Recent Advances in the Rigorous Integration of Flows of ODEs with Taylor Models

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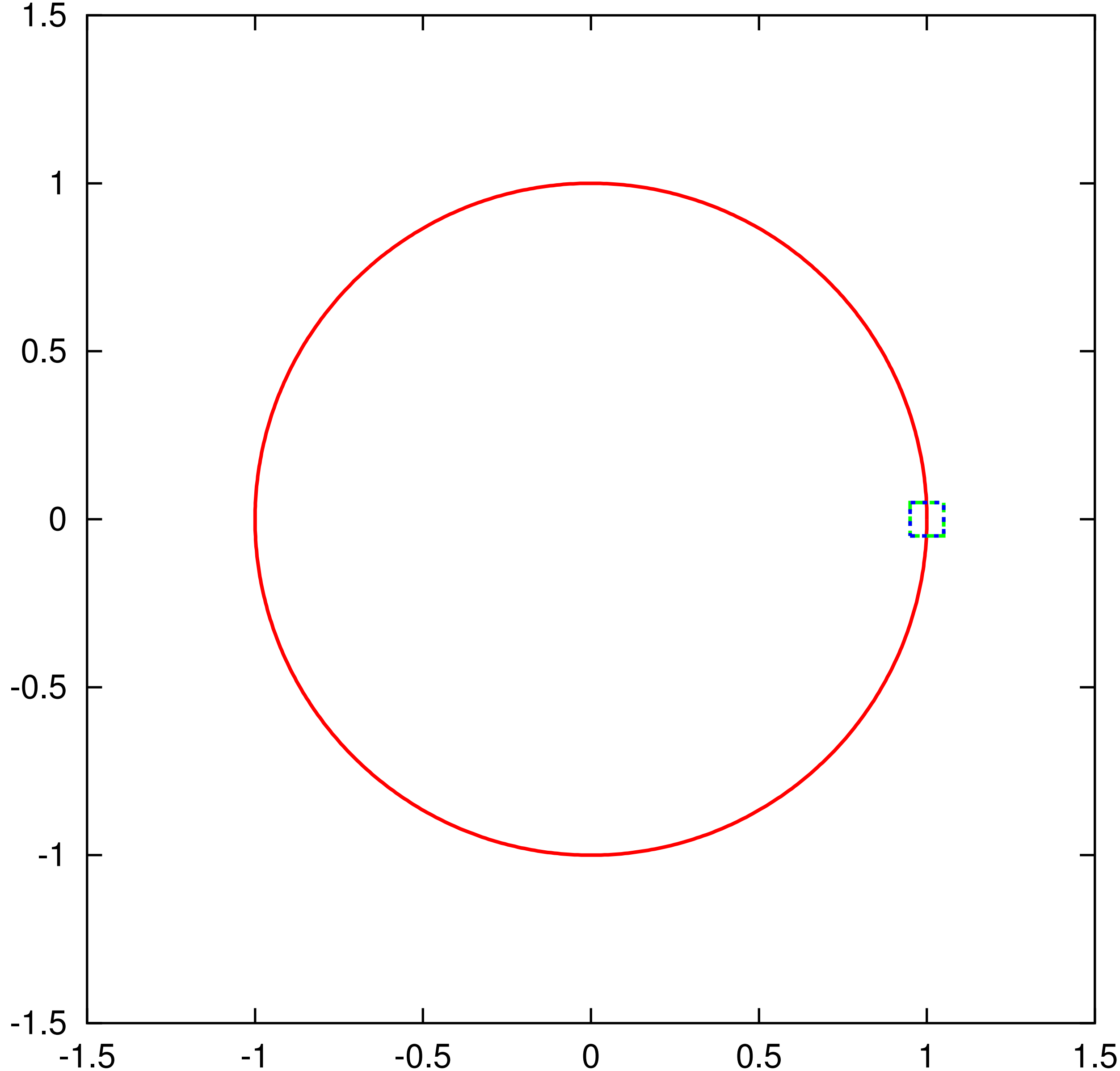
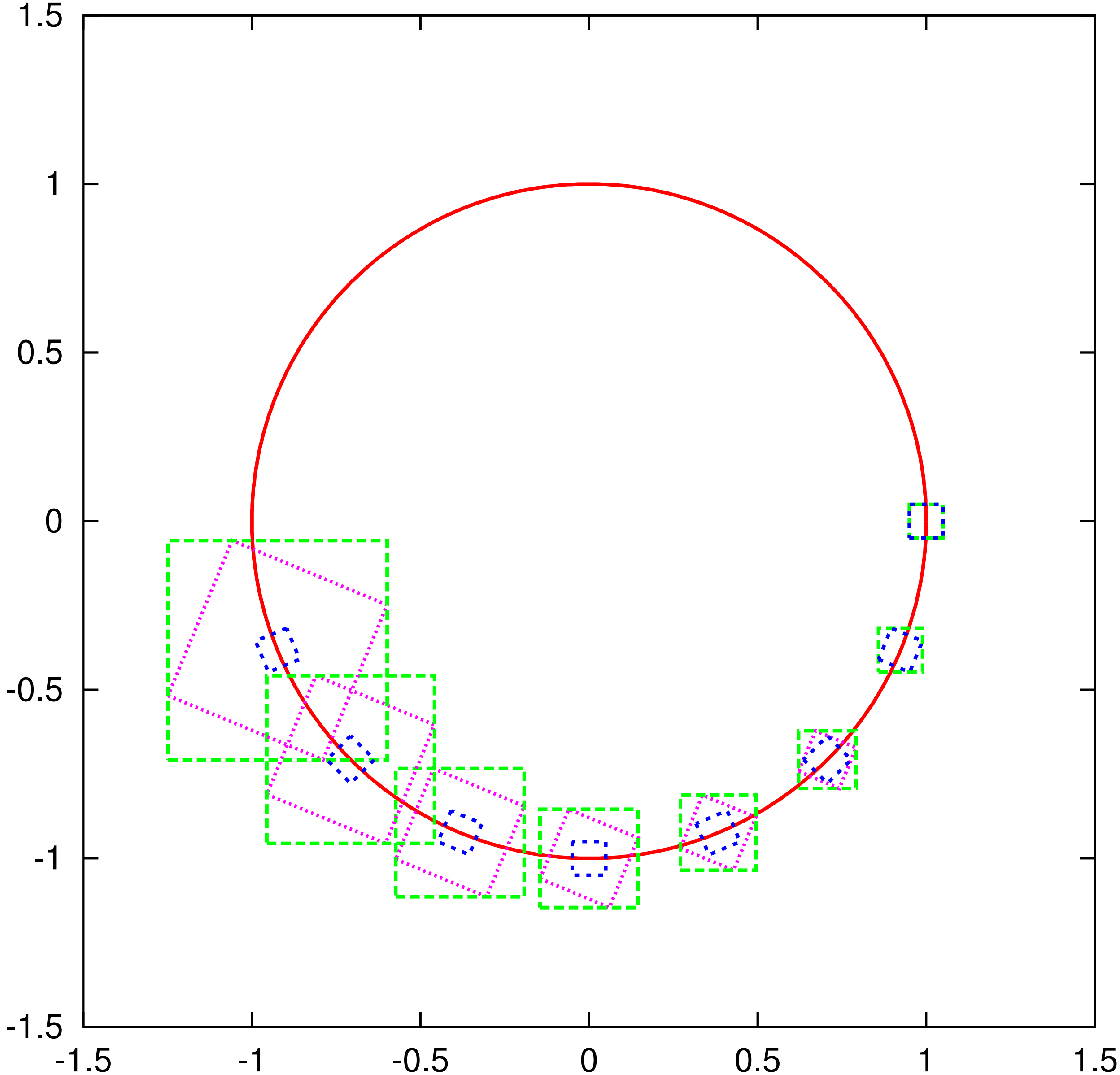
Outline

1. Review of the old version of COSY-VI
2. The Reference Trajectory and the Flow Operator
3. Step Size Control
4. Error Parametrization of Taylor Models
5. Dynamic Domain Decomposition
6. Examples

To transport a large phase space volume with validation,



Over Estimation has to be controlled.



Review of the Old Version of COSY-VI

Version 2 (2004)

Key Features and Algorithms of COSY-VI

- High order expansion not only in time t but also in transversal variables $\vec{x}$.
- Capability of weighted order computation, allowing to suppress the expansion order in transversal variables $\vec{x}$.
- Shrink wrapping algorithm including blunting to control ill-conditioned cases.
- Pre-conditioning algorithms based on the Curvilinear, QR decomposition, and blunting pre-conditioners.
- Resulting data is available in various levels including graphics output.

The Volterra Equation

Describe dynamics of two conflicting populations

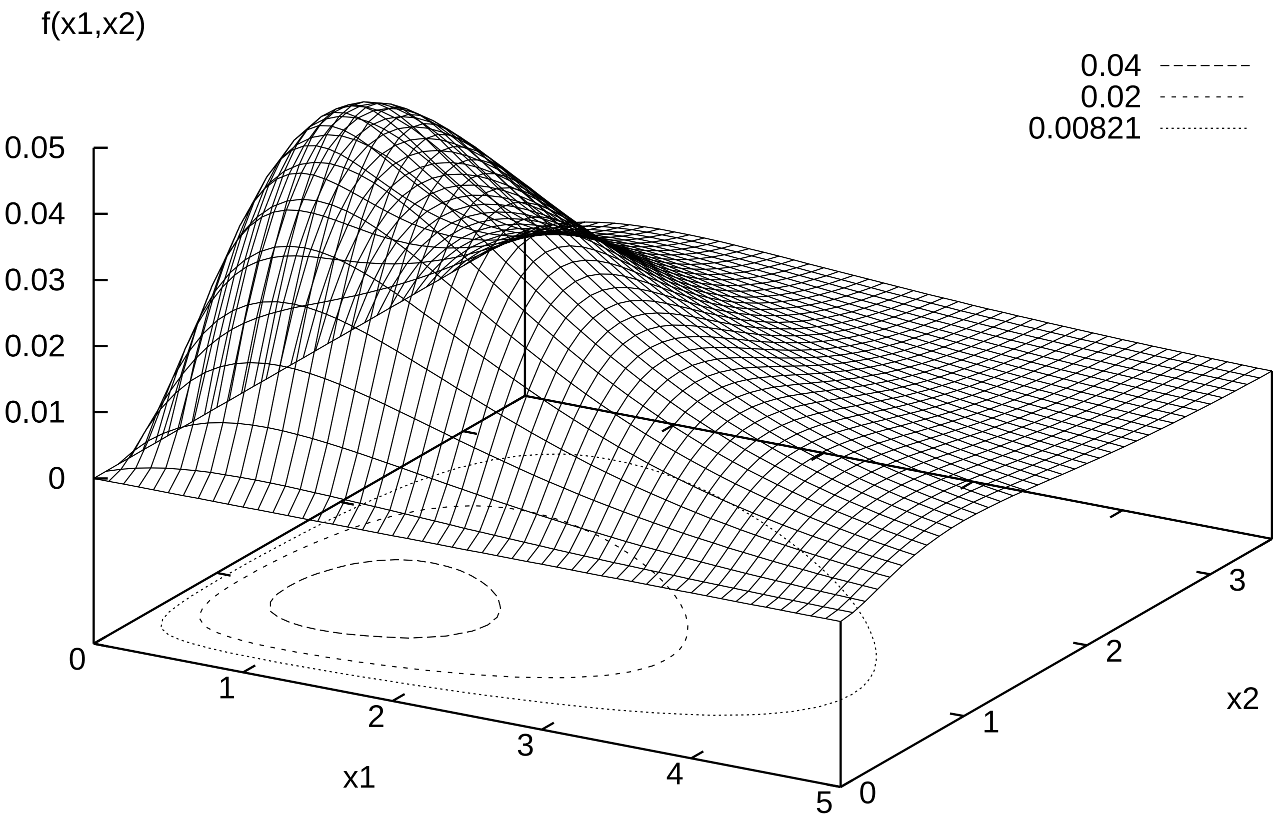
$$\frac{dx_1}{dt} = 2x_1(1 - x_2), \quad \frac{dx_2}{dt} = -x_2(1 - x_1)$$

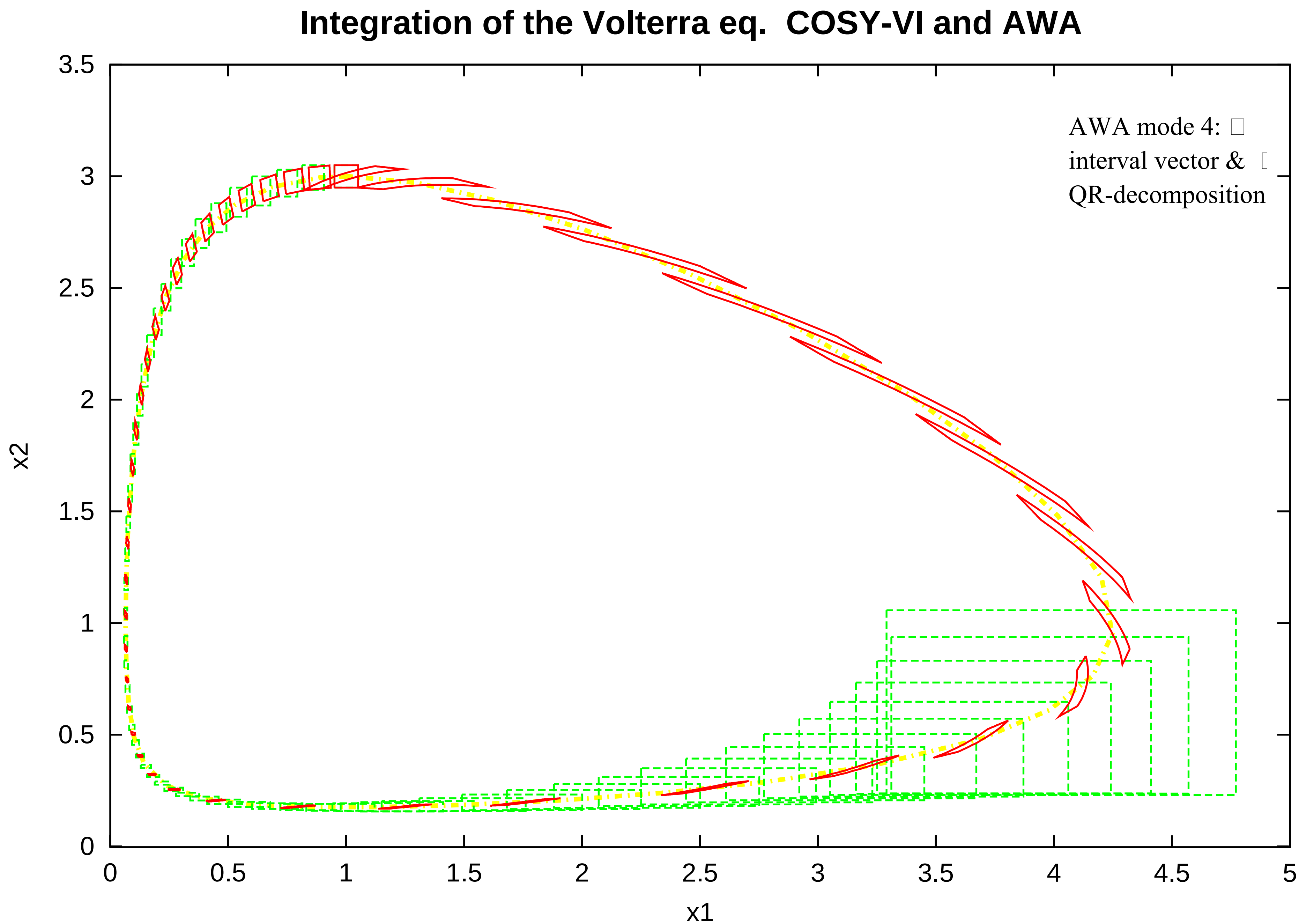
Interested in initial condition

$$x_{01} \in 1 + [-0.05, 0.05], \quad x_{02} \in 3 + [-0.05, 0.05] \quad \text{at } t = 0.$$

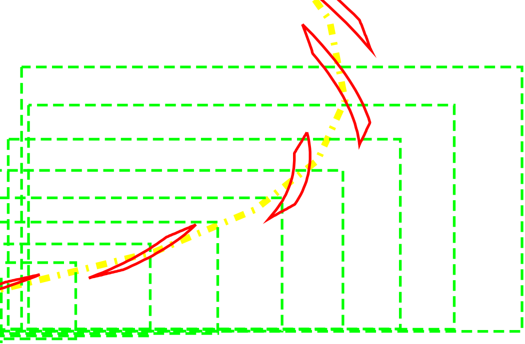
Satisfies constraint condition

$$C(x_1, x_2) = x_1 x_2^2 e^{-x_1 - 2x_2} = \text{Constant}$$


f(x1,x2)
0.04
0.02
0.00821
0.05
0.04
0.03
0.02
0.01
0
0
1
2
3
4
5
x1
0
1
2
3
x2



Integration of the Volterra eq. COSY-VI and AWA
AWA mode 4:
interval vector &
QR-decomposition
x1
x2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



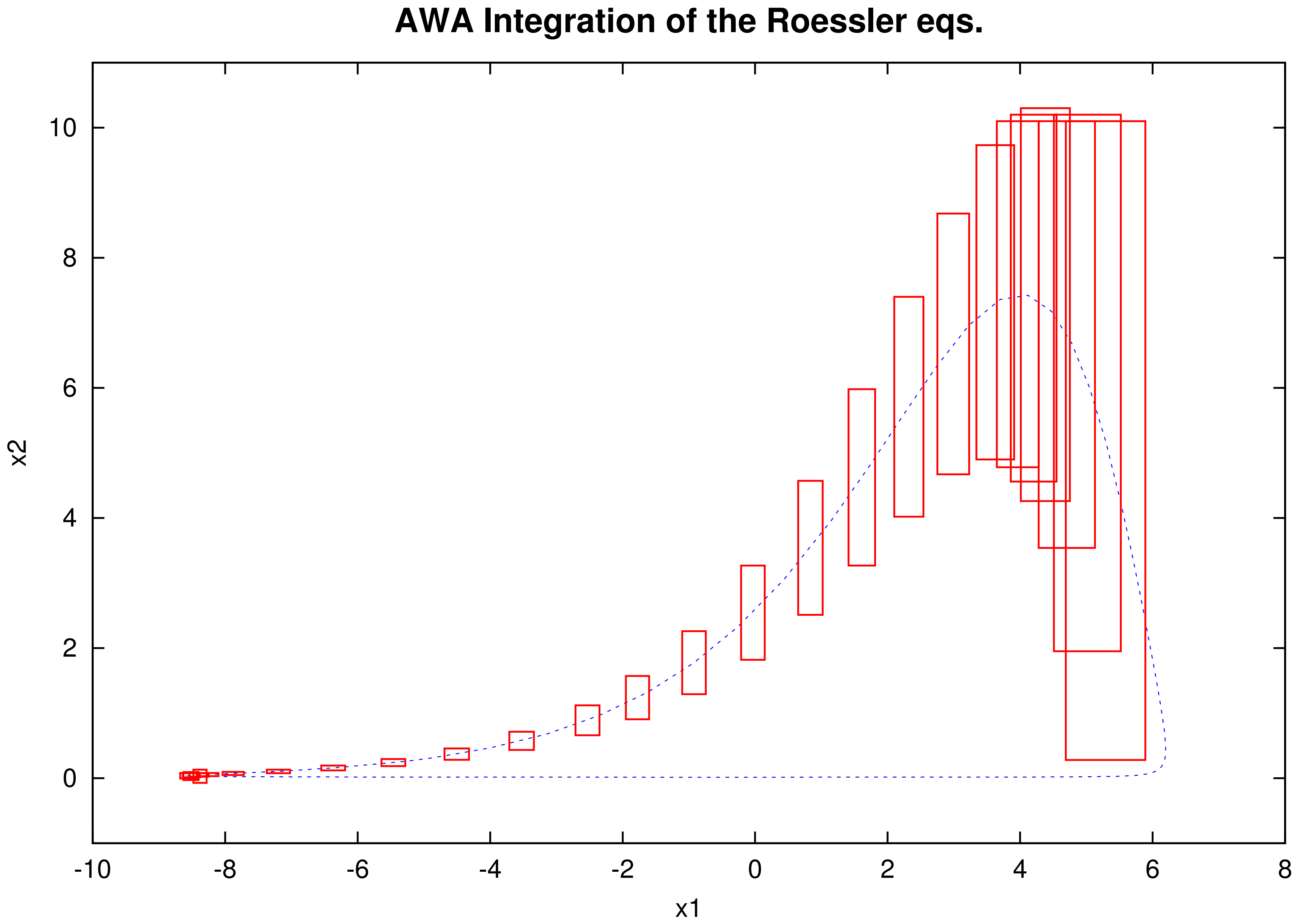
2 Rössler equations

The Rössler equations are given by

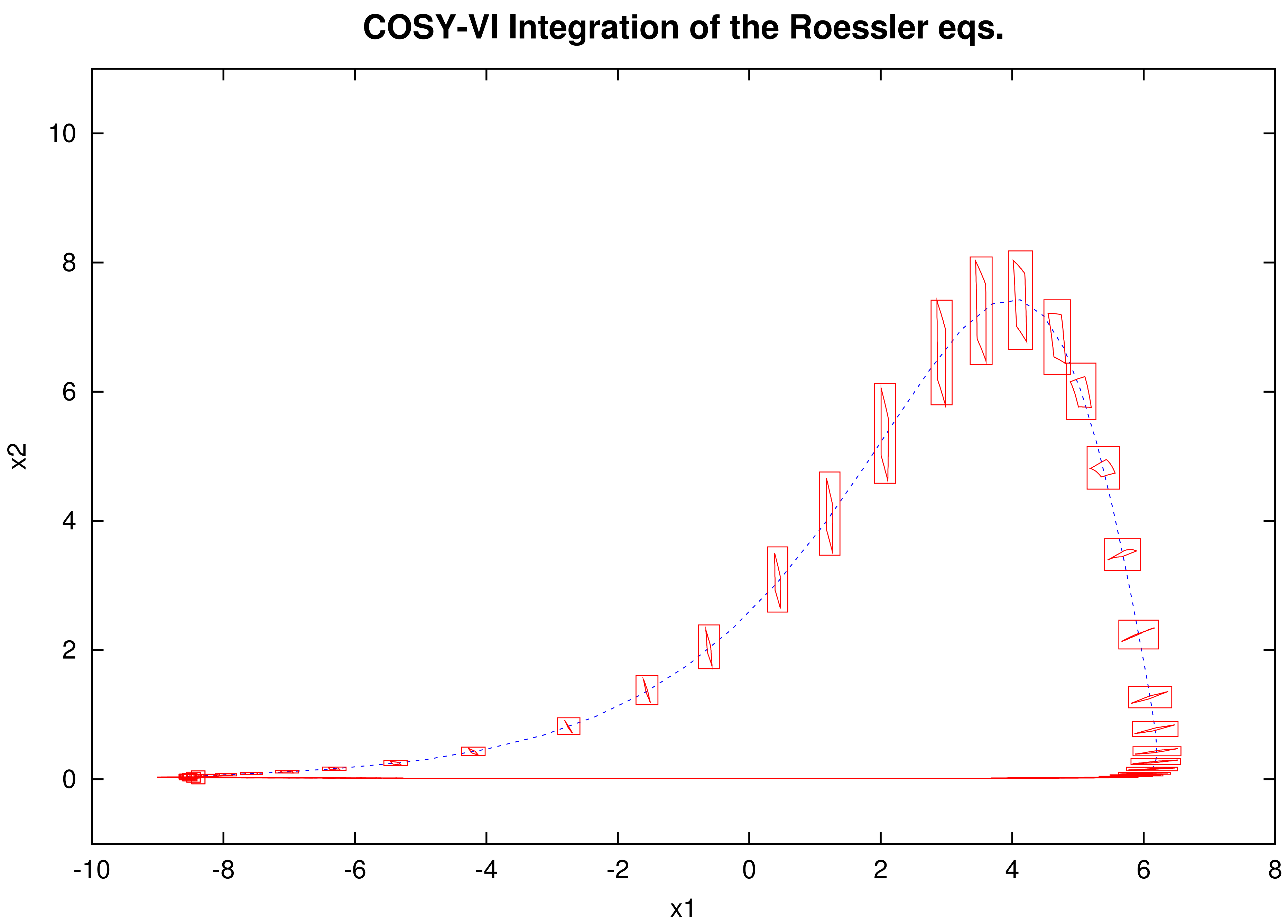
$$\begin{aligned} x' &= -(y+z) \\ y' &= x + 0.2y \\ z' &= 0.2 + z(x-a), \end{aligned} \tag{4}$$

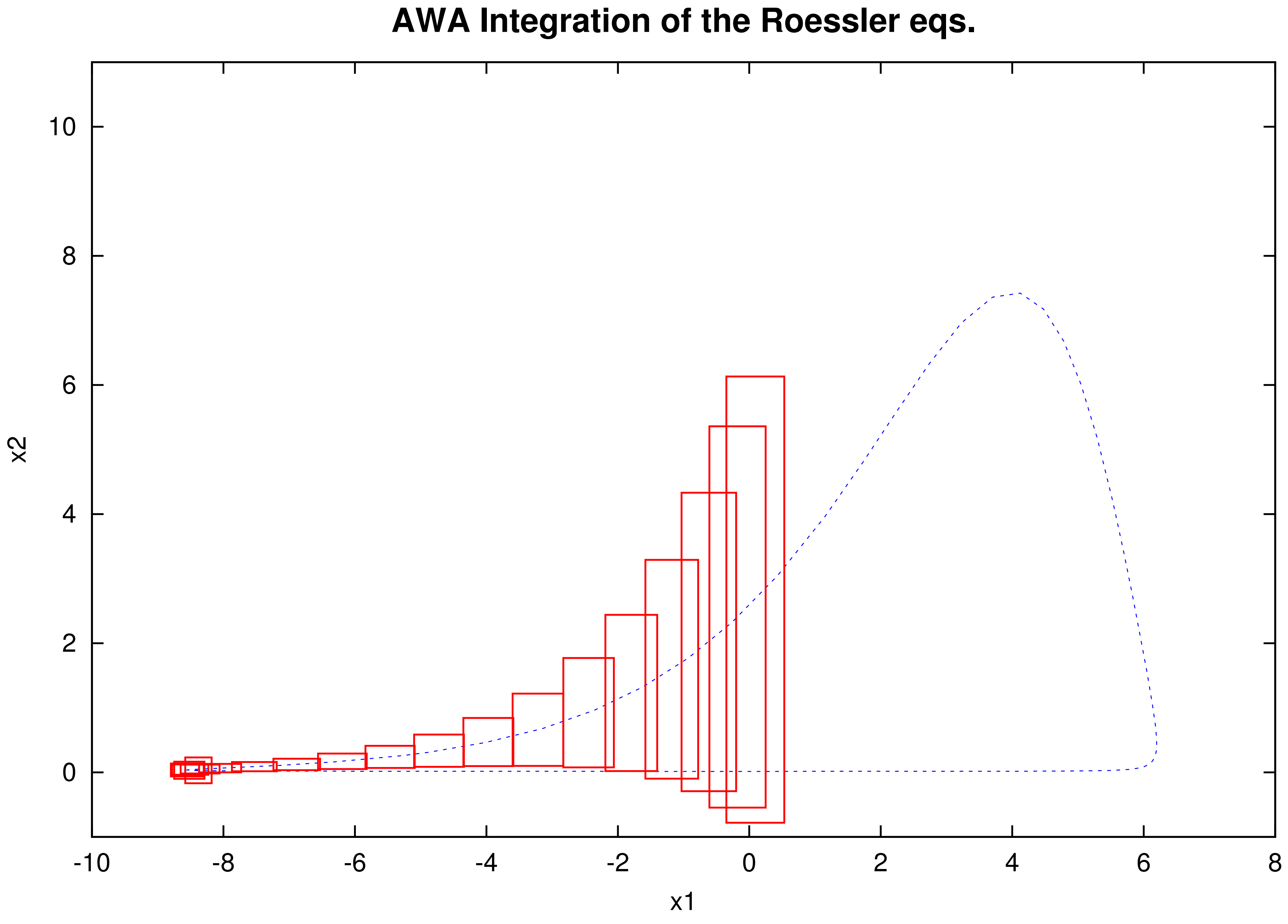
where a is a real parameter. We focus here at the value of $a = 5.7$, where numerical simulations suggest an existence of a strange attractor.

