iterations to get closer to the real fixed point.
2. Send small box (about 10^{-5}) around fixed point through the map (in eigen-coordinates)
3. Calculate rigorous interval enclosure of the left boundary of the result and compare to interval enclosure of the original box.
4. Do similar tests for the other boundaries.

Successful verification



Cartesian Coordinates: blue: original box (10^{-5}); red: mapped box

Successful verification

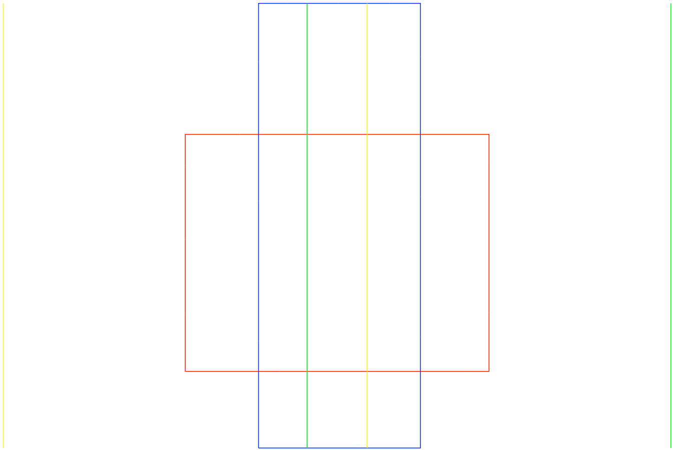


Cartesian Coordinates: blue: original box (10^{-5}); red: mapped box

Effects of floating point precision

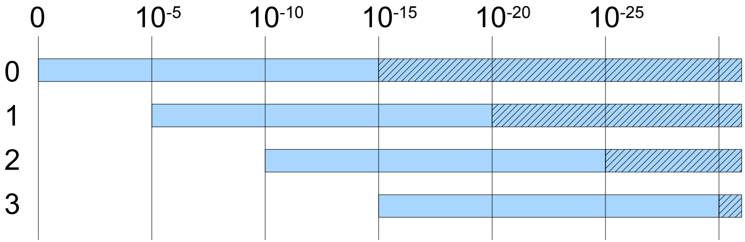
Of course Taylor Models also suffer from limited floating point precision. However here an interesting effect happens.
Our result becomes *worse* as we *reduce* the size of the initial box.
In fact, choosing a box of size 10^{-10} causes the verification to fail

Failed verification



blue: original box (10^{-10}); red: mapped box; yellow, green: interval enclosures of the boundaries.

This curious effect is due to a loss of effective expansion order in the Taylor series:



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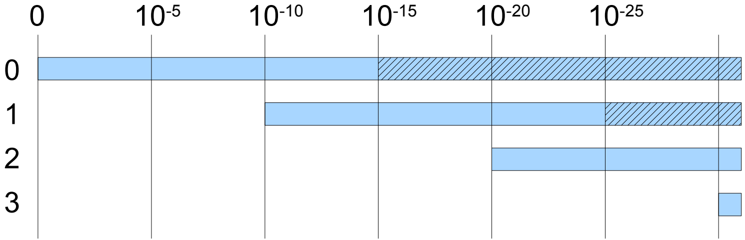
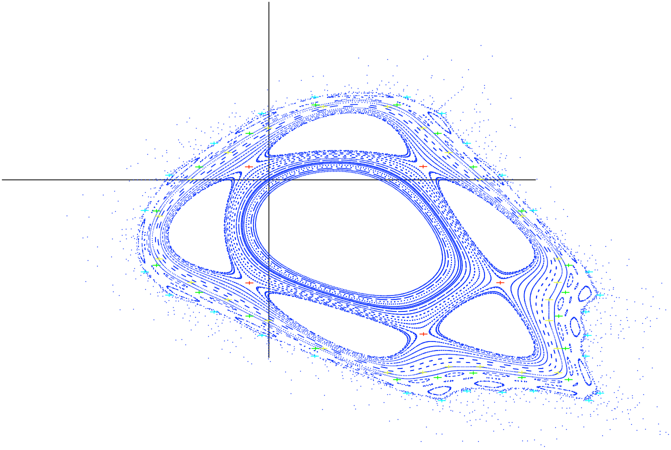


Other fixed points



Hyperbolic fixed points of orders 5, 21, 26, 27 in this map

Elliptic fixed points

For each hyperbolic fixed point there is an associated elliptic fixed point of same order in each of the islands.

- Currently we do not have code to verify those.
- An approach to proving those is to construct a Newton step like operator

$$\mathcal{N}(x) = x - (E - Df(x))^{-1}(x - f(x))$$

and show that this operator has a fixed point.

Further research

1. Develop a unified method to prove the existence of any type of fixed point by constructing a Newton-method like operator and show that it has a fixed point.
2. Automate the process of finding, classifying, and verifying fixed points of a map using global optimisation.
3. Study properties of fixed points under small variations of the parameters of the map (such as period doubling).

Conclusion

1. We (and by that I mean Sheldon) found several high order fixed points in different variations of the Hénon map.
2. We were able to verify those points using the Miranda and Schauder fixed point theorems in combination with Interval methods and Taylor models.
3. Limitations in the floating point accuracy of the current COSY Taylor Model implementation currently prohibit tight enclosures of the fixed points.

Thank you.

Thank you for your attention.