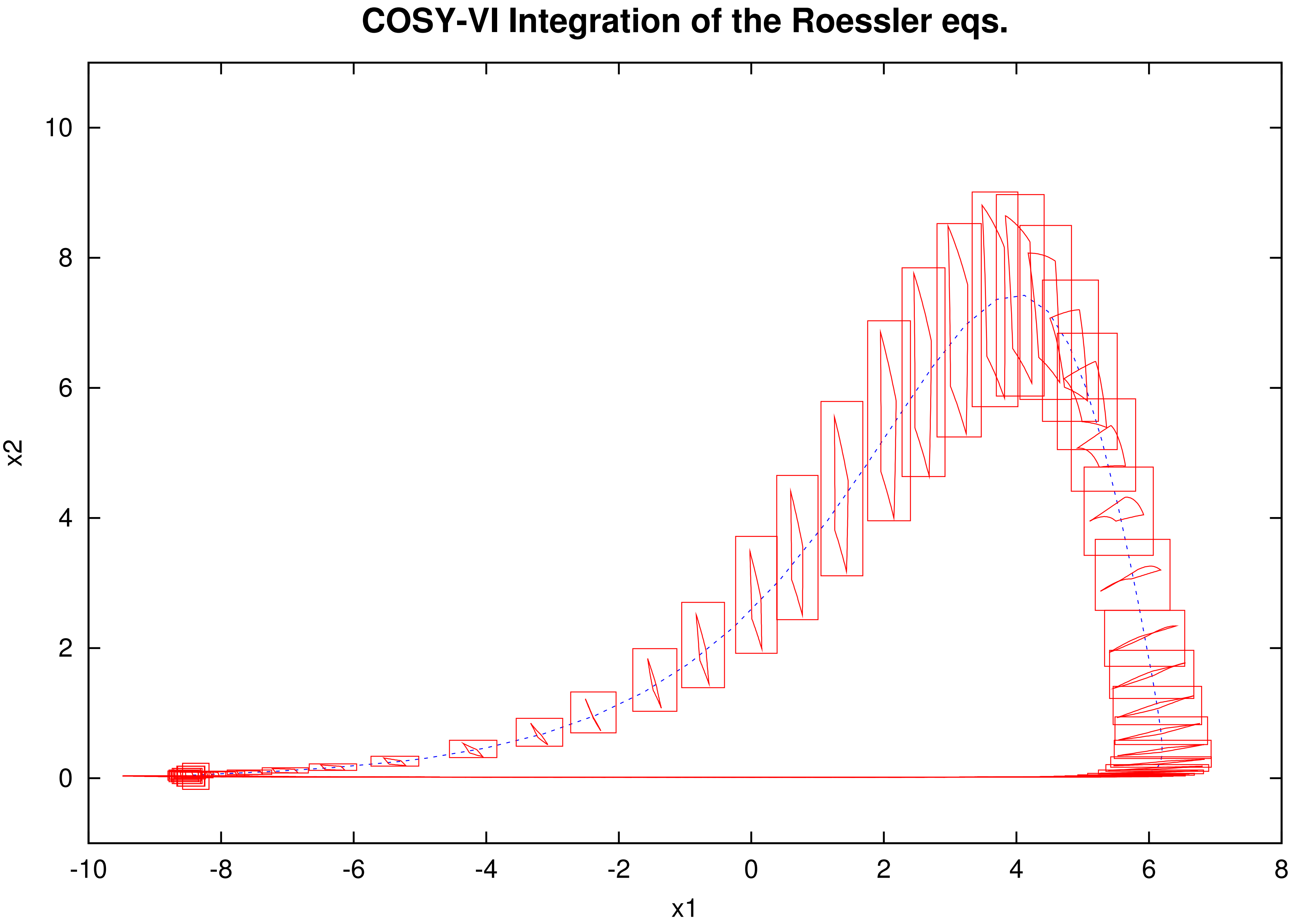
On section $x = 0$ we consider the following initial condition $(y, z) \in (-8.38095, 0.0295902) + [-\delta, \delta]^2$, where δ should be considerably larger than 10^{-3}. The integration time should be around $T = 6$.

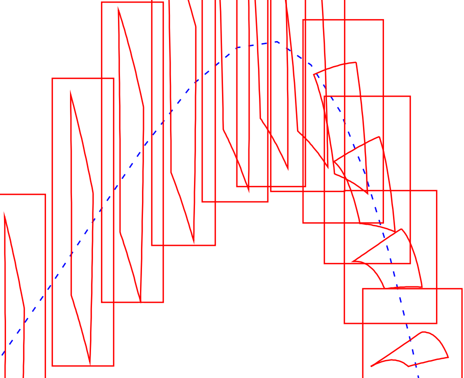


AWA Integration of the Roessler eqs.
x2
x1
10
8
6
4
2
0
-10
-8
-6
-4
-2
0
2
4
6
8


COSY-VI Integration of the Roessler eqs.
x2
x1
10
8
6
4
2
0
-10
-8
-6
-4
-2
0
2
4
6
8


AWA Integration of the Roessler eqs.
x2
x1
10
8
6
4
2
0
-10
-8
-6
-4
-2
0
2
4
6
8


COSY-VI Integration of the Roessler eqs.
x2
x1
10
8
6
4
2
0
-10
-8
-6
-4
-2
0
2
4
6
8



The Henon Map

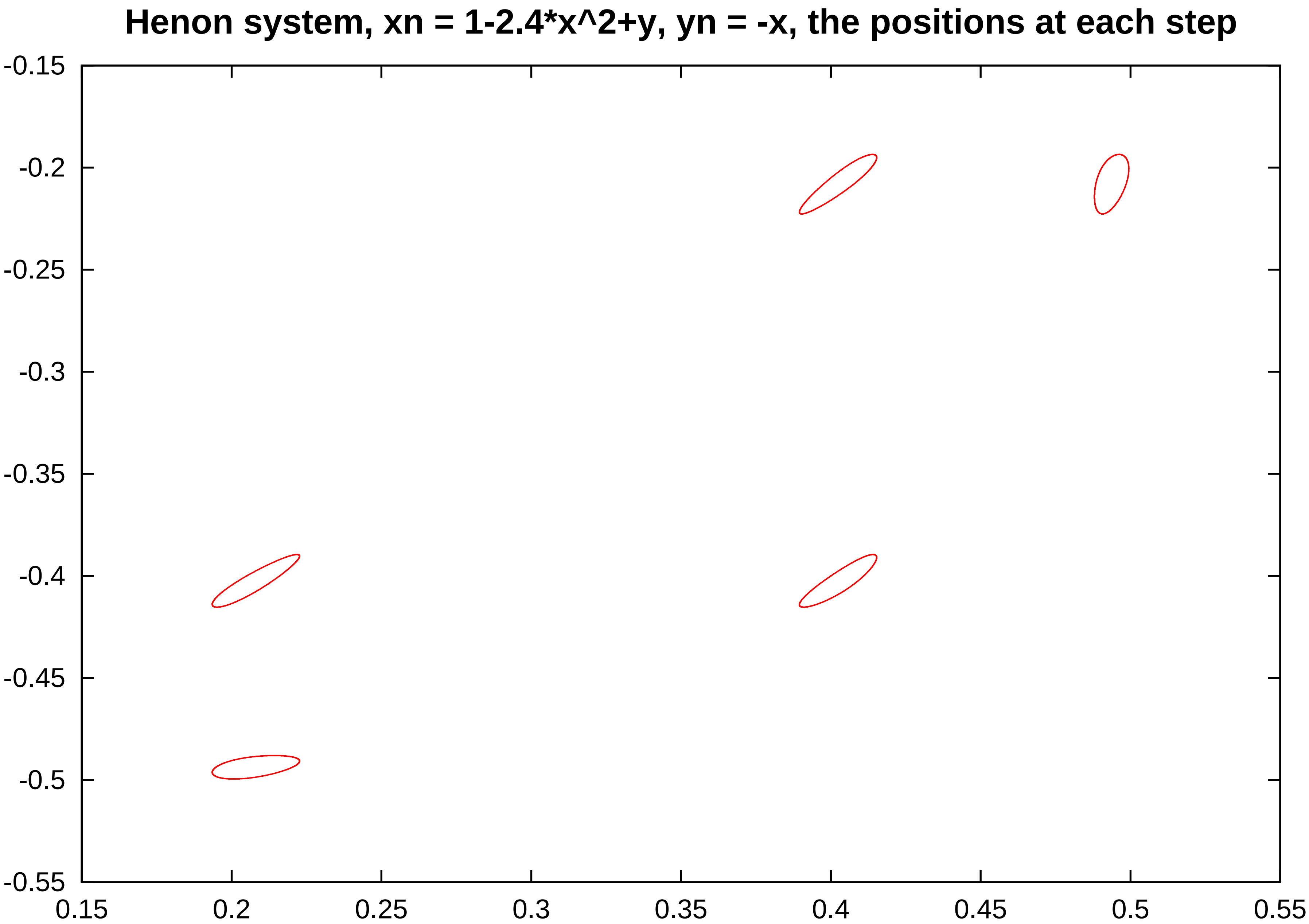
Henon Map: frequently used elementary example that exhibits many of the well-known effects of nonlinear dynamics, including chaos, periodic fixed points, islands and symplectic motion. The dynamics is two-dimensional, and given by

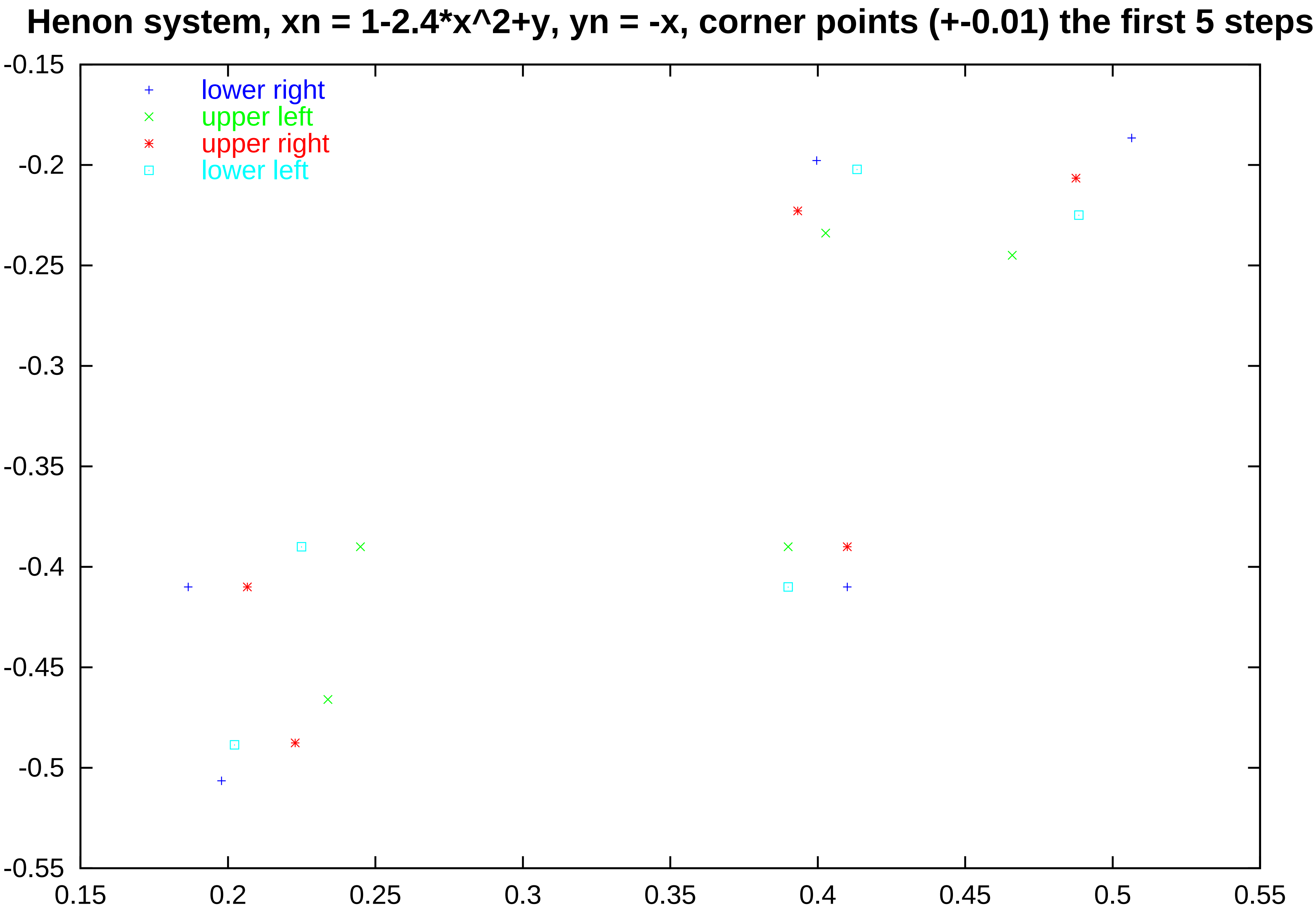
$$\begin{aligned} x_{n+1} &= 1 - \alpha x_n^2 + y_n \\ y_{n+1} &= \beta x_n. \end{aligned}$$

It can easily be seen that the motion is area preserving for $|\beta| = 1$.We consider

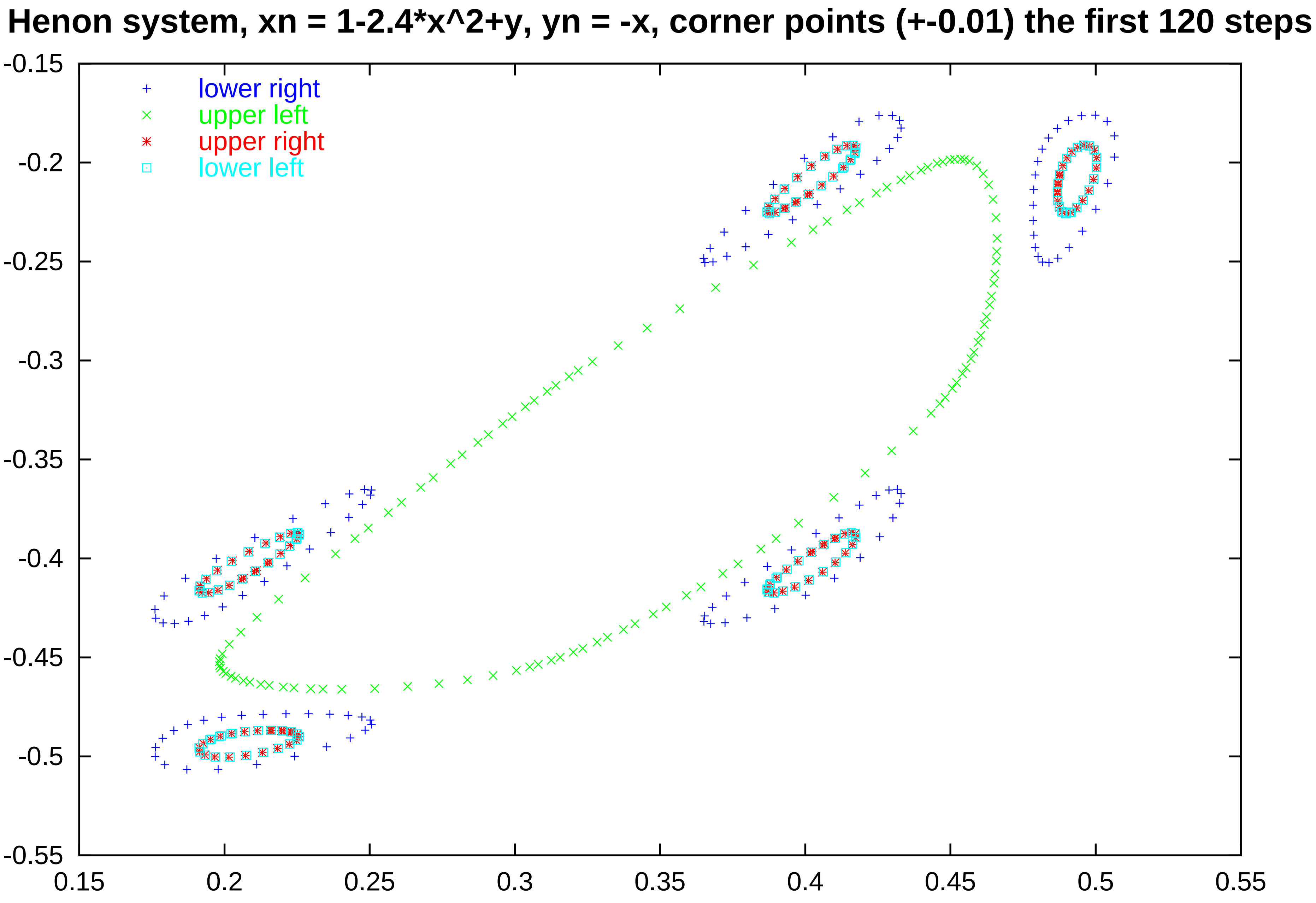
$$\alpha = 2.4 \text{ and } \beta = -1,$$

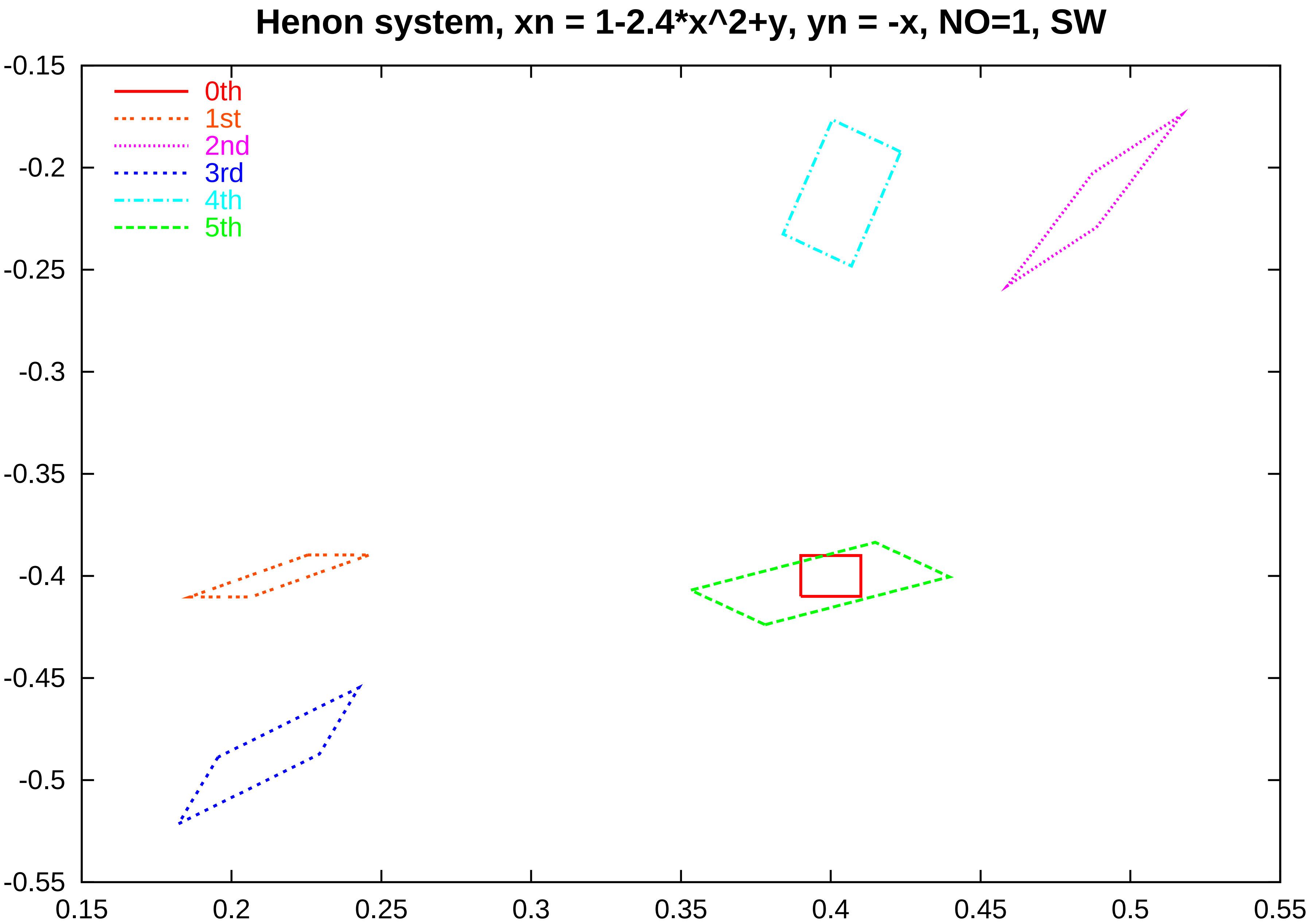
and concentrate on initial boxes of the from $(x_0, y_0) \in (0.4, -0.4) + [-d, d]^2$.

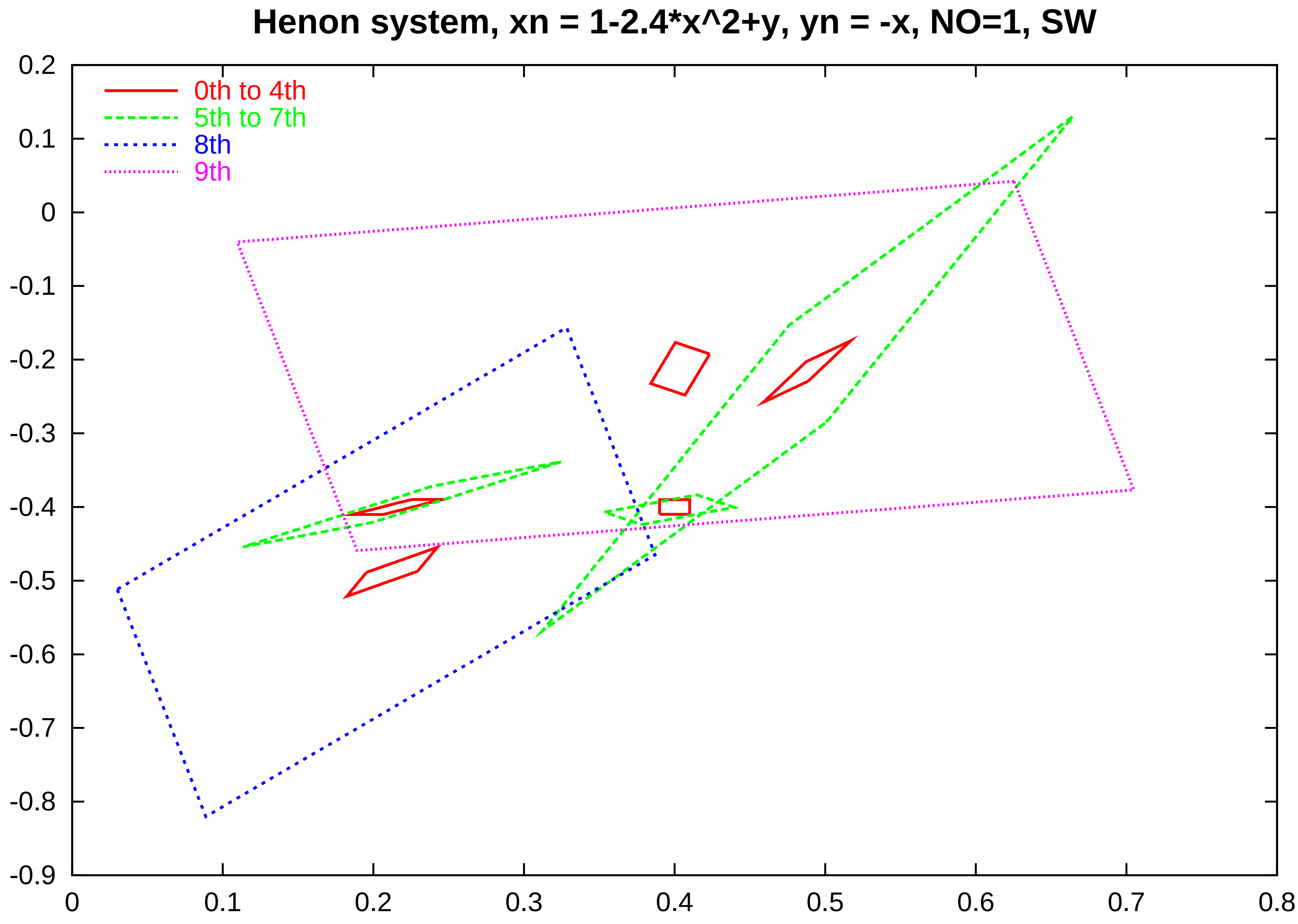

Henon system, xn = 1-2.4*x^2+y, yn = -x, the positions at each step
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

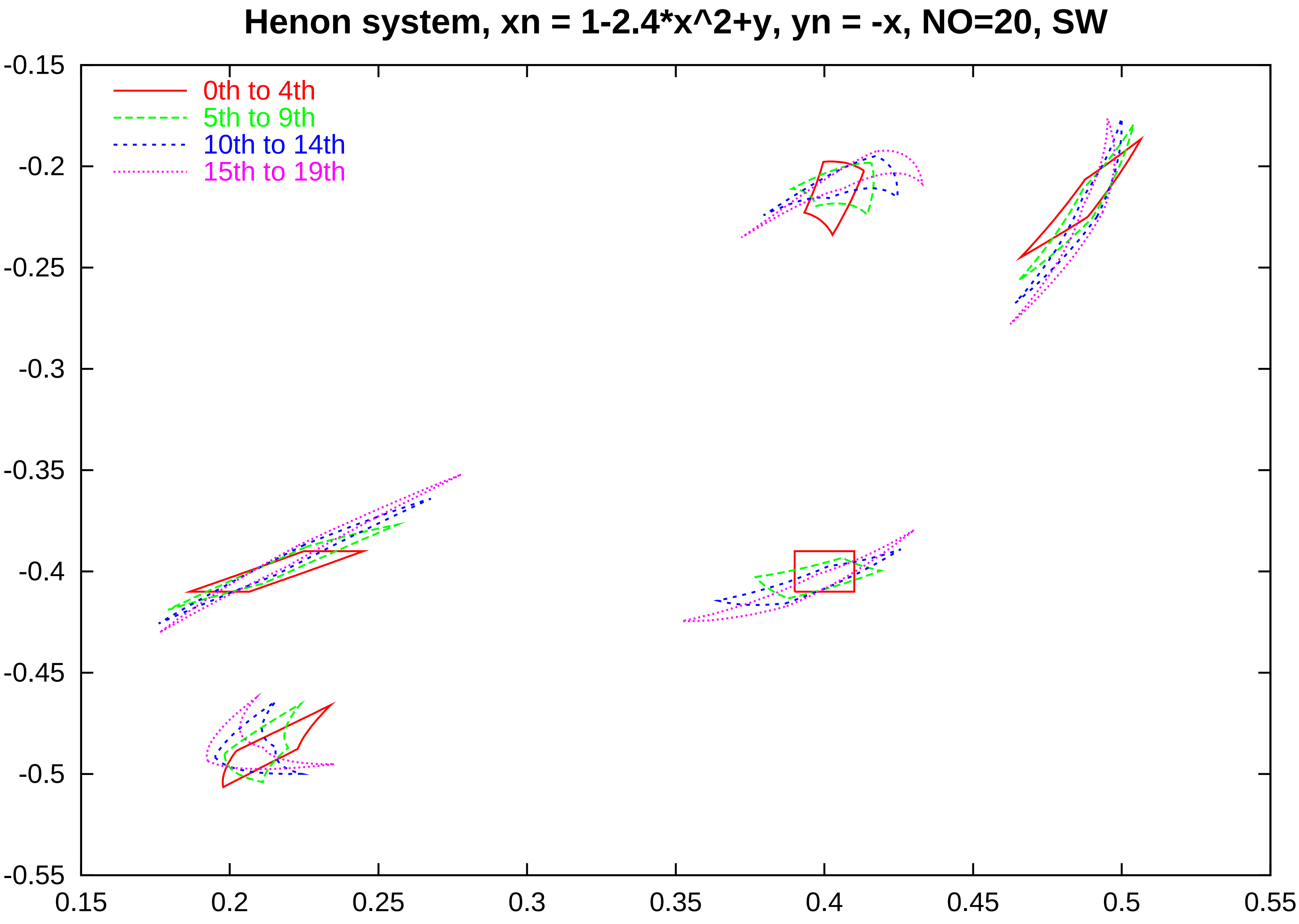


Henon system, xn = 1-2.4*x^2+y, yn = -x, corner points (+-0.01) the first 5 steps
lower right
upper left
upper right
lower left
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

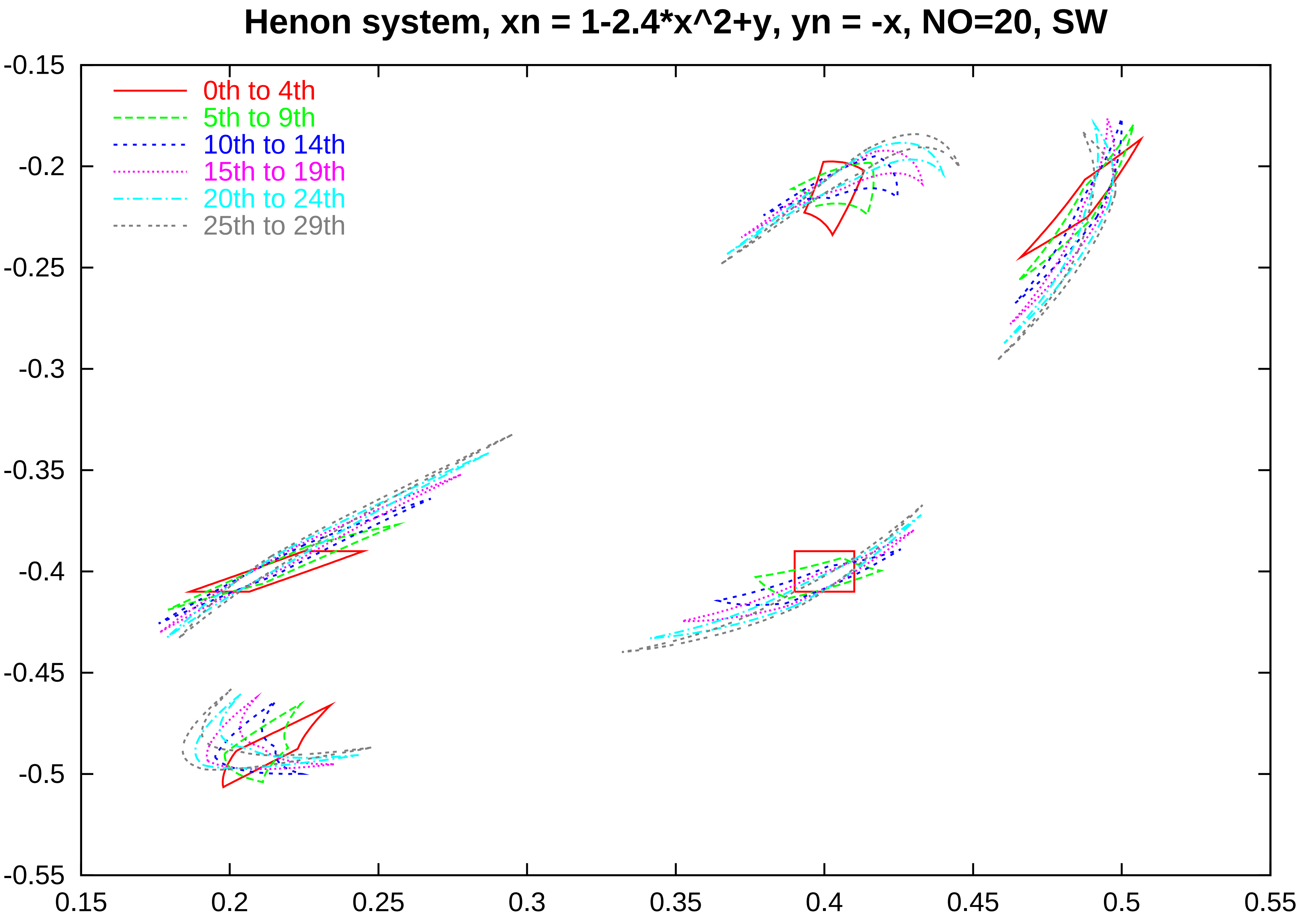

Henon system, xn = 1-2.4*x^2+y, yn = -x, corner points (+-0.01) the first 120 steps
lower right
upper left
upper right
lower left
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55


Henon system, xn = 1-2.4*x^2+y, yn = -x, NO=1, SW
0th
1st
2nd
3rd
4th
5th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55


Henon system, xn = 1-2.4*x^2+y, yn = -x, NO=1, SW
0th to 4th
5th to 7th
8th
9th
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8



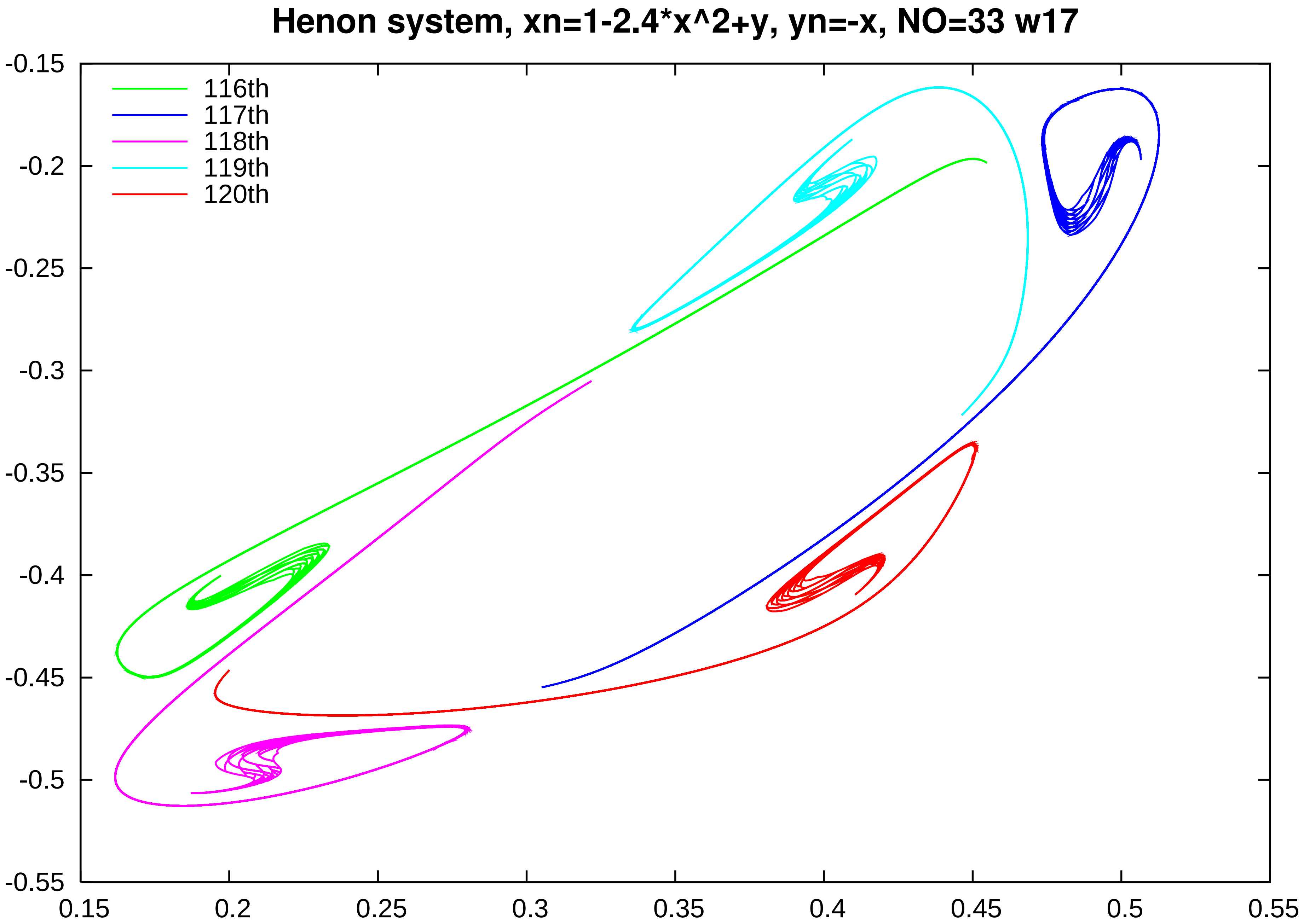
Henon system, xn = 1-2.4*x^2+y, yn = -x, NO=20, SW
0th to 4th
5th to 9th
10th to 14th
15th to 19th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55



Henon system, xn = 1-2.4*x^2+y, yn = -x, NO=20, SW
0th to 4th
5th to 9th
10th to 14th
15th to 19th
20th to 24th
25th to 29th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

Review of the New Features

- The Reference Trajectory and the Flow Operator
- Step Size Control
- Error Parametrization of Taylor Models
- Dynamic Domain Decomposition


Henon system, xn=1-2.4*x^2+y, yn=-x, NO=33 w17
116th
117th
118th
119th
120th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

The Reference Trajectory

First Step: Obtain Taylor expansion in time of solution of ODE of center point c, i.e. obtain

$$c(t) = c_0 + c_1 \cdot (t - t_0) + c_2 \cdot (t - t_0)^2 + ... + c_n \cdot (t - t_0)^n$$

Very well known from day one how to do this with automatic differentiation. Rather convenient way: can be done by n iterations of the Picard Operator

$$c(t) = c_0 + \int_0^t f(r(t'), t) dt'$$

in one-dimensional Taylor arithmetic. Each iteration raises the order by one; so in each iteration i, only need to do Taylor arithmetic in order i. In either way, this step is **cheap** since it involves only **one-dimensional** operations.

The Nonlinear Flow

Second Step: The goal is to obtain Taylor expansion in time to order n **and** initial conditions to order k. Note:

1. This is usually the most **expensive** step. In the original Taylor model-based algorithm, it is done by n **iterations** of the Picard Operator in multi-dimensional Taylor arithmetic, where c_0 is now a polynomial in initial conditions.
2. The case $k = 1$ has been known for a long time. Traditionally solved by setting up **ODEs for sensitivities** and solving these as before.
3. The case of higher k goes back to Beam Physics (M. Berz, Particle Accelerators 1988)
4. Newest Taylor model arithmetic naturally supports different expansions orders k for initial conditions and n for time.

Goal: Obtain flow with one **single evaluation** of right hand side.

The Nonlinear Relative ODE

We now develop a better way for second step.
First: introduce new "perturbation" variables $\tilde{r}$ such that

$$r(t) = c(t) + A \cdot \tilde{r}(t).$$

The matrix A provides **preconditioning**. ODE for $\tilde{r}(t)$:

$$\tilde{r}' = A^{-1} \left[f(c(t) + A \cdot \tilde{r}(t)) - c'(t) \right]$$

Second: evaluate ODE for $\tilde{r}'$ in Taylor arithmetic. Obtain a Taylor expansion of the ODE, i.e.

$$\tilde{r}' = P(\tilde{r}, t)$$

up to order n in time and k in $\tilde{r}$. **Very important** for later use: the polynomial P will have no constant part, i.e.

$$P(0, t) = 0.$$

Reminder: The Lie Derivative

Let

$$r' = f(r,t)$$

be a dynamical system. Let g be a variable in state space, and let us study $g(r(t))$, i.e. along a solution of the ODE. We have

$$\frac{d}{dt} g(t) = f \cdot \nabla g + \frac{\partial g}{\partial t}$$

Introducing the **Lie Derivative** $L_f = f \cdot \nabla + \partial/\partial t$, we have

$$\frac{d^n}{dt^n} g = L_f^n g \text{ and } g(t) \approx \sum_{i=0}^{n} \frac{(t-t_0)^i}{i!} \, L_f^i g \Big/_{t=t_0}$$

Differential Algebras on Taylor Polynomial Spaces

Consider space ${}_nD_v$ of Taylor polynomials in v variables and order n with truncation multiplication. Formally: introduce **equivalence relation** on space of smooth functions

$$f =_n g$$

if all derivatives from 0 to n agree at 0. **Class** of f is denoted $[f]$. This induces addition, multiplication and scalar multiplication on classes. The resulting structure forms an algebra.

An algebra is a **Differential Algebra** if there is an operation ∂, called a derivation, that satisfies

$$\partial(s \cdot a + t \cdot b) = s \cdot \partial a + t \cdot \partial b \text{ and}$$
$$\partial(a \cdot b) = a \cdot (\partial b) + (\partial a) \cdot b$$

for any vectors a and b and scalars s and t. **Unfortunately**, the **natural partial derivative** operations $[f] \to [\partial_i f]$ does **not** introduce a differential algebra, because of loss of highest order.

Differential Algebras on Taylor Polynomial Spaces

However, consider the modified operation

$$\partial_f \text{ with } \partial_f g = f \cdot \nabla g$$

If f is origin preserving, i.e. $f(0) = 0$, then ∂_f is a derivation on the space ${}_nD_v$. Why?

- Each derivative operation in the gradient ∇g looses the highest order;
- but since $f(0) = 0$, the missing order in ∇g **does not matter** since it does not contribute to the product $f \cdot \nabla g$.

Polynomial Flow from Lie Derivative

Remember the ODE for $\tilde{r}'$:

$$\tilde{r}' = P(\tilde{r}, t)$$

up to order n in time and k in $\tilde{r}$. And remember $P(0,t) = 0$. Thus we can obtain the n-th order expansion of the flow as

$$\tilde{r}(t) = \sum_{i=0}^{n} \frac{(t-t_0)^i}{i!} \cdot \left(P \cdot \nabla + \frac{\partial}{\partial t} \right)^i \tilde{r}_0 \Big/_{t=t_0}$$

- The fact that $P(0,t) = 0$ restores the derivatives lost in ∇
- The fact that $\partial/\partial t$ appears without origin-preserving factor limits the expansion to order n.

Performance of Lie Derivative Flow Methods

Apparently we have the following:

- Each term in the Lie derivative sum requires $v + 1$ derivations (very cheap, just re-shuffling of coefficients)
- Each term requires v multiplications
- We need **one** evaluation of f in ${}_nD_v$ (to set up ODE)

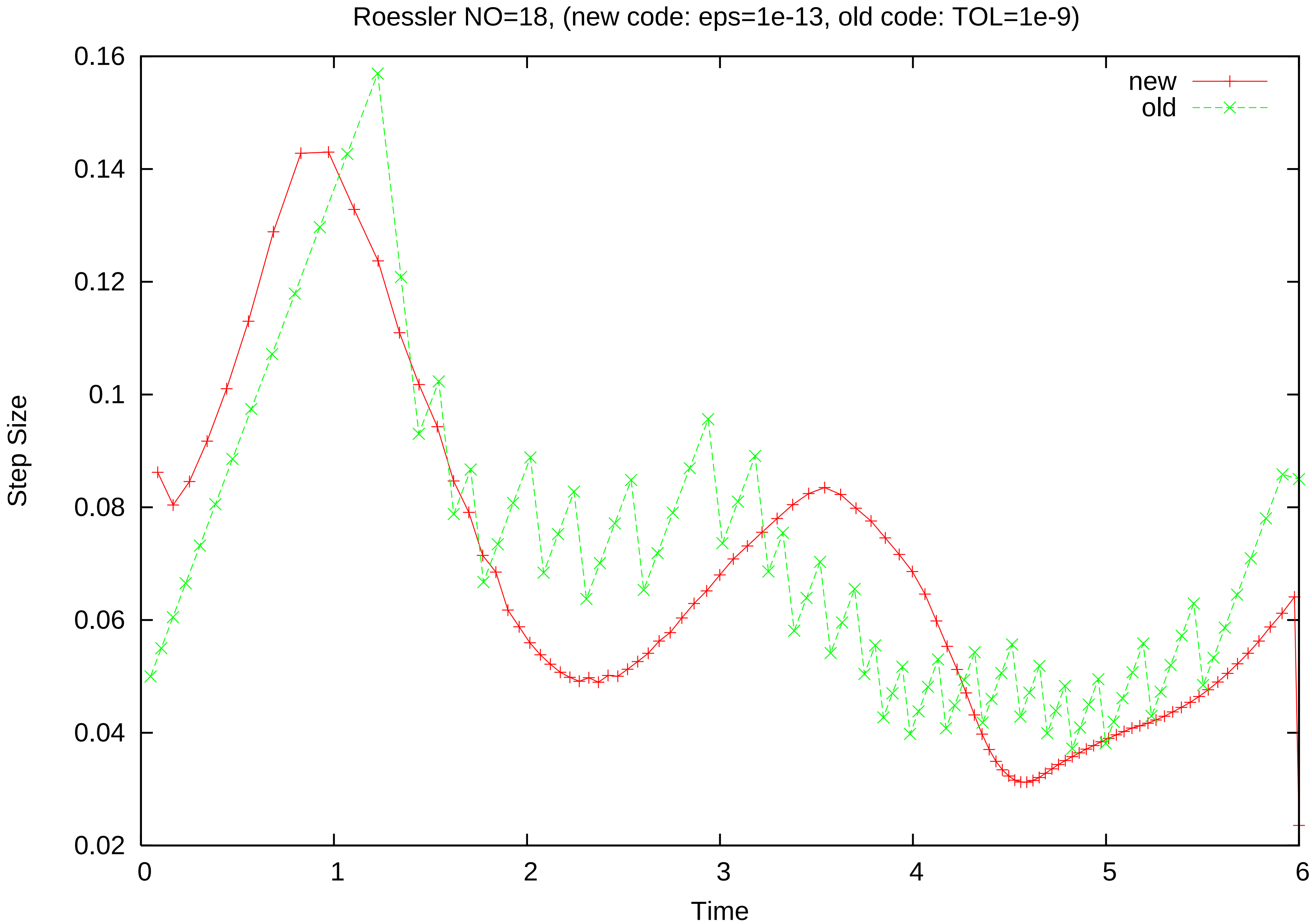
Compare this with the conventional algorithm, which requires n evaluations of the function f of the right hand side. Thus, roughly, if the evaluation of f requires more than v multiplications, the new method is more efficient.

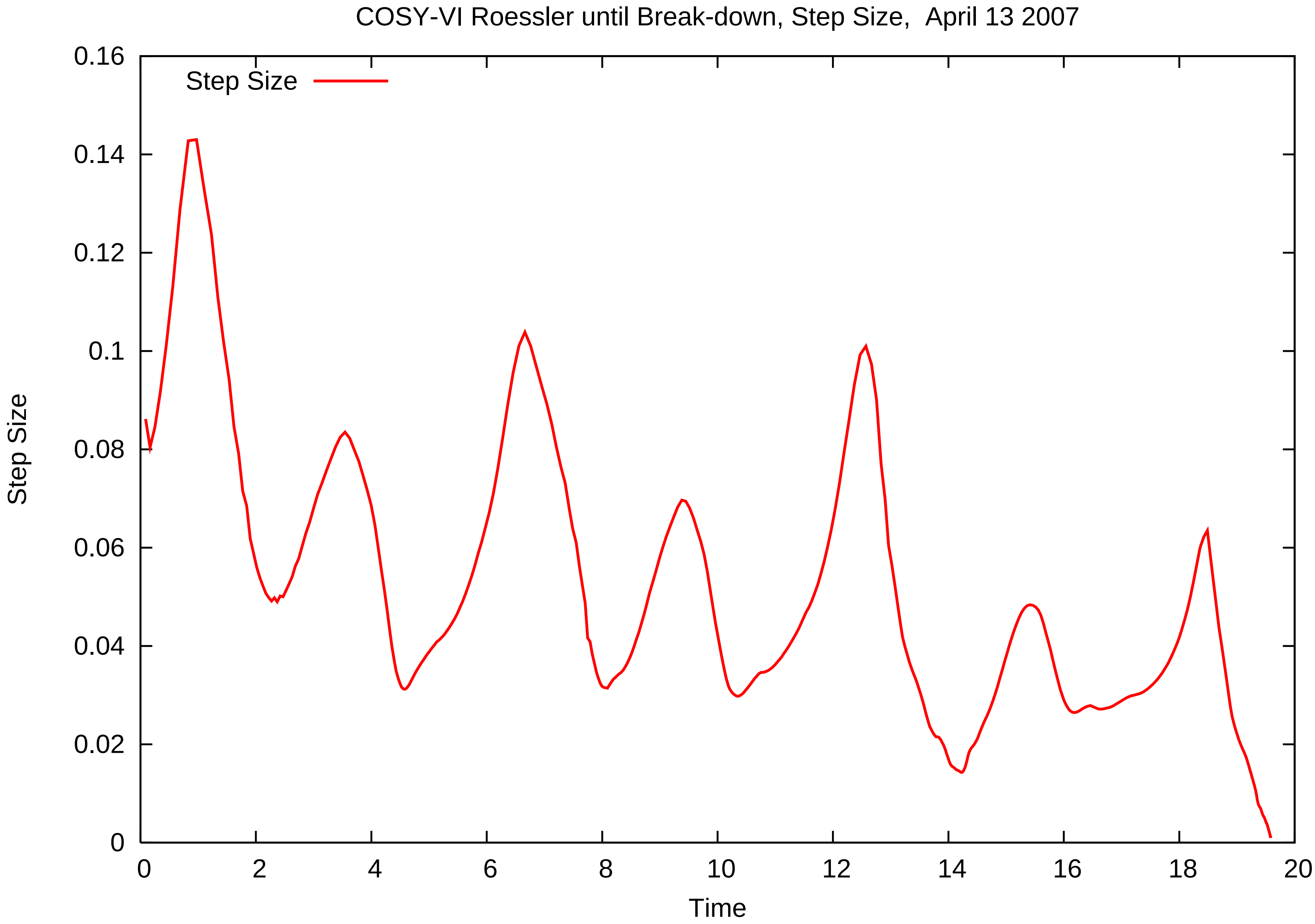
- Many practically appearing right hand sides f satisfy this.
- But on the other hand, if the function f does not satisfy this (for example for the linear case), then also P will be simple (in the linear case: P will be linear), and thus less operations appear

Step Size Control

Step size control to maintain approximate error ε in each step. Based on a suite of tests:

1. Utilize the **Reference Orbit.** Extrapolate the size of coefficients for estimate of remainder error, scale so that it reaches and get Δt_1. Goes back to Moore in 1960s. This is one of conveniences when using Taylor integrators.
2. Utilize the **Flow.** Compute flow time step with Δt_1.Extrapolate the contributions of each order of flow for estimate of remainder error to get update Δt_2.
3. Utilize a **Correction factor** c to account for overestimation in TM arithmetic as $c = \sqrt[n+1]{|R|/\varepsilon}$. Largely a measure of complexity of ODE. Dynamically update the correction factor.
4. Perform verification attempt for $\Delta t_3 = c \cdot \Delta t_2$


Roessler NO=18, (new code: eps=1e-13, old code: TOL=1e-9)
Step Size
Time
new
old
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
1
2
3
4
5
6


COSY-VI Roessler until Break-down, Step Size, April 13 2007
Step Size
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
2
4
6
8
10
12
14
16
18
20
Time
Step Size

Error Parametrization of Taylor models

Motivation: Is it possible to absorb the remainder error bound intervals of Taylor models into the polynomial parts using additional parameters?

Phrase the question as the following problem:

1. Have Taylor models with 0 remainder error interval, which depend on the independent variables $\vec{x}$ and the parameters $\vec{\alpha}$.

$$\vec{T}_0 = \vec{P}_0(\vec{x}, \vec{\alpha}) + \overrightarrow{[0, 0]}.$$

2. Perform Taylor model arithmetic on $\vec{T}_0$, namely $\vec{F}(\vec{T}_0)$

$$\vec{F}(\vec{T}_0) = \vec{P}(\vec{x}, \vec{\alpha}) + \vec{I}_F, \text{ where } \vec{I}_F \neq \overrightarrow{[0, 0]}.$$

3. Try to absorb $\vec{I}_F$ into the polynomial part that depends on $\vec{\alpha}$

$$\vec{P}(\vec{x}, \vec{\alpha}) + \vec{I}_F \subseteq \vec{P}'(\vec{x}, \vec{\alpha}) + \overrightarrow{[0, 0]}. \qquad \text{(A)}$$

Observe

$$\vec{P}(\vec{x},\vec{\alpha}) = \underbrace{\vec{P}(\vec{x},0)}_{\vec{\alpha}\text{-indep.}} + \underbrace{\vec{P}(\vec{x},\vec{\alpha}) - \vec{P}(\vec{x},0)}_{\vec{\alpha}\text{-dependent}} = \vec{P}(\vec{x},0) + \vec{P}_\alpha(\vec{x},\vec{\alpha})$$

The size of $\vec{P}(\vec{x},0)$ is much larger than the rest, because the rest is essentially errors. The process of (A) does not alter $\vec{P}(\vec{x},0)$, so set the $\vec{\alpha}$-independent part $\vec{P}(\vec{x},0)$ aside from the whole process, which helps the numerical stability of the process.

The task is now

$$\vec{P}_\alpha(\vec{x},\vec{\alpha}) + \vec{I}_F \subseteq \vec{P}'_\alpha(\vec{x},\vec{\alpha}) + \overrightarrow{[0,0]}.$$

We limit $\vec{P}_\alpha(\vec{x},\vec{\alpha})$ to be only **linearly** dependent on $\vec{\alpha}$.

$$\vec{P}_\alpha(\vec{x},\vec{\alpha}) + \vec{I}_F = \left(\widehat{M} + \widehat{\bar{M}}(\vec{x})\right) \cdot \vec{\alpha} + \vec{I}_F.$$

Express $\vec{I}_F$ by the matrix form using additional parameters $\vec{\beta}$

$$\vec{I}_F \subseteq \left(\widehat{I}_F + \widehat{\bar{I}}_F(\vec{x})\right) \cdot \vec{\beta}.$$

where $\widehat{\bar{I}}_F(\vec{x}) = 0$ and $\left(\widehat{I}_F\right)_{ii} = |I_{Fi}|$.

$$\vec{P}_\alpha(\vec{x}, \vec{\alpha}) + \vec{I}_F \subseteq \left(\widehat{M} + \widehat{\bar{M}}(\vec{x})\right) \cdot \vec{\alpha} + \left(\widehat{I}_F + \widehat{\bar{I}}_F(\vec{x})\right) \cdot \vec{\beta}.$$

View this as a collection of $2 \cdot v$ column vectors associated to $2 \cdot v$ parameters $\vec{\alpha}$ and $\vec{\beta}$. Recall a matrix, or a collection of v column vectors, represent a parallelepiped. The problem is now to find a **set sum of two parallelepipeds**.

Psum Algorithm for choosing column vectors

Task: Choose v vectors out of n vectors $\vec{s}_i$, $i = 1, ..., n$, $n \geq v$.

1. Choose the longest vector $\vec{s}_k$, and assign it as $\vec{t}_1$. Normalize it as $\vec{e}_1 = \vec{t}_1 / \left|\vec{t}_1\right|$.
2. Out of the remaining vectors $\vec{s}_i$, choose the j-th vector $\vec{t}_j = \vec{s}_k$ such that
$$\frac{\left|\vec{s}_k\right|^2 - \sum_{m=1}^{j-1} \left|\vec{s}_k \cdot \vec{e}_m\right|^2}{\left|\vec{s}_k\right|^{2p}}$$
is largest. Compute $\vec{e}_j$, the orthonormalized vector of $\vec{t}_j$ to $\vec{e}_1, ..., \vec{e}_{j-1}$. (Gram-Schmidt)
3. Repeat the process 2 until $j = v$.

Experimentally, $p = 0.5$ is found to be efficient and robust for obtaining a set sum of two parallelepipeds

Psum Algorithm for two parallelepipeds

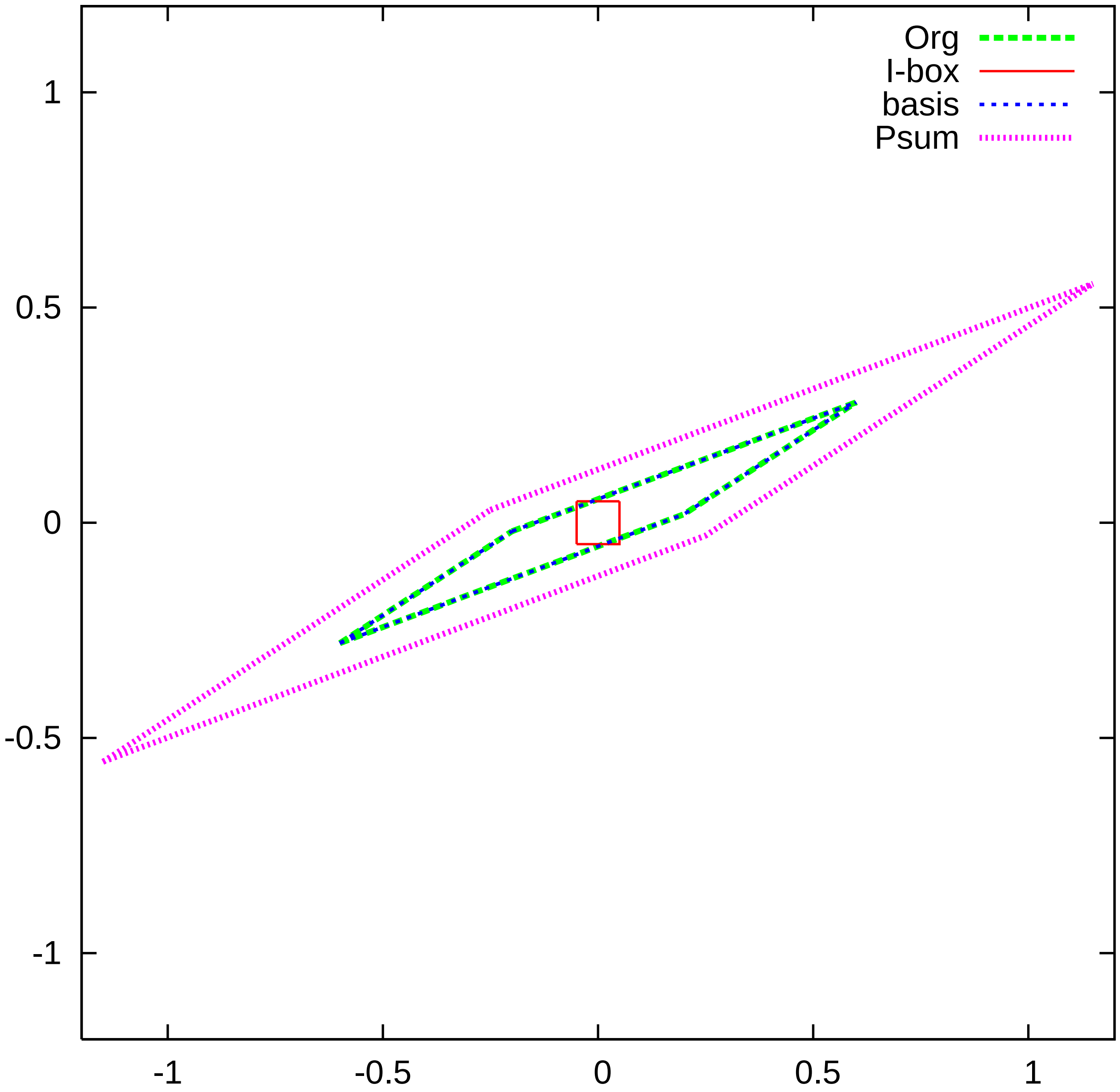
Task: Obtain a set sum of two parallelepipeds $\widehat{M_1}$ and $\widehat{M_2}$.

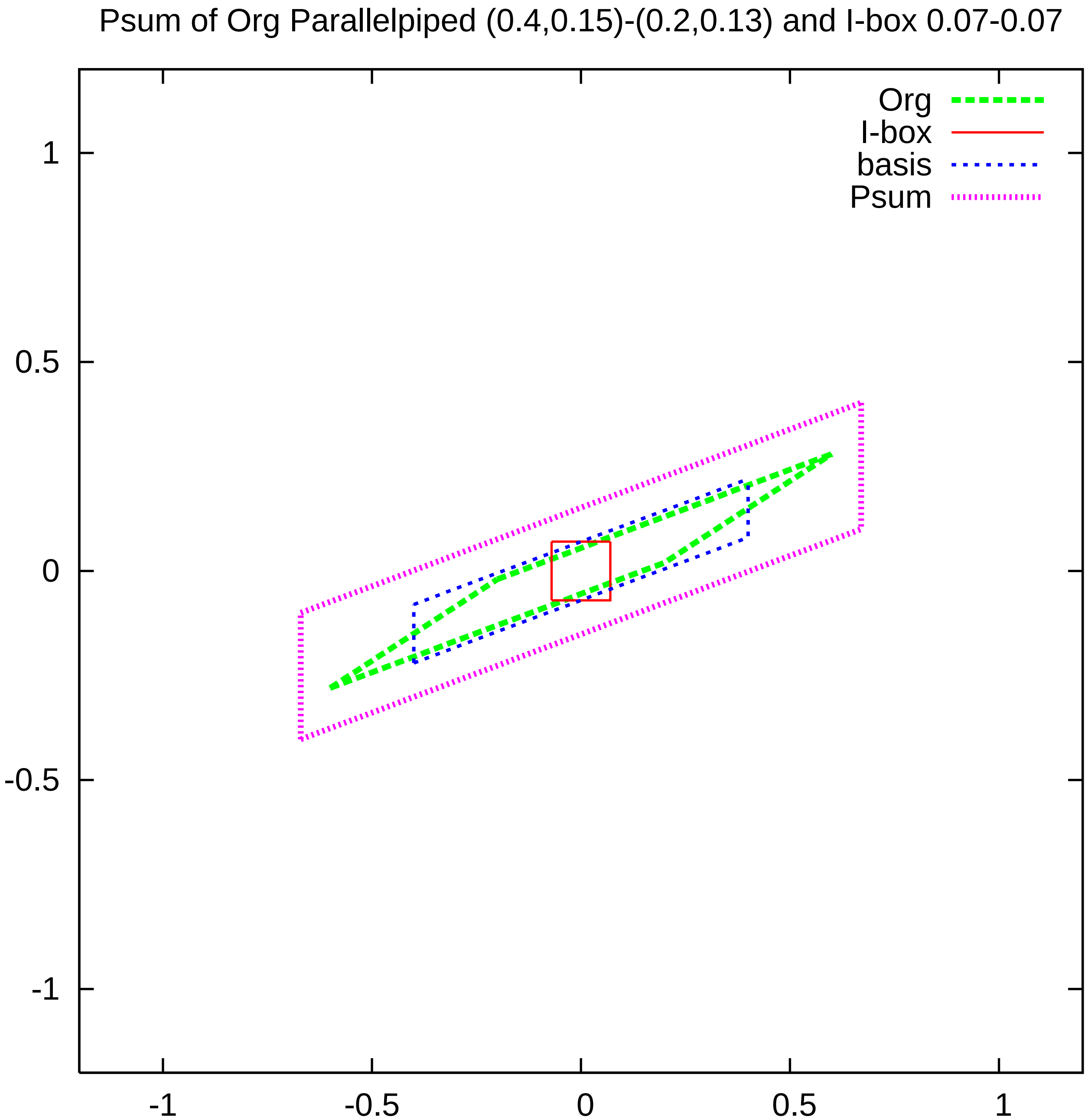
1. Prepare the basis $\widehat{M_b}$ using the Psum algorithm for choosing v column vectors out of $2 \cdot v$ column vectors from $\widehat{M_1}$ and $\widehat{M_2}$.
2. Compute conditioned parallelepipeds $\widehat{M_b}^{-1} \cdot \widehat{M_1}$ and $\widehat{M_b}^{-1} \cdot \widehat{M_2}$.
3. Confine the conditioned parallelepipeds by bounding them.

$$\vec{B}_1 = \text{bound}\left(\widehat{M_b}^{-1} \cdot \widehat{M_1}\right) \text{ and } \vec{B}_2 = \text{bound}\left(\widehat{M_b}^{-1} \cdot \widehat{M_2}\right).$$

4. Compute the interval sum $\vec{B} = \vec{B}_1 + \vec{B}_2$. $\vec{B}$ confines the conditioned set sum of the conditioned parallelepipeds.
5. From $\vec{B}$, set up a parallelepiped as a box $\widehat{B} = \begin{pmatrix} |B_1| & & 0 \\ & \ddots & \\ 0 & & |B_v| \end{pmatrix}$.
6. Compute $\widehat{M_b} \cdot \widehat{B}$, which is a set sum of $\widehat{M_1}$ and $\widehat{M_2}$ under $\widehat{M_b}$.

Psum of Org Parallelpiped (0.4,0.15)-(0.2,0.13) and I-box 0.05-0.05





Psum of Org Parallelpiped (0.4,0.15)-(0.2,0.13) and I-box 0.07-0.07
Org
I-box
basis
Psum
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

Error Absorption

We now chose a favoured collection of v column vectors $\widehat{L}+\widehat{\bar{L}}(\vec{x})$ using the Psum algorithm. Collect the left over v column vectors to $\widehat{E}+\widehat{\bar{E}}(\vec{x})$. Associate them to $2 \cdot v$ parameters $\vec{\alpha}'$ and $\vec{\beta}'$.

$$\vec{P}_{\alpha}(\vec{x},\vec{\alpha}) + \vec{I}_F \subseteq \left(\widehat{L} + \widehat{\bar{L}}(\vec{x})\right) \cdot \vec{\alpha}' + \left(\widehat{E} + \widehat{\bar{E}}(\vec{x})\right) \cdot \vec{\beta}'.$$

Since $\vec{\alpha}$ and $\vec{\beta}$ do not appear anymore, we can rename $\vec{\alpha}'$ and $\vec{\beta}'$ as $\vec{\alpha}$ and $\vec{\beta}$ for the simplicity.

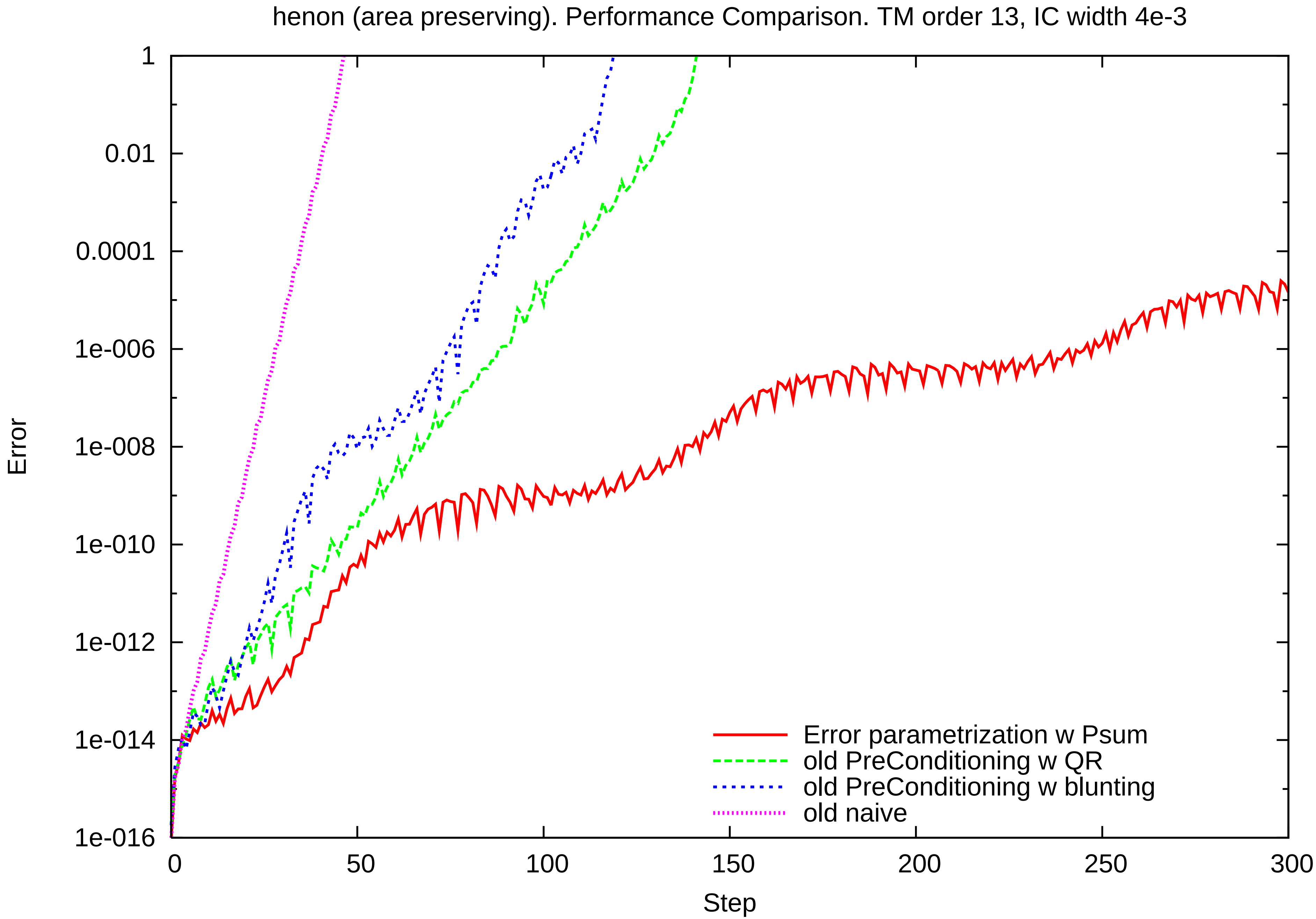
$$\begin{aligned}
\vec{P}_{\alpha}(\vec{x},\vec{\alpha}) + \vec{I}_F &\subseteq \left(\widehat{L} + \widehat{\bar{L}}(\vec{x})\right) \cdot \vec{\alpha} + \left(\widehat{E} + \widehat{\bar{E}}(\vec{x})\right) \cdot \vec{\beta} \\
&= \widehat{L} \circ \left[\widehat{L}^{-1} \circ \left(\widehat{L} + \widehat{\bar{L}}(\vec{x})\right) \cdot \vec{\alpha} + \widehat{L}^{-1} \circ \left(\widehat{E} + \widehat{\bar{E}}(\vec{x})\right) \cdot \vec{\beta}\right] \\
&\subseteq \widehat{L} \circ \left[\left(\widehat{I} + \widehat{L}^{-1} \circ \widehat{\bar{L}}(\vec{x})\right) \cdot \vec{\alpha} + \widehat{B} \cdot \vec{\beta}\right]
\end{aligned}$$

where $\widehat{B}$ is a diagonal matrix with the i-th element is $|B_i|$ and $\vec{B}$ =bound$\left(\widehat{L}^{-1} \circ \left(\widehat{E} + \widehat{\bar{E}}(\vec{x})\right) \cdot \vec{\beta}\right)$.

If the **diagonal terms** of $\left(\widehat{I} + \widehat{L}^{-1} \circ \widehat{\overline{\overline{L}}}(\vec{x})\right)$ are **positive**,

$$\begin{aligned}
\vec{P}_\alpha(\vec{x}, \vec{\alpha}) + \vec{I}_F &\subseteq \widehat{L} \circ \left[\left(\widehat{I} + \widehat{L}^{-1} \circ \widehat{\overline{\overline{L}}}(\vec{x})\right) \cdot \vec{\alpha} + \widehat{B} \cdot \vec{\alpha}\right] \\
&= \widehat{L} \circ \left(\widehat{I} + \widehat{L}^{-1} \circ \widehat{\overline{\overline{L}}}(\vec{x})\right) \cdot \vec{\alpha} + \widehat{L} \circ \widehat{B} \cdot \vec{\alpha} \\
&= \left(\widehat{L} + \widehat{\overline{\overline{L}}}(\vec{x}) + \widehat{L} \circ \widehat{B}\right) \cdot \vec{\alpha}.
\end{aligned}$$

Note: A modification to use $\widehat{A}$ instead of $\widehat{L}$, when $\widehat{A} \approx \widehat{L}$, is done easily. This involves bounding of $\widehat{A}^{-1} \circ \left(\widehat{L} - \widehat{A}\right) \cdot \vec{\alpha}$ and the diagonal terms to be checked positive are those of $\left(\widehat{I} + \widehat{A}^{-1} \circ \widehat{\overline{\overline{L}}}(\vec{x})\right)$.


henon (area preserving). Performance Comparison. TM order 13, IC width 4e-3
Error
Step
1
0.01
0.0001
1e-006
1e-008
1e-010
1e-012
1e-014
1e-016
0
50
100
150
200
250
300
Error parametrization w Psum
old PreConditioning w QR
old PreConditioning w blunting
old naive

Cost of Additional Parameters

For a v dimensional system, we need v parameters $\vec{\alpha}$ to absorb Taylor model remainder error bound intervals. The dependence on $\vec{\alpha}$ is limited to **linear**. So, we use weighted DA. Choose an appropriate weight order w for $\vec{\alpha}$.

- The dependence on $\vec{\alpha}$ has to be kept linear. Namely $2 \cdot w > n$, where n is the computational order of Taylor models. Choose

$$w = \text{Int}\left(\frac{n}{2}\right) + 1.$$

Maximum size necessary for DA and TM for $v = 2$.

n	v	DA	TM	v	DA	TM		w	v_w	DA	TM
13	2	105	140	$2+2$	2380	2419	$\Rightarrow$	7	$2+2_w$	161	200
21	2	253	304	$2+2$	12650	12705		11	$2+2_w$	385	440
33	2	595	670	$2+2$	66045	66124		17	$2+2_w$	901	980

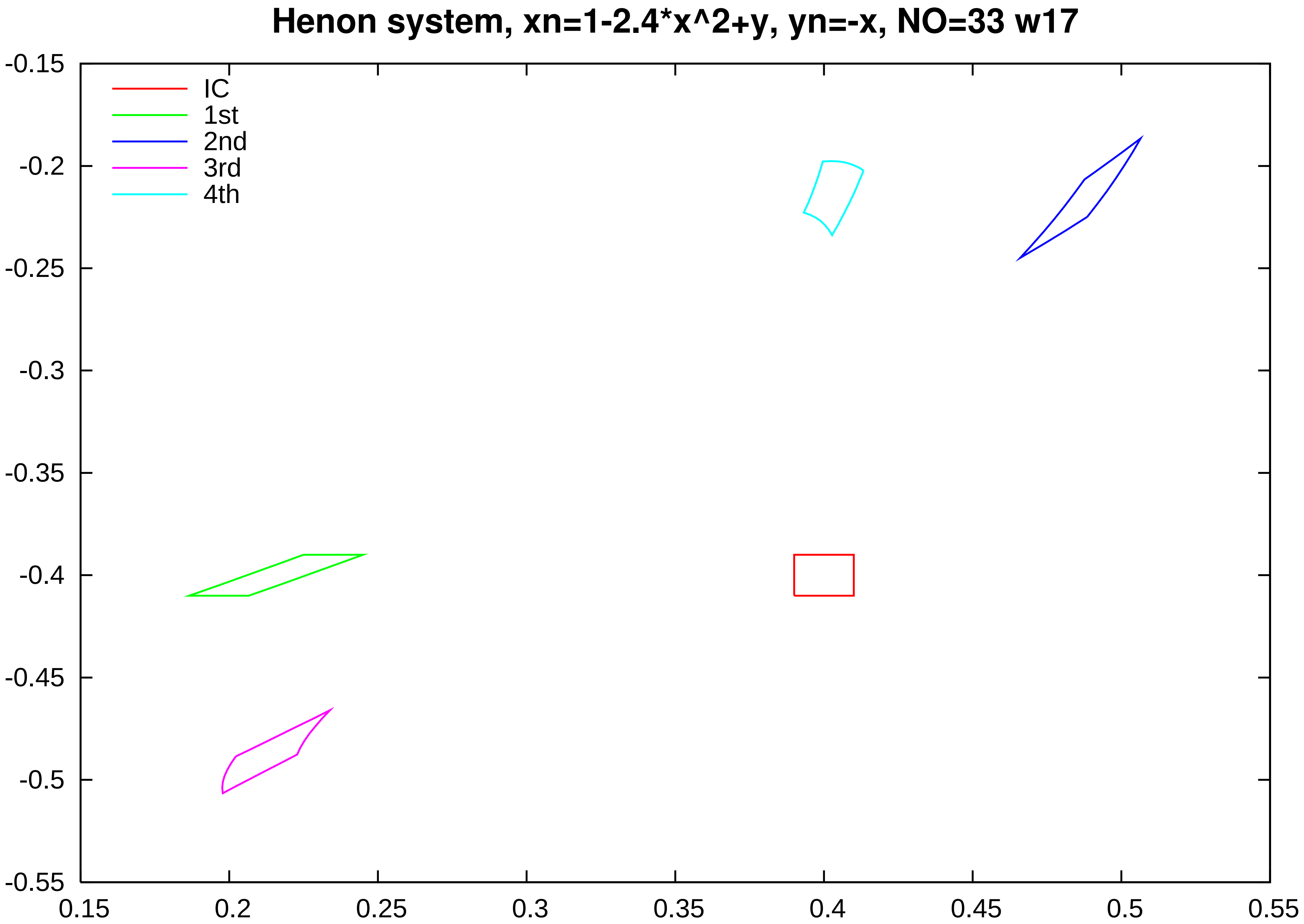
Dynamic Domain Decomposition

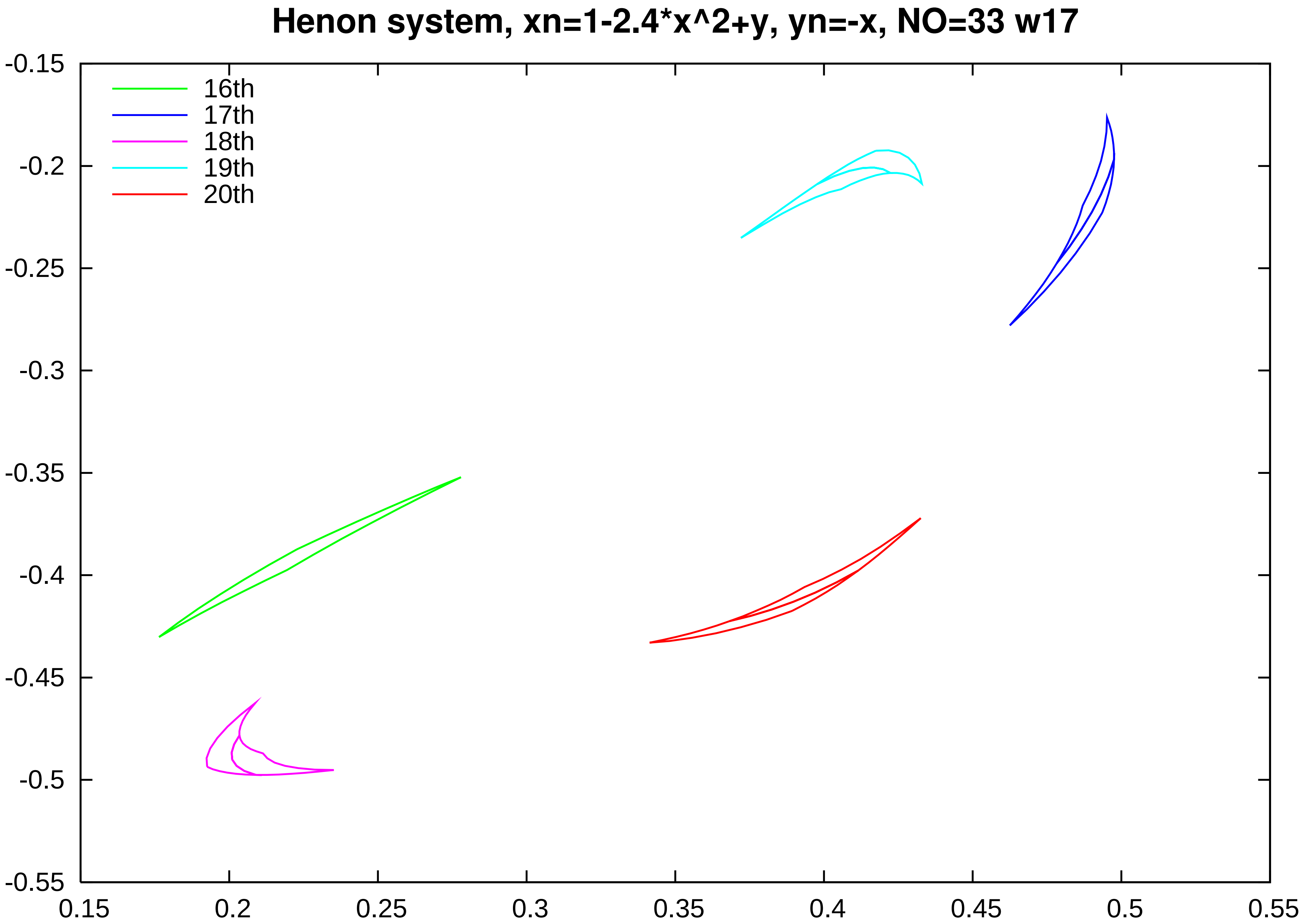
For extended domains, this is **natural equivalent** to step size control. Similarity to what's done in global optimization.

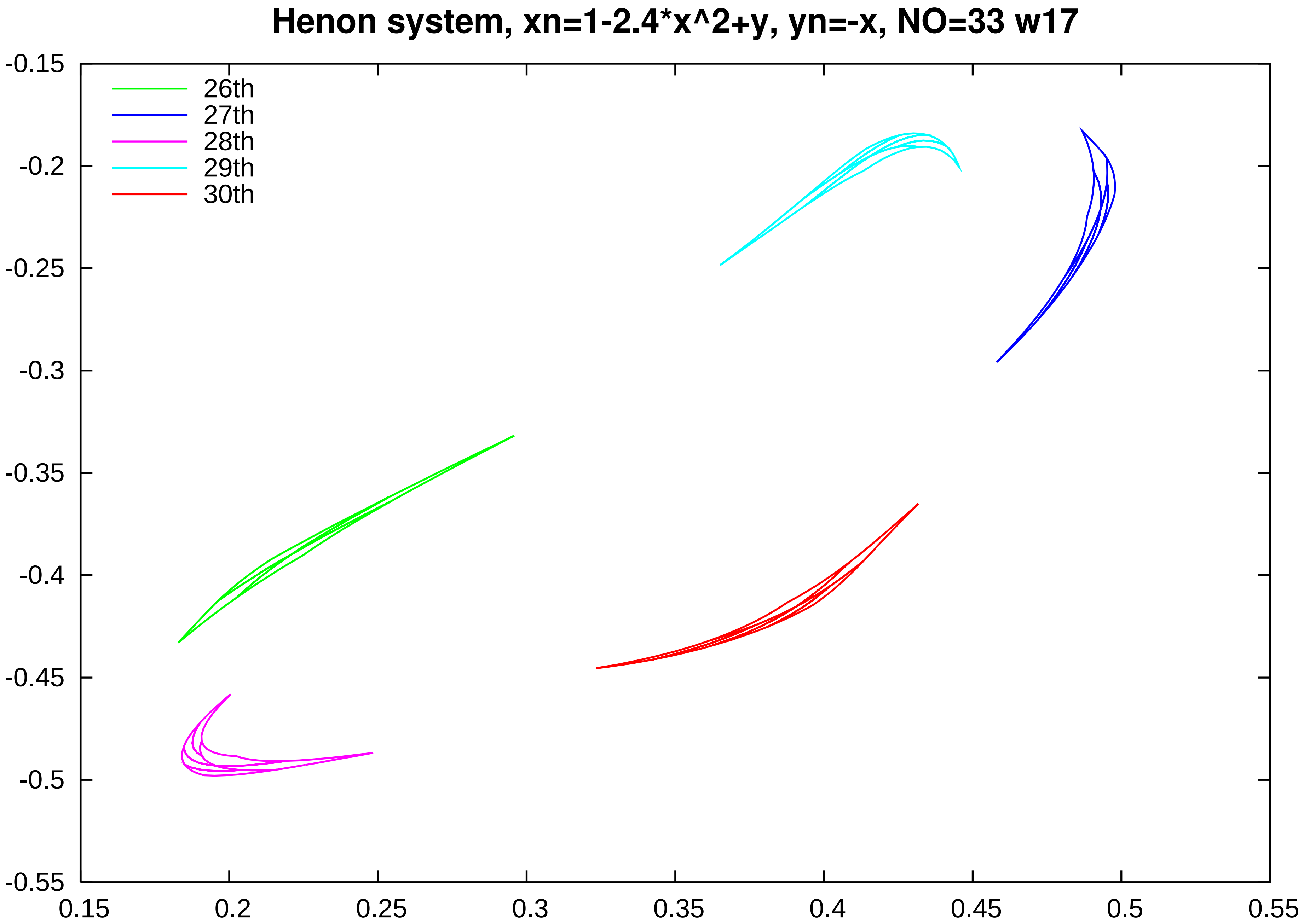
1. Evaluate ODE for $\Delta t = 0$ for current flow.
2. If resulting remainder bound R greater than ε, split the domain along variable leading to longest axis.
3. Absorb R in the TM polynomial part using the error parametrization method. If it fails, split the domain along variable leading to largest x dependence of the error.
4. Put one half of the box on stack for future work.

Things to consider:

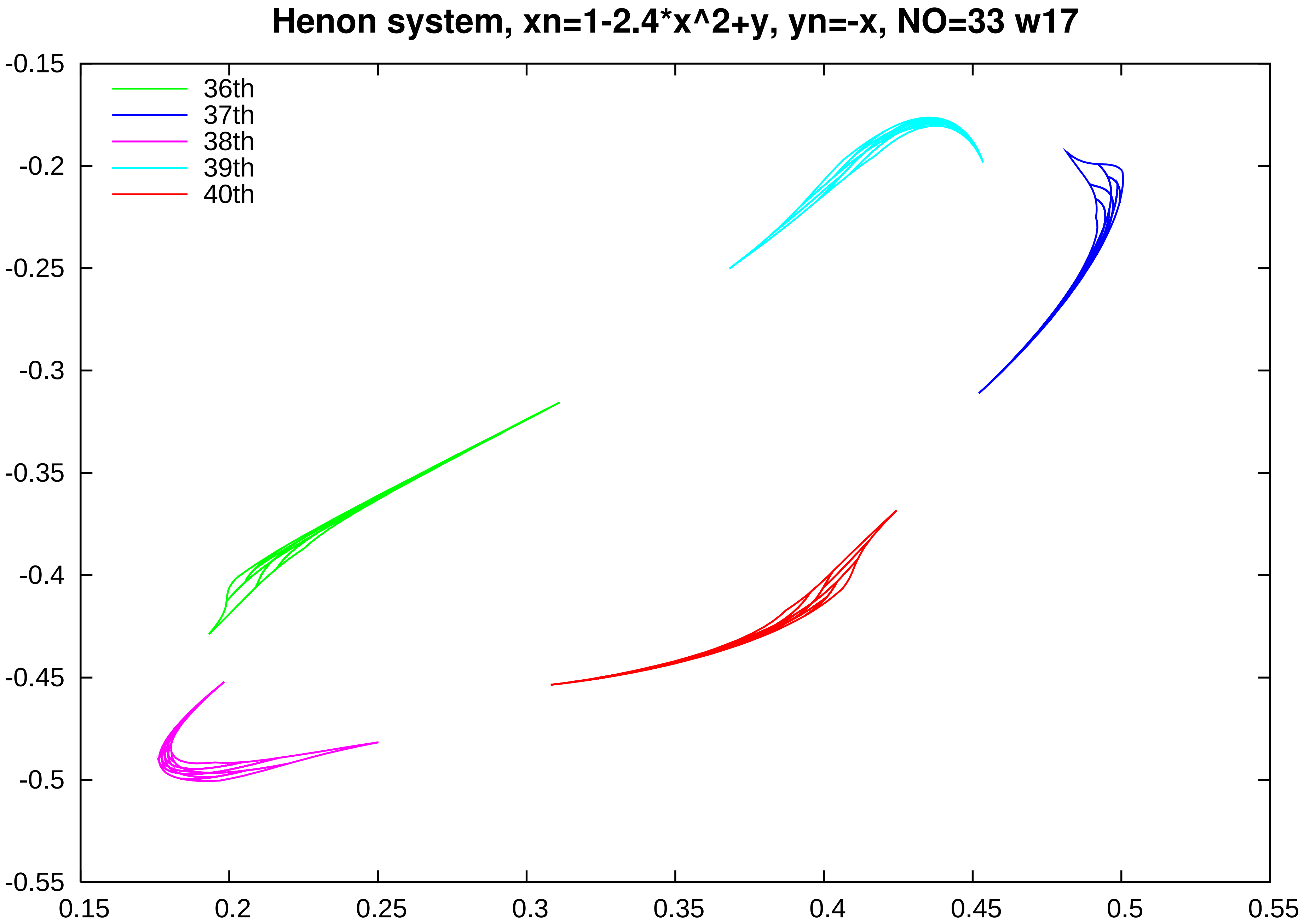
- Utilize "First-in-last-out" stack; minimizes stack length. Special adjustments for stack management in a parallel environment, including load balancing.
- Outlook: also dynamic order control for dependence on initial conditions

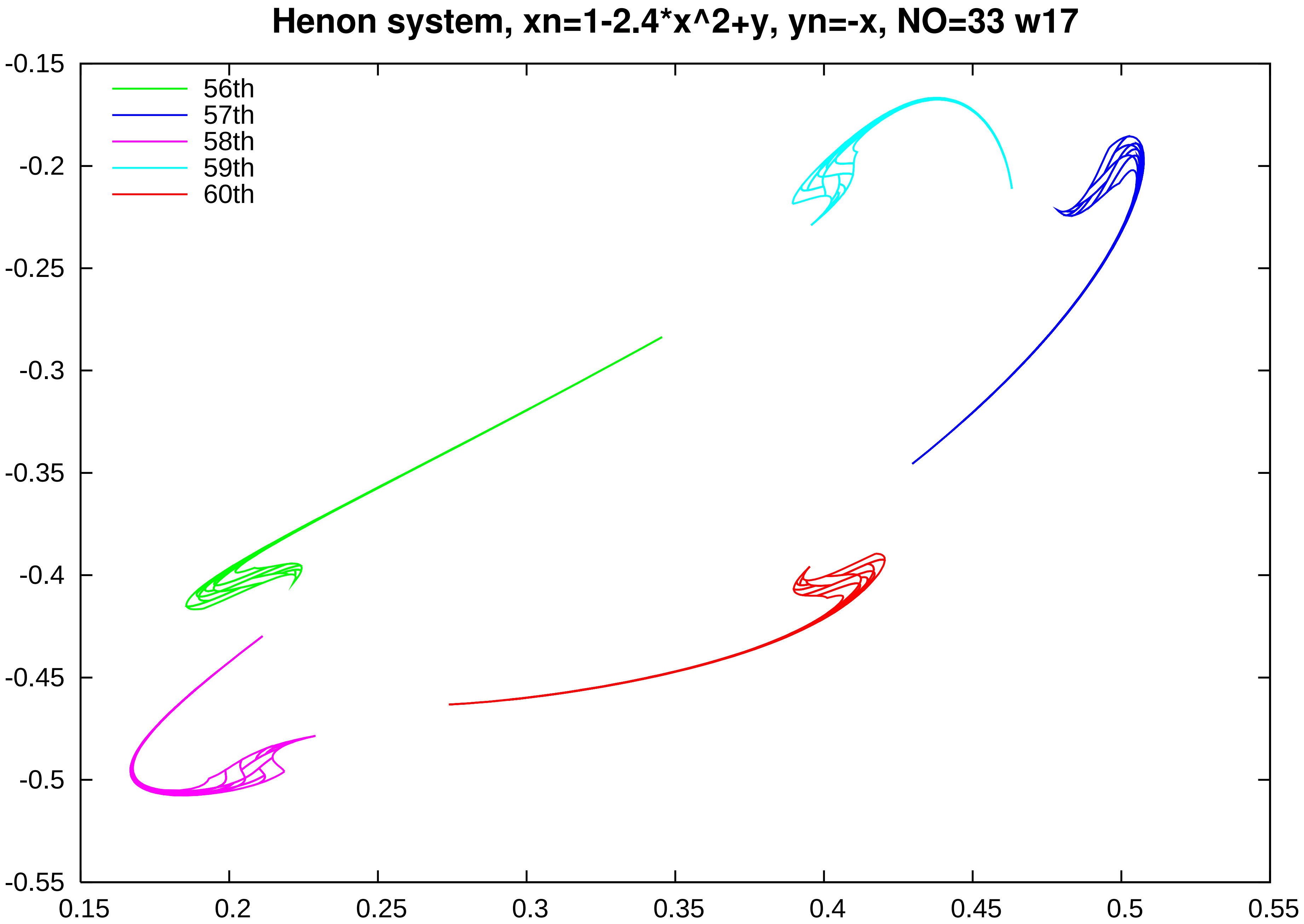

Henon system, xn=1-2.4*x^2+y, yn=-x, NO=33 w17
IC
1st
2nd
3rd
4th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55


Henon system, xn=1-2.4*x^2+y, yn=-x, NO=33 w17
16th
17th
18th
19th
20th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

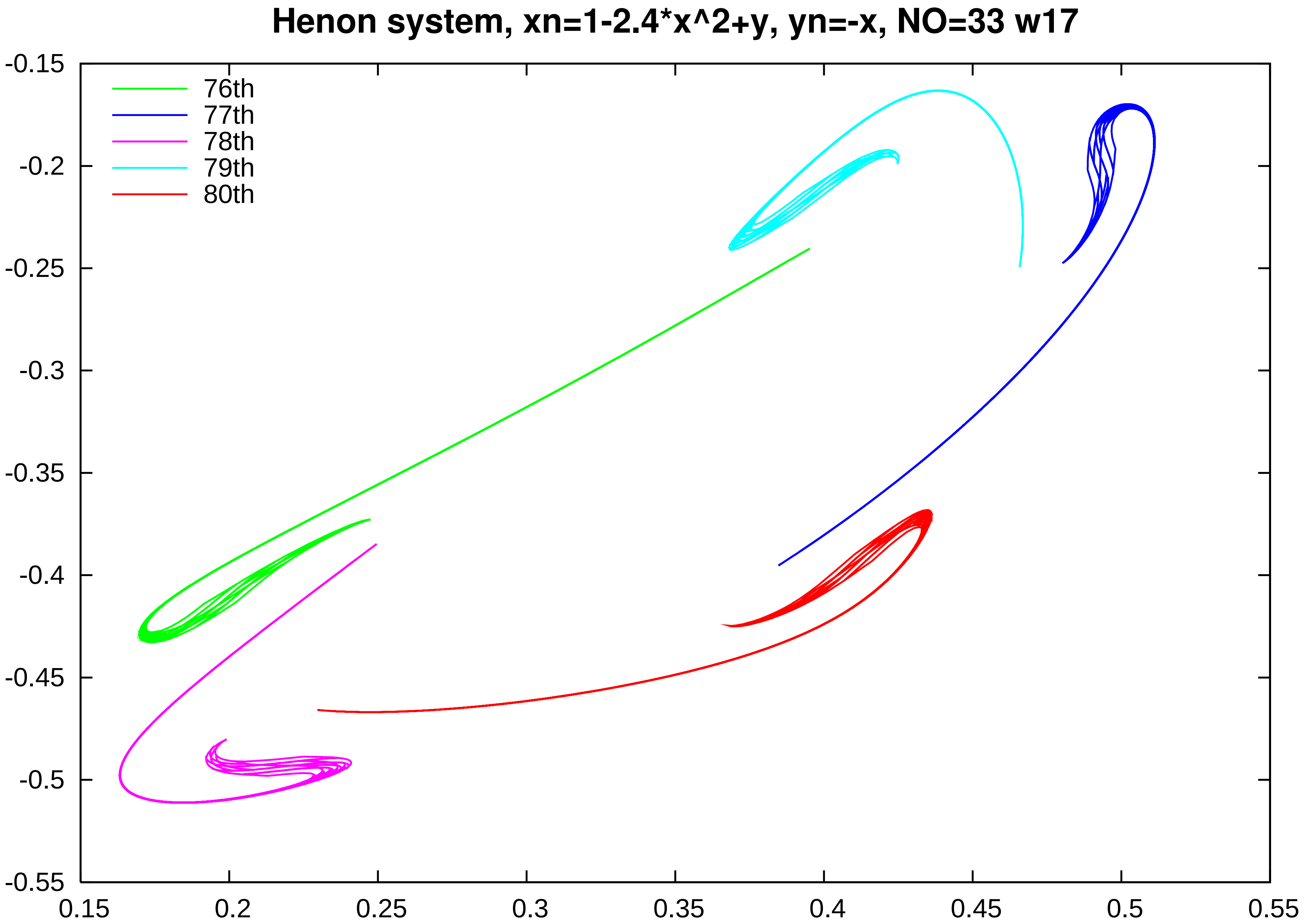


Henon system, xn=1-2.4*x^2+y, yn=-x, NO=33 w17
26th
27th
28th
29th
30th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

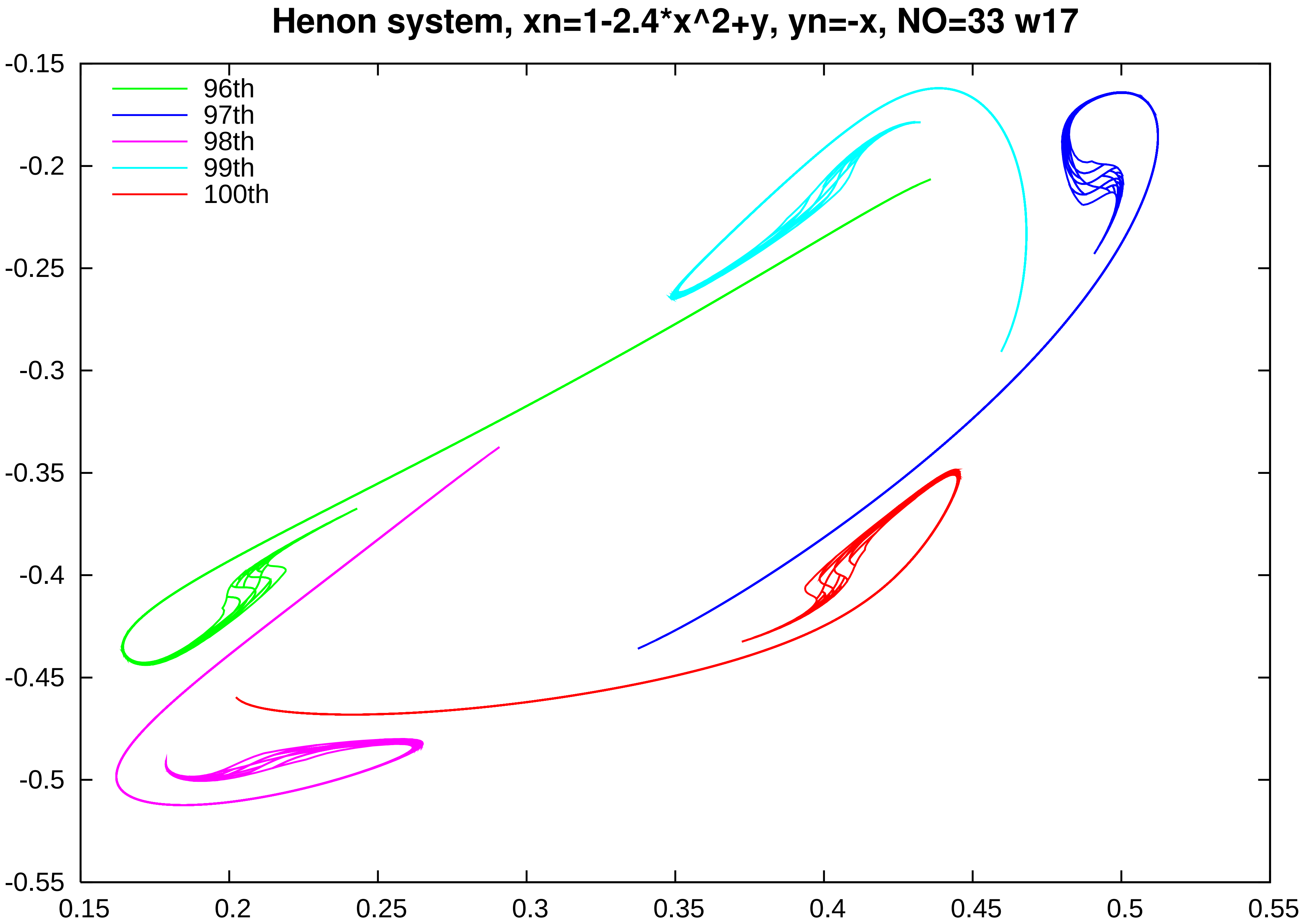

Henon system, xn=1-2.4*x^2+y, yn=-x, NO=33 w17
36th
37th
38th
39th
40th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55



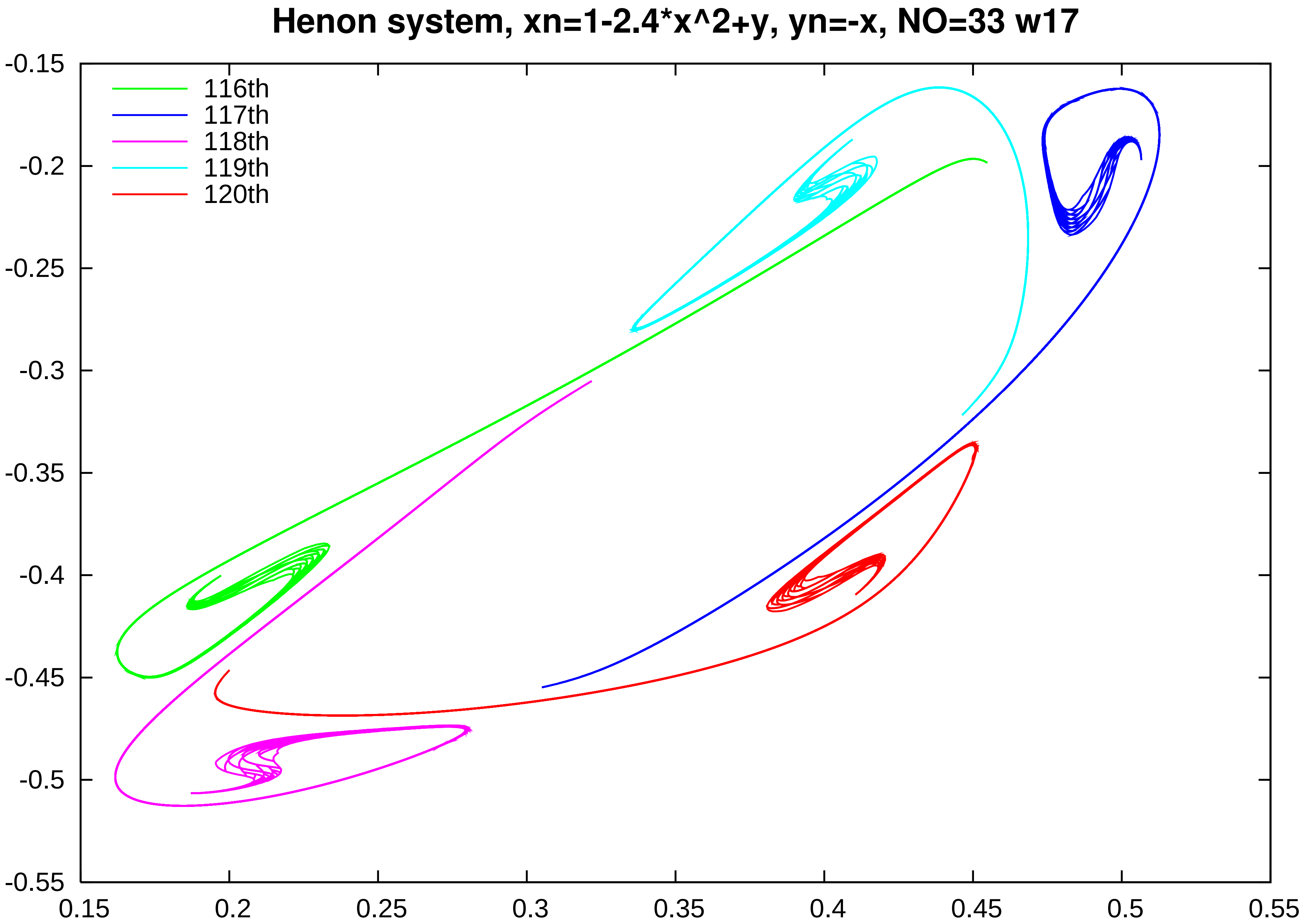
Henon system, xn=1-2.4*x^2+y, yn=-x, NO=33 w17
56th
57th
58th
59th
60th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

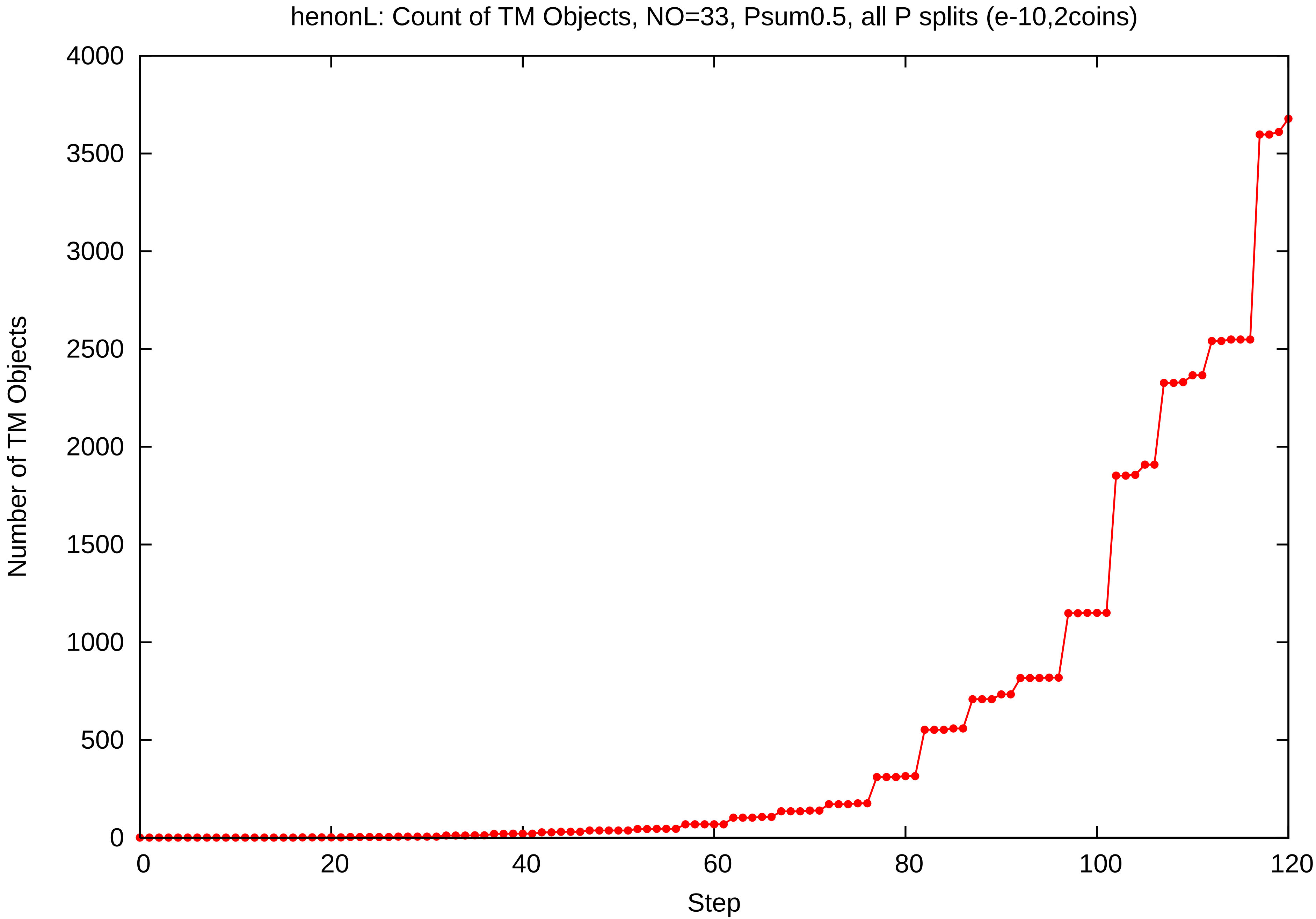


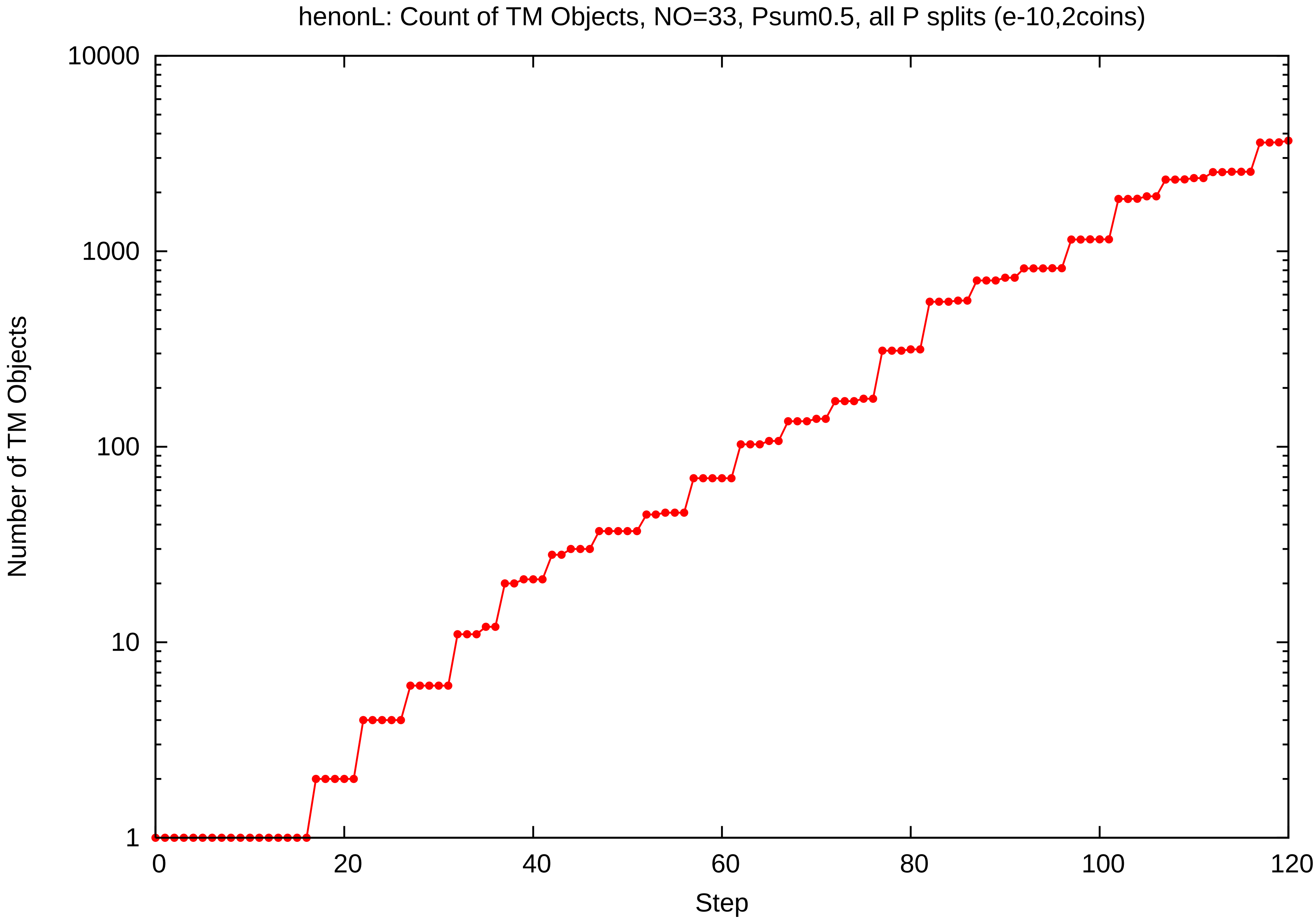
Henon system, xn=1-2.4*x^2+y, yn=-x, NO=33 w17
76th
77th
78th
79th
80th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55



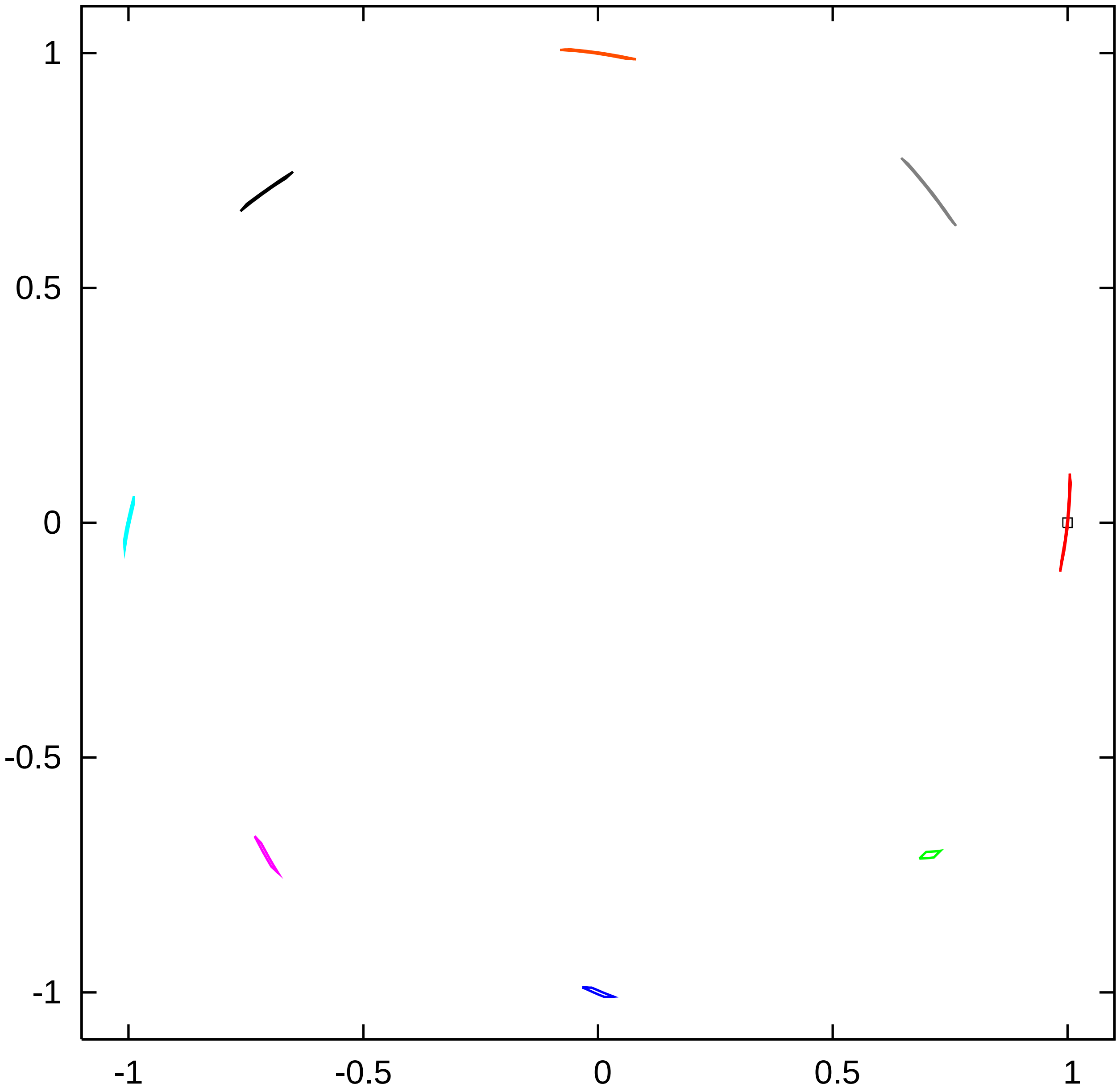
Henon system, xn=1-2.4*x^2+y, yn=-x, NO=33 w17
96th
97th
98th
99th
100th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55


Henon system, xn=1-2.4*x^2+y, yn=-x, NO=33 w17
116th
117th
118th
119th
120th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

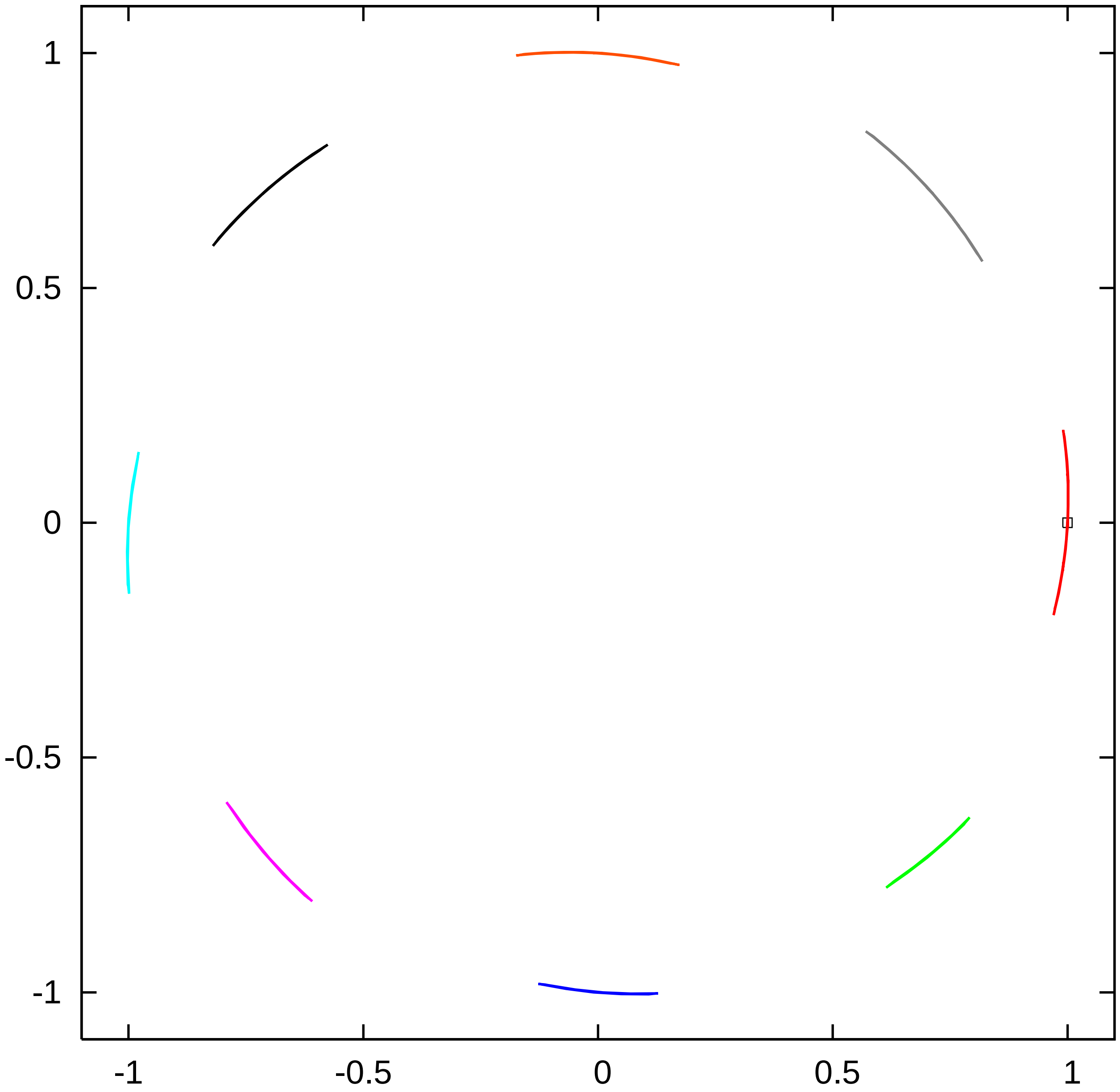

henonL: Count of TM Objects, NO=33, Psum0.5, all P splits (e-10,2coins)
Number of TM Objects
Step
0
500
1000
1500
2000
2500
3000
3500
4000
0
20
40
60
80
100
120

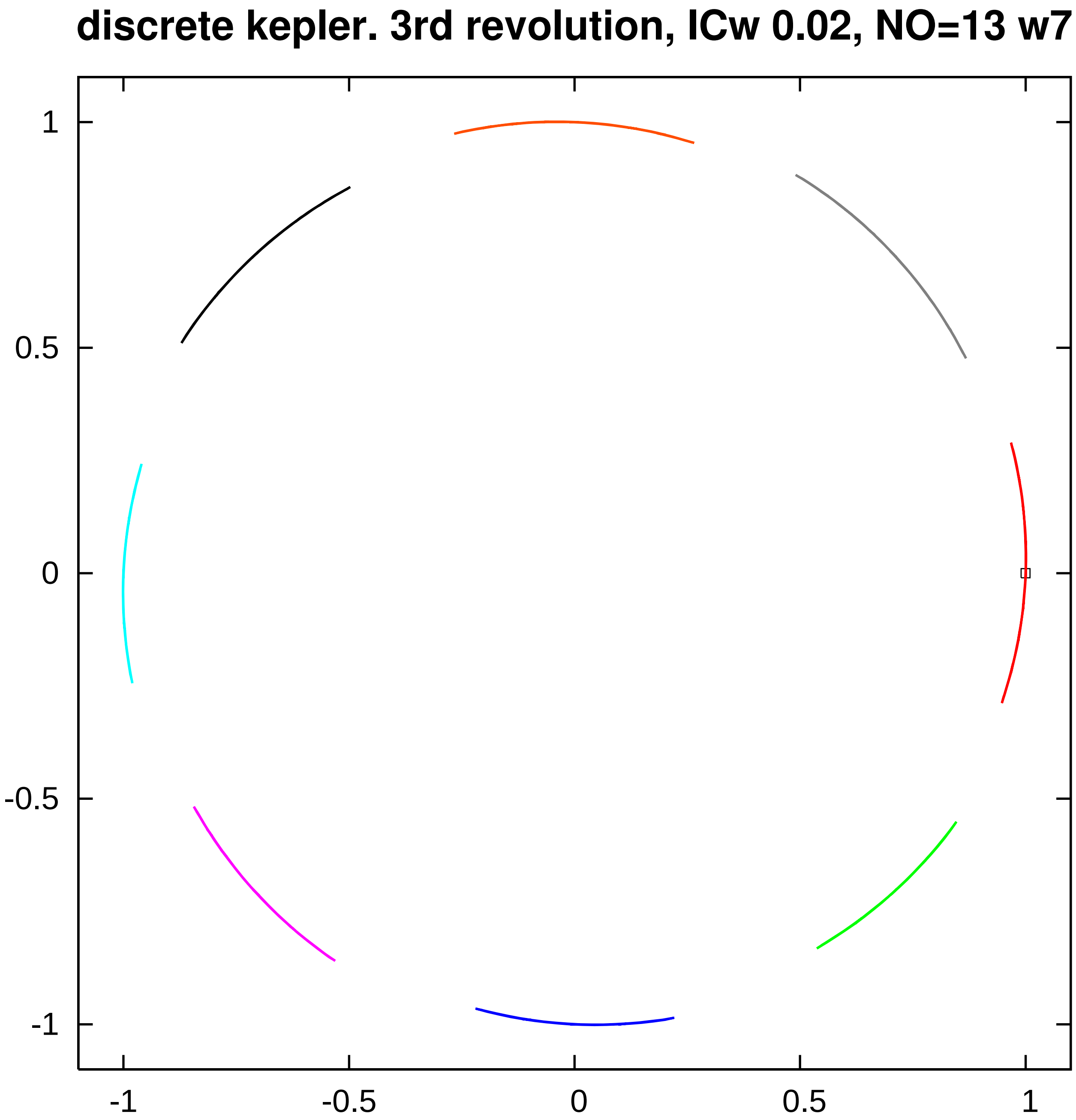

henonL: Count of TM Objects, NO=33, Psum0.5, all P splits (e-10,2coins)
Number of TM Objects
10000
1000
100
10
1
0
20
40
60
80
100
120
Step

discrete kepler. 1st revolution, ICw 0.02, NO=13 w7



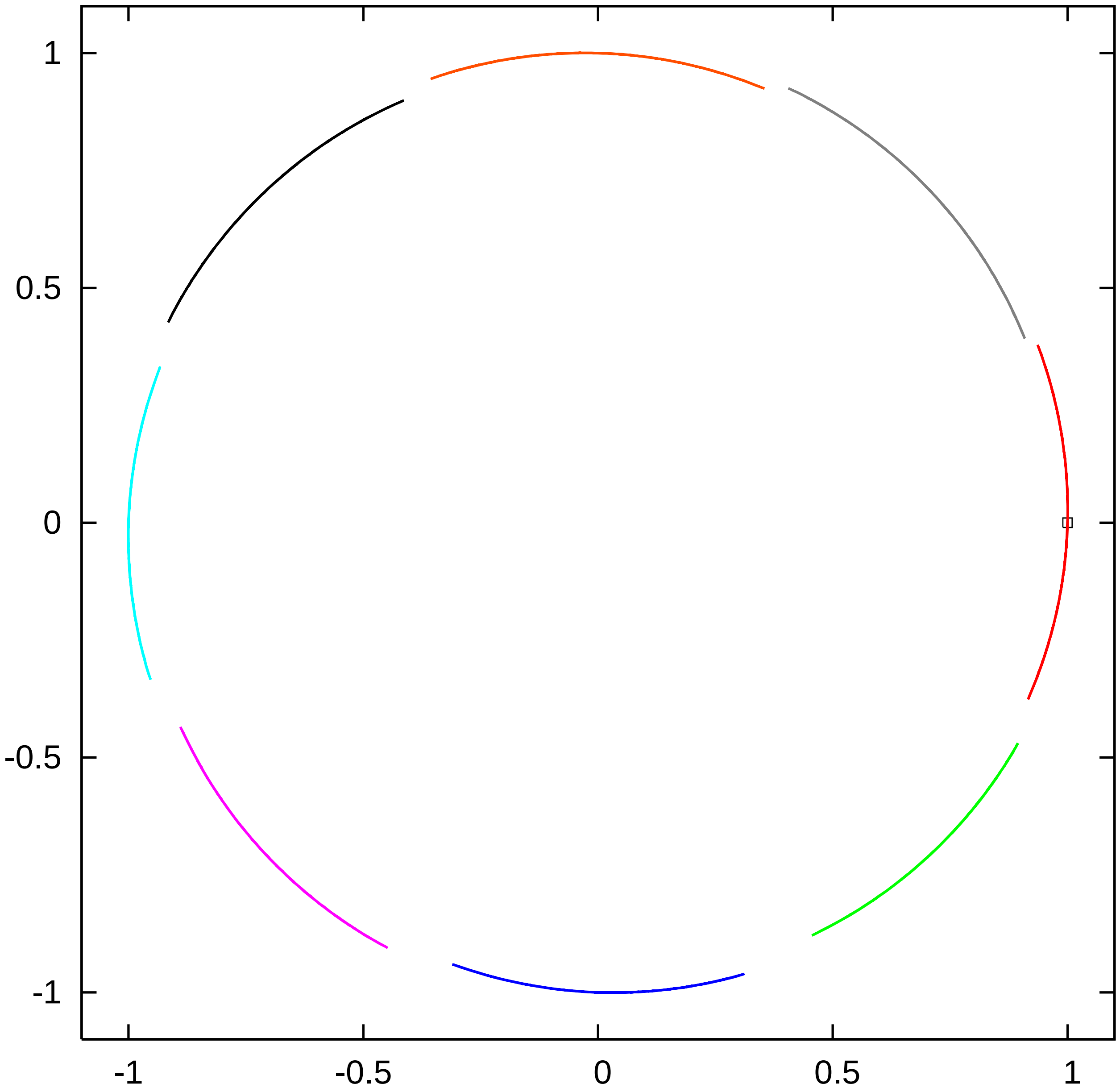
discrete kepler. 2nd revolution, ICw 0.02, NO=13 w7



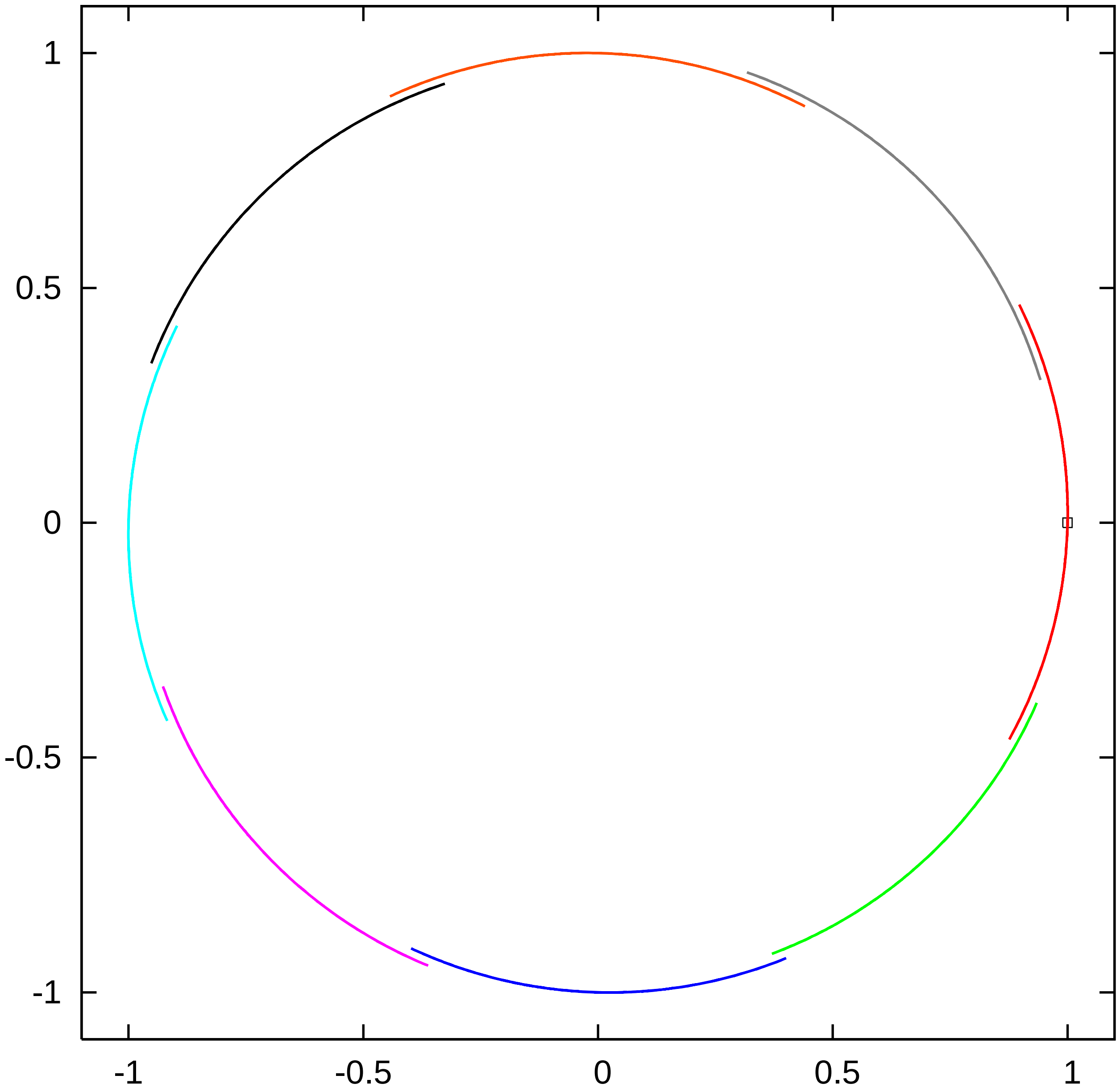


discrete kepler. 3rd revolution, ICw 0.02, NO=13 w7
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

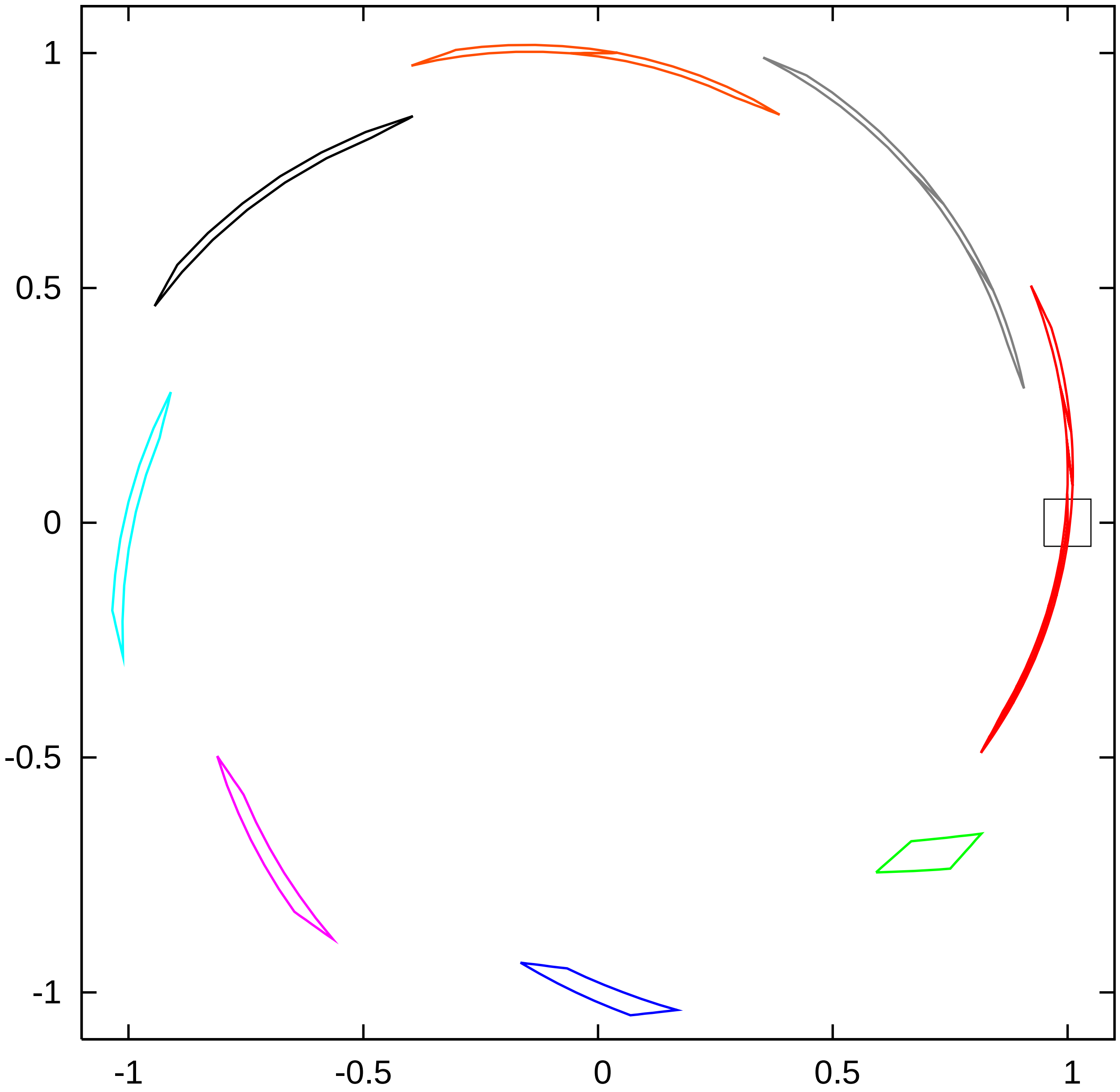
discrete kepler. 4th revolution, ICw 0.02, NO=13 w7



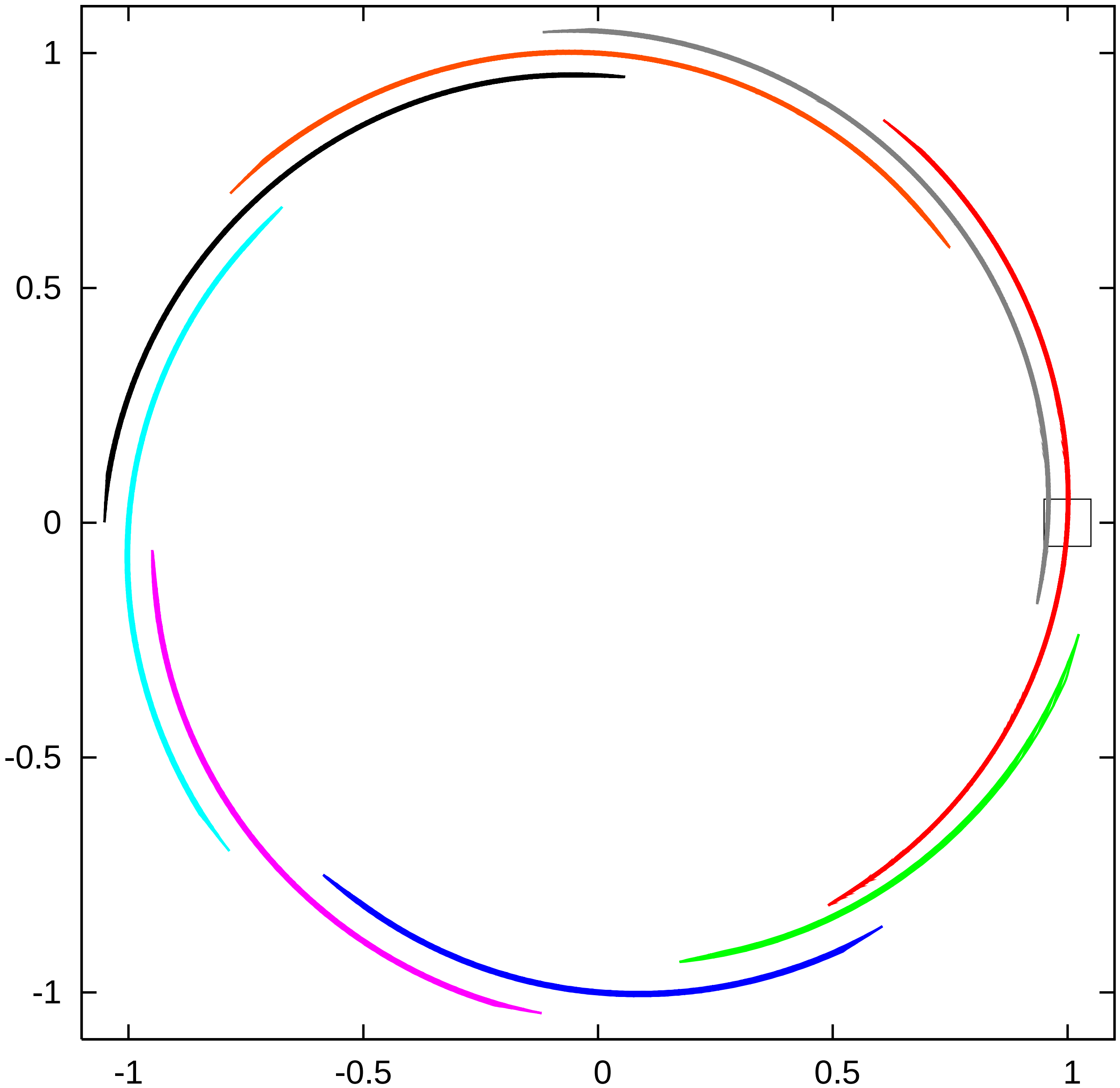
discrete kepler. 5th revolution, ICw 0.02, NO=13 w7



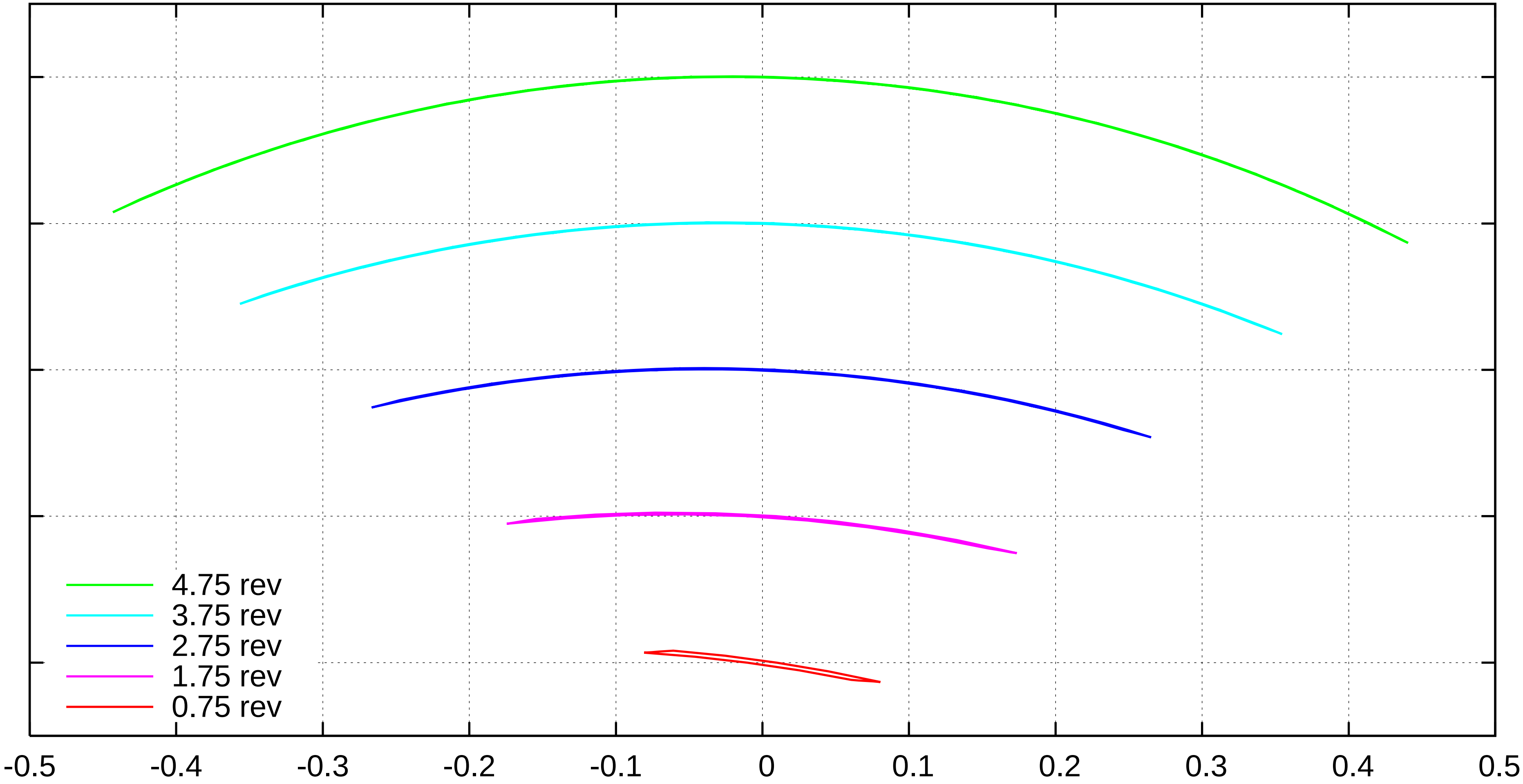
discrete kepler. 1st revolution, ICw 0.1, NO=13 w7



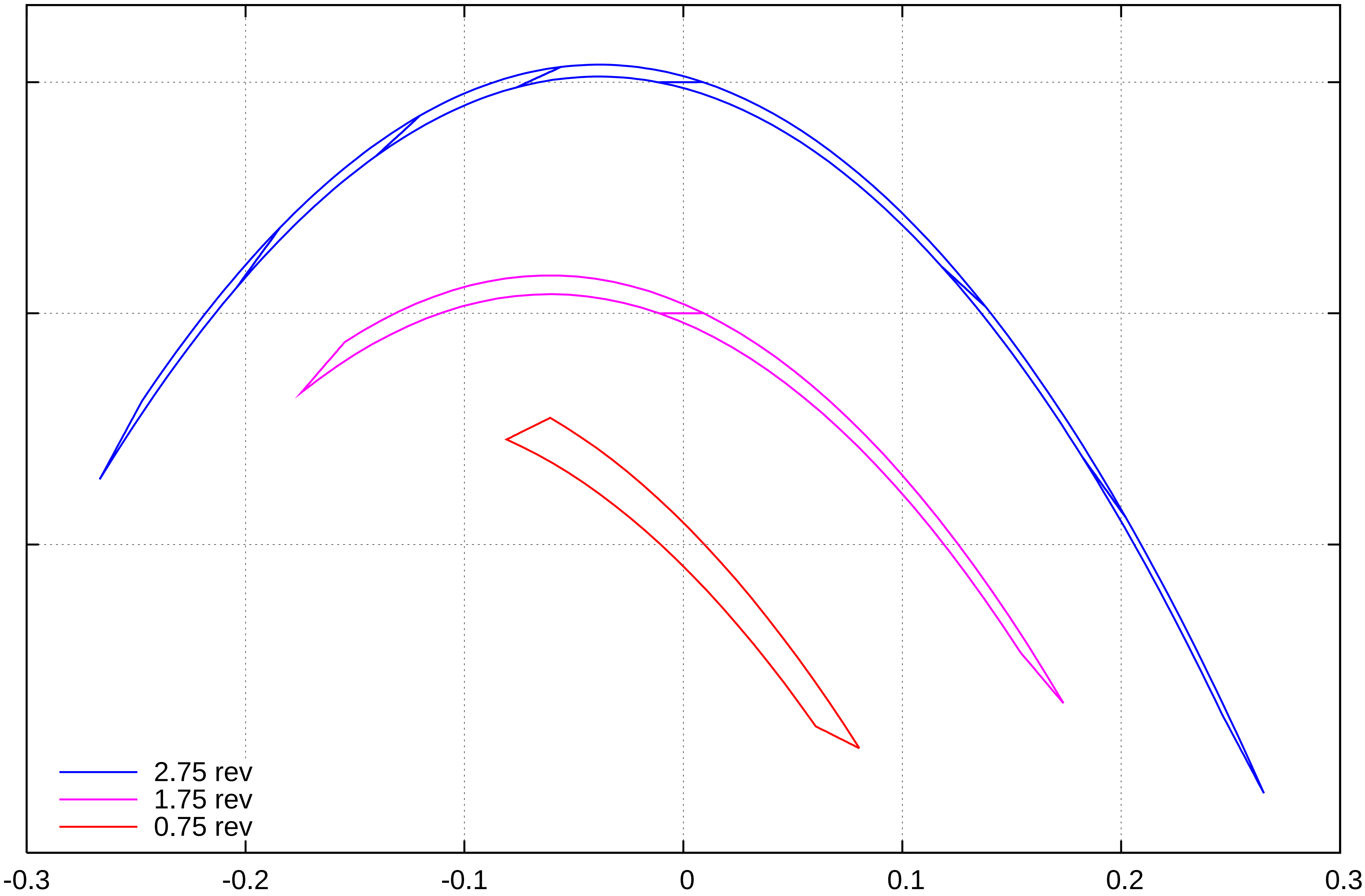
discrete kepler. 2nd revolution, ICw 0.1, NO=13 w7



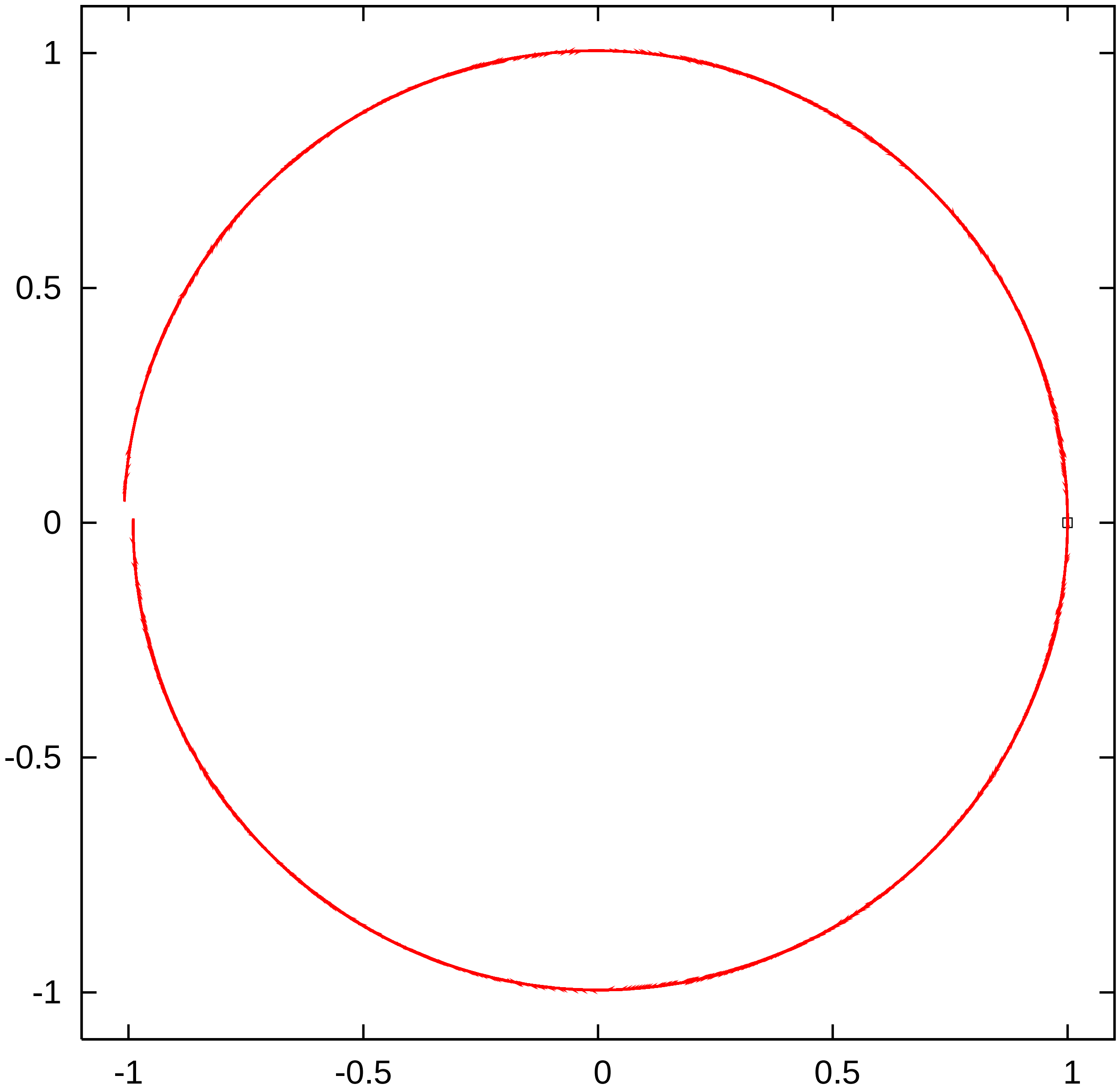
discrete kepler. NO=13 w7

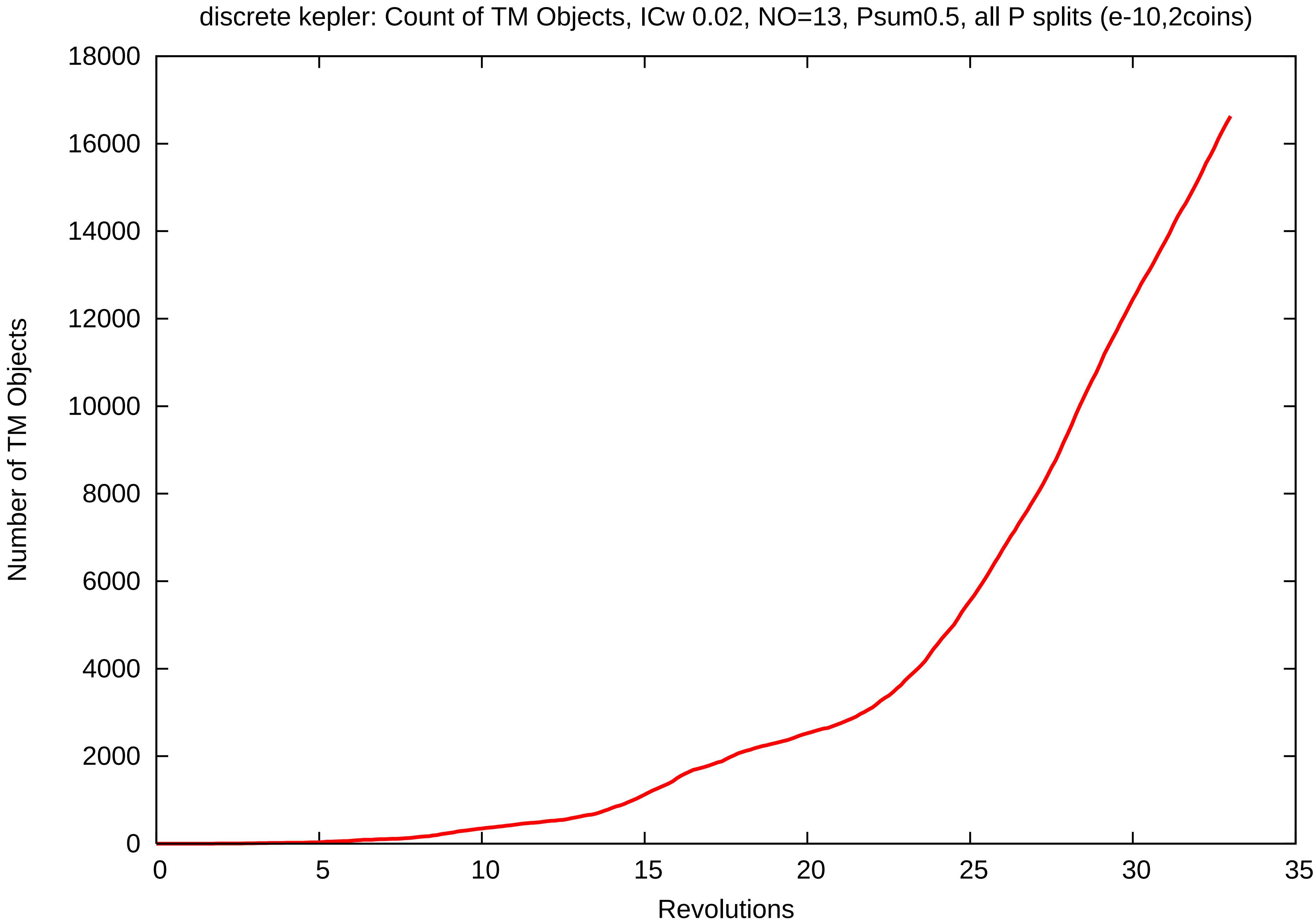


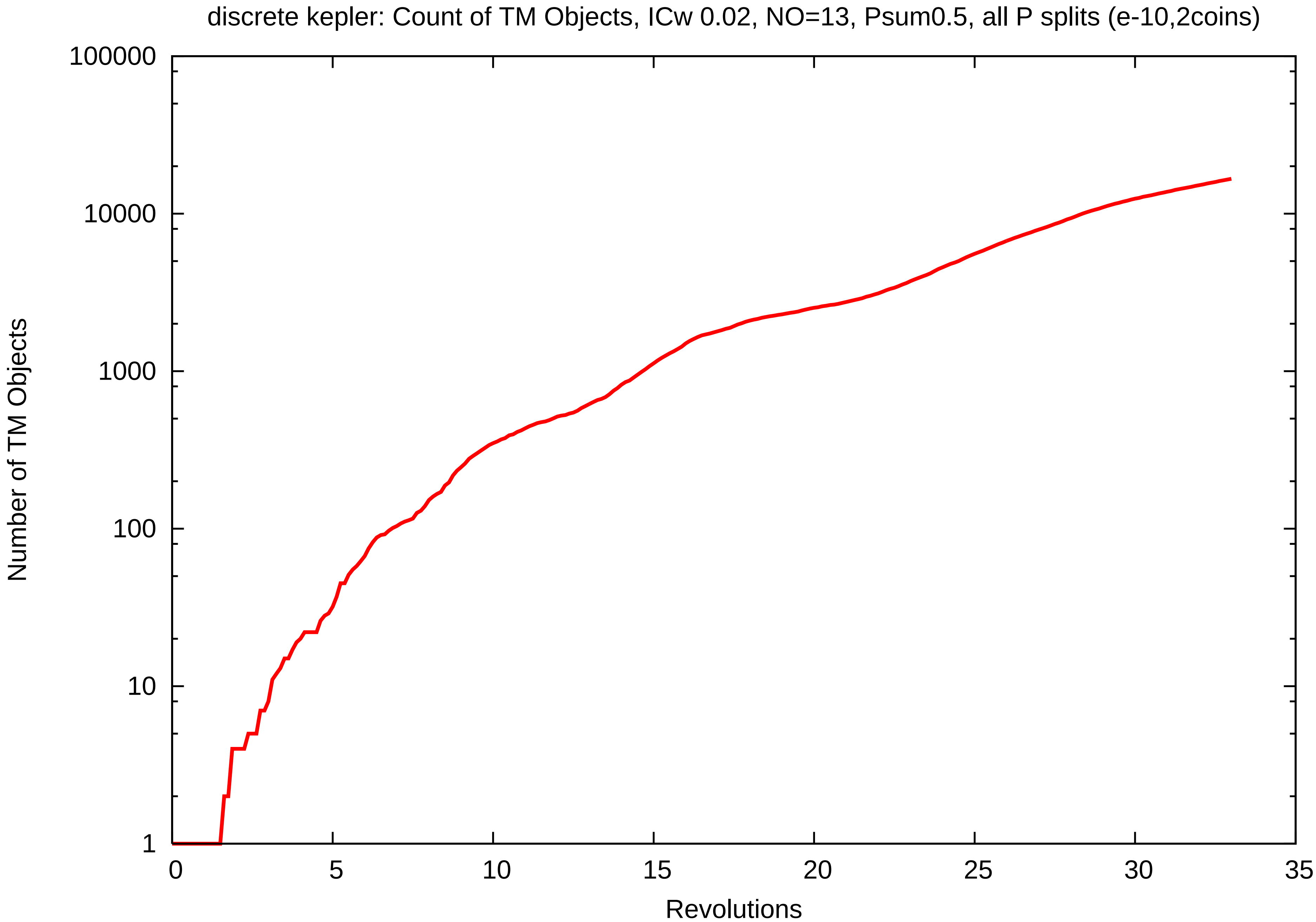
discrete kepler. NO=13 w7



discrete kepler. 33rd revolution, ICw 0.02, NO=13 w7




discrete kepler: Count of TM Objects, ICw 0.02, NO=13, Psum0.5, all P splits (e-10,2coins)
Number of TM Objects
Revolutions
0
2000
4000
6000
8000
10000
12000
14000
16000
18000
0
5
10
15
20
25
30
35

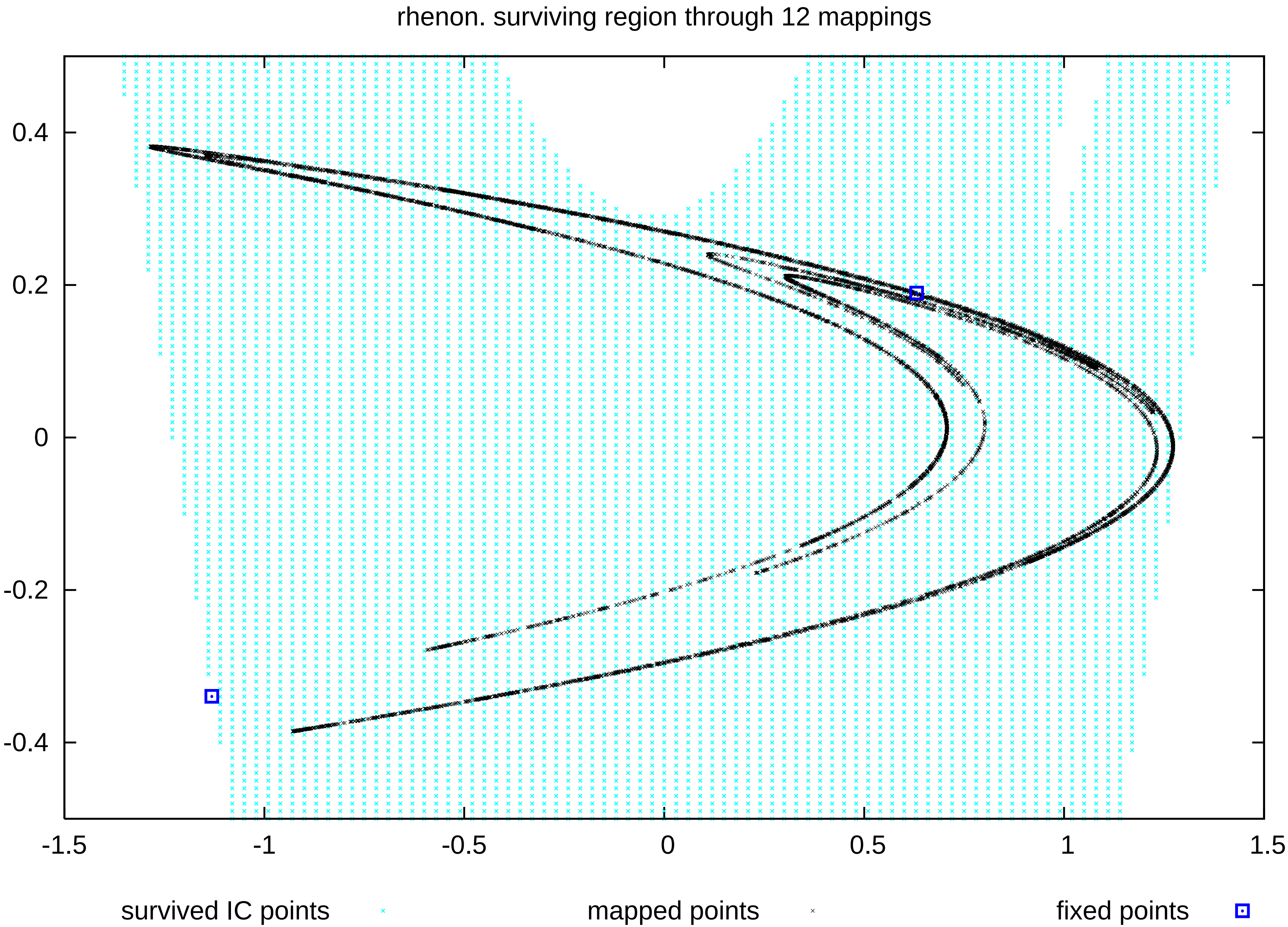

discrete kepler: Count of TM Objects, ICw 0.02, NO=13, Psum0.5, all P splits (e-10,2coins)
Number of TM Objects
100000
10000
1000
100
10
1
0
5
10
15
20
25
30
35
Revolutions

The Henon Map

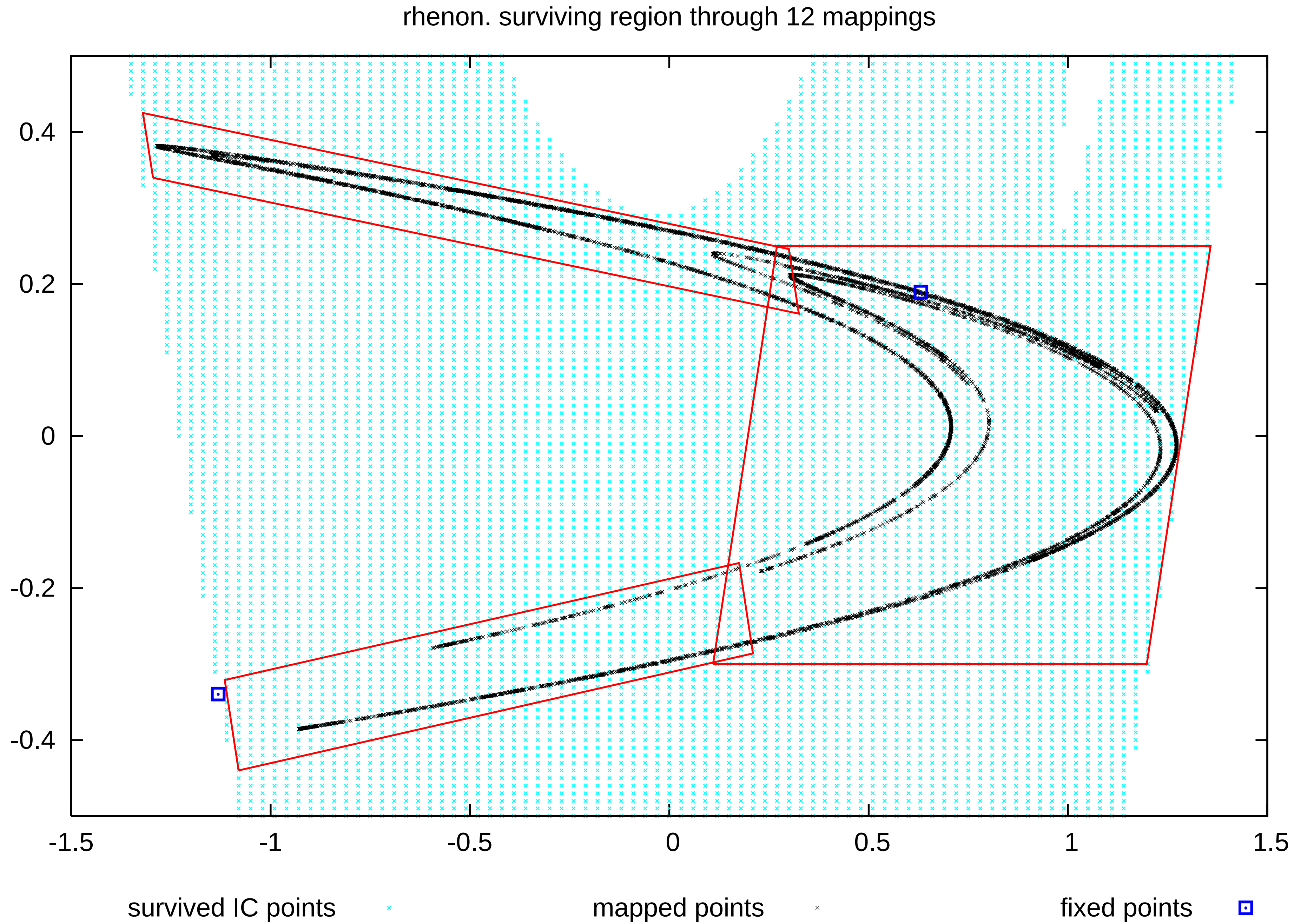
$$H(x, y) = (1 - ax^2 + y, bx).$$

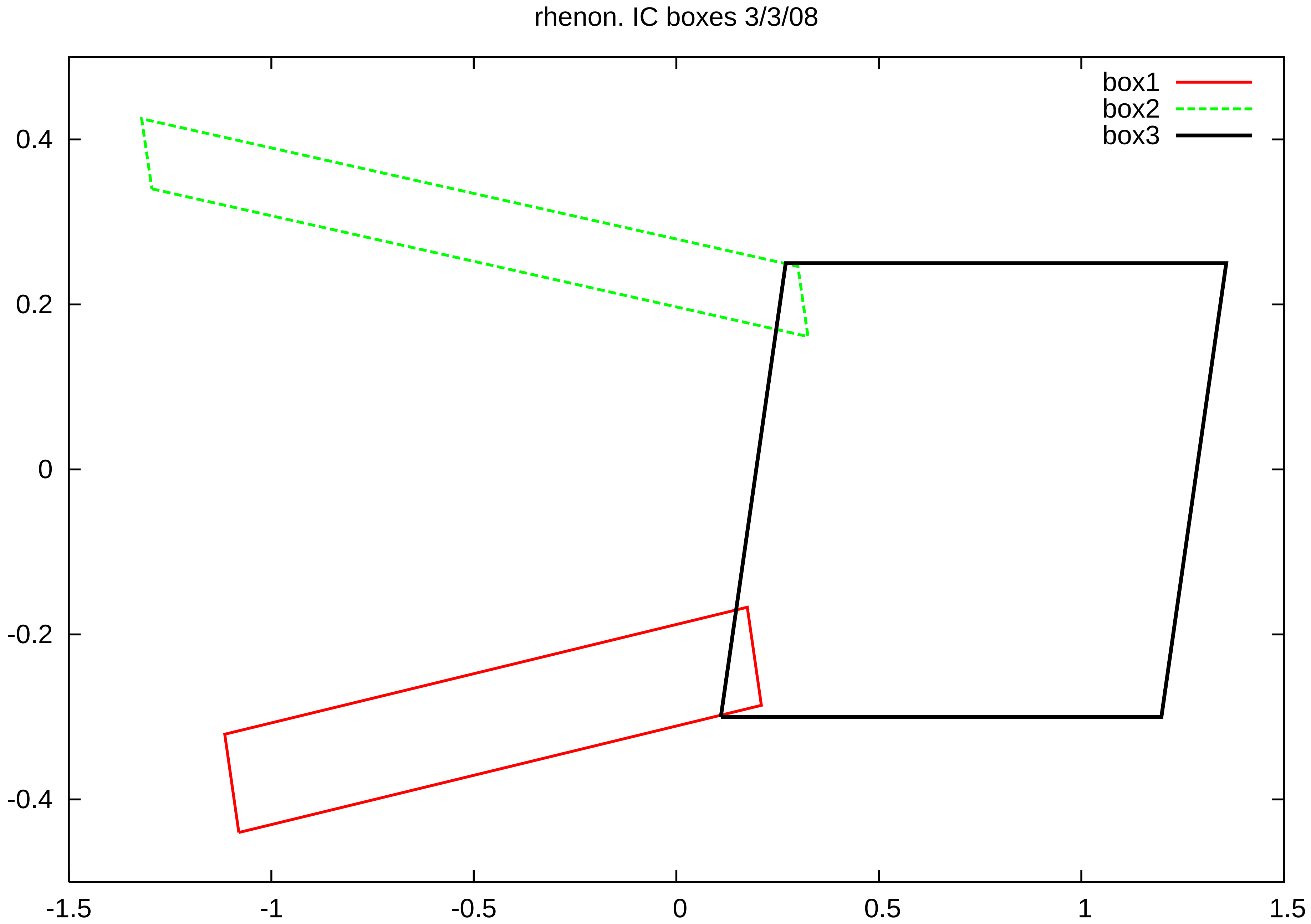
We set the parameters $a = 1.4$ and $b = 0.3$, which are originally considered by Henon. The map H has two fixed points.

$$\vec{p}_1 = (0.63135, 0.18940) \quad \text{and} \quad \vec{p}_2 = (-1.13135, -0.33941).$$

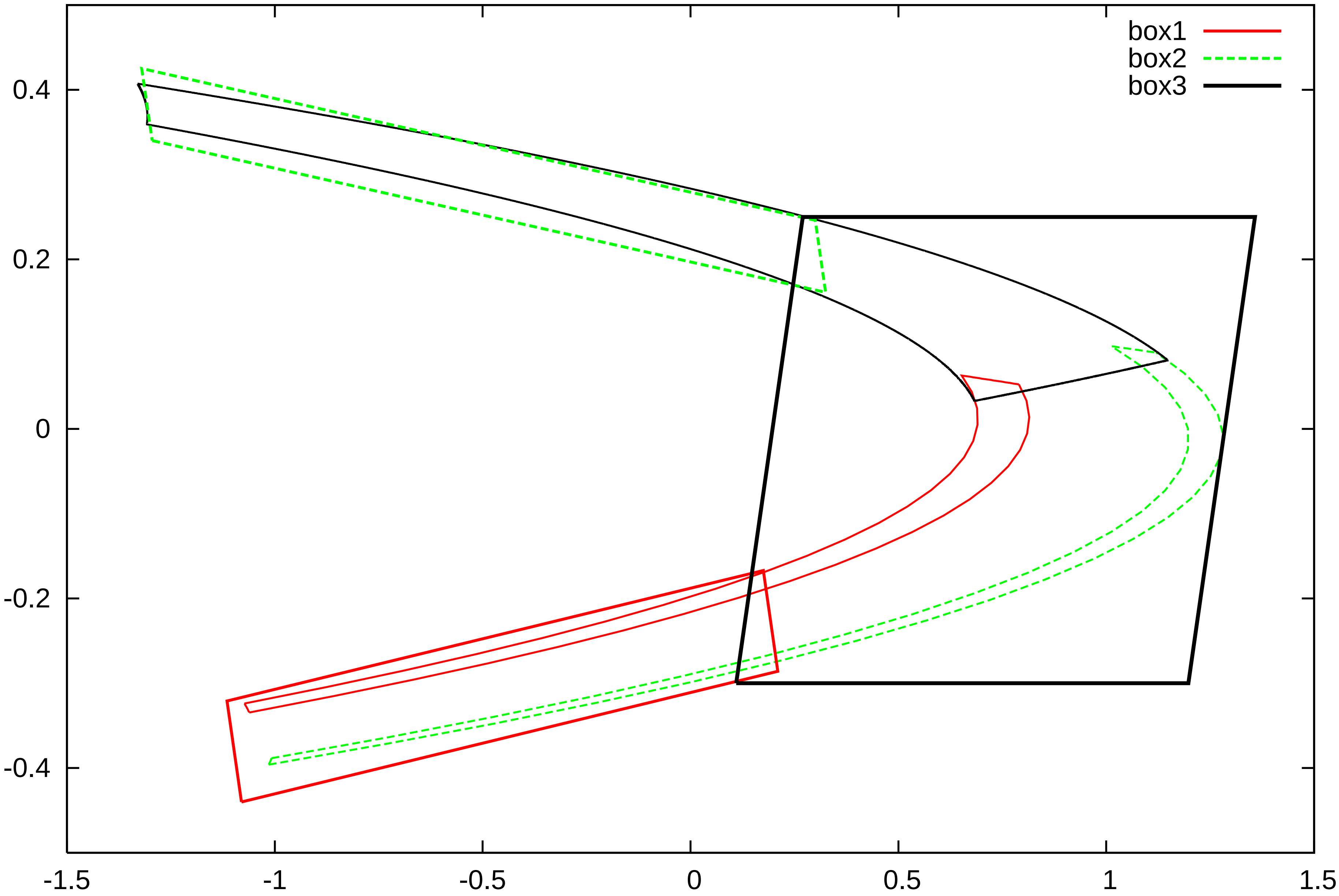


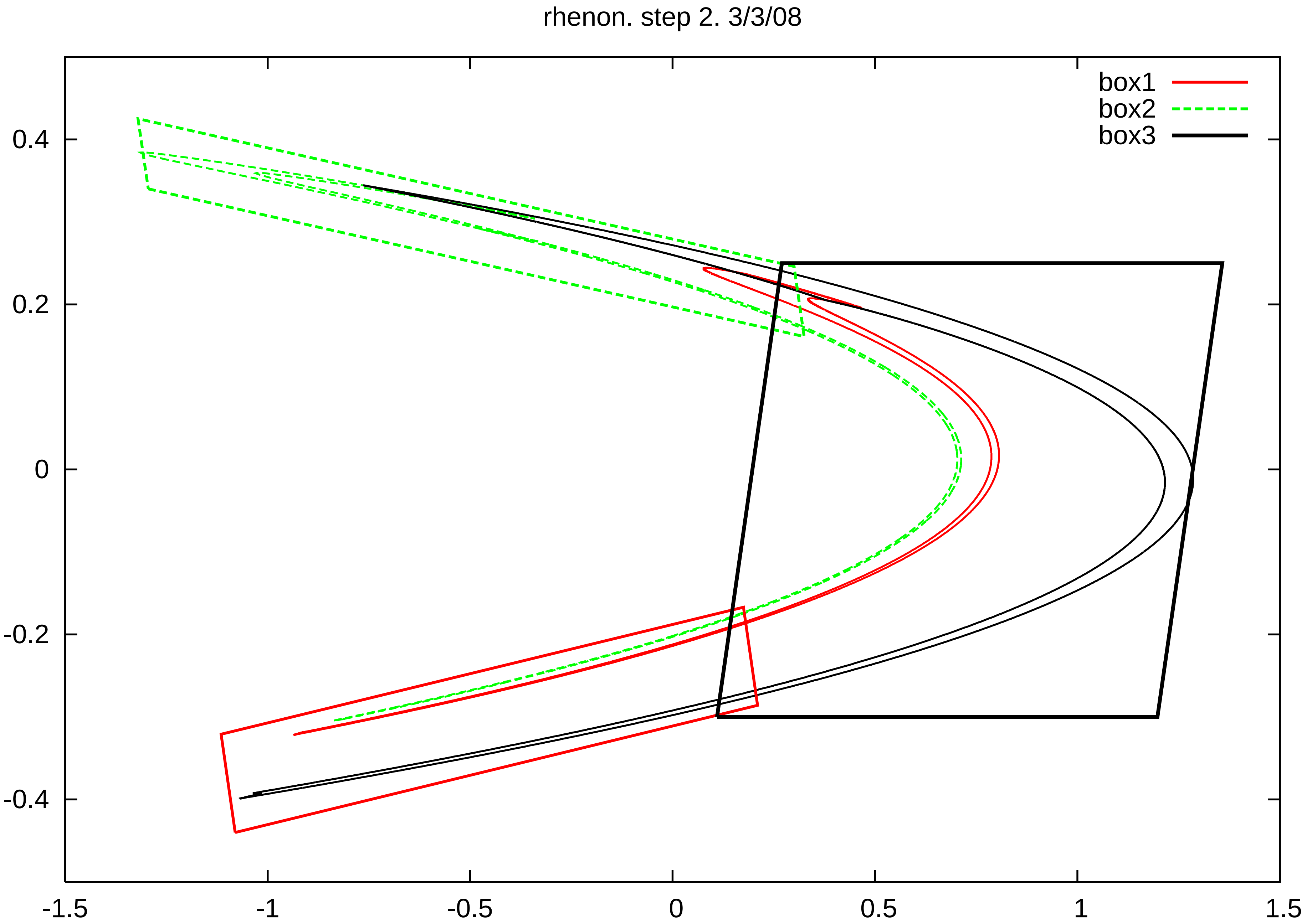
rhenon. surviving region through 12 mappings
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
survived IC points
mapped points
fixed points

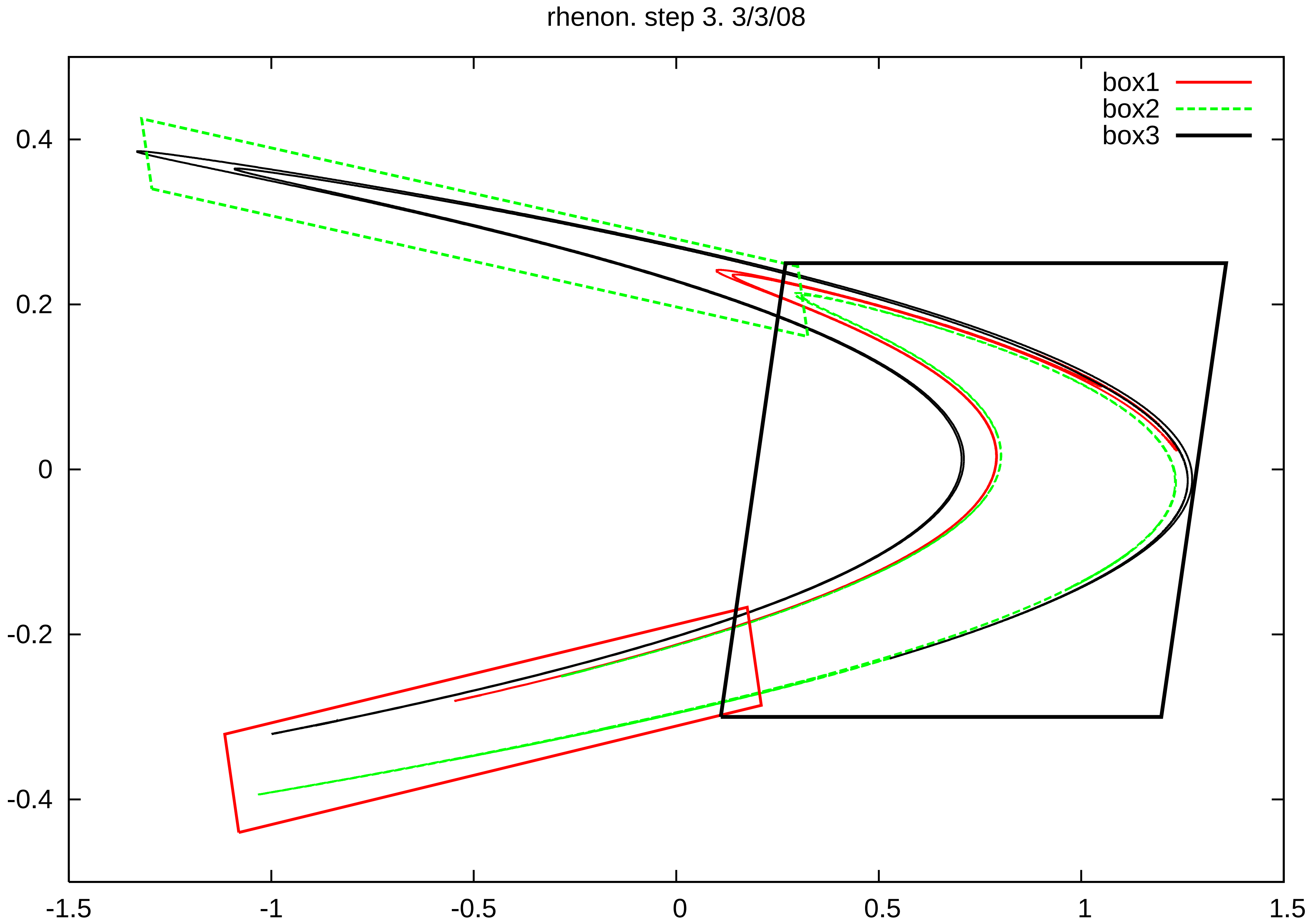

rhenon. surviving region through 12 mappings
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
survived IC points
mapped points
fixed points


rhenon. IC boxes 3/3/08
box1
box2
box3
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5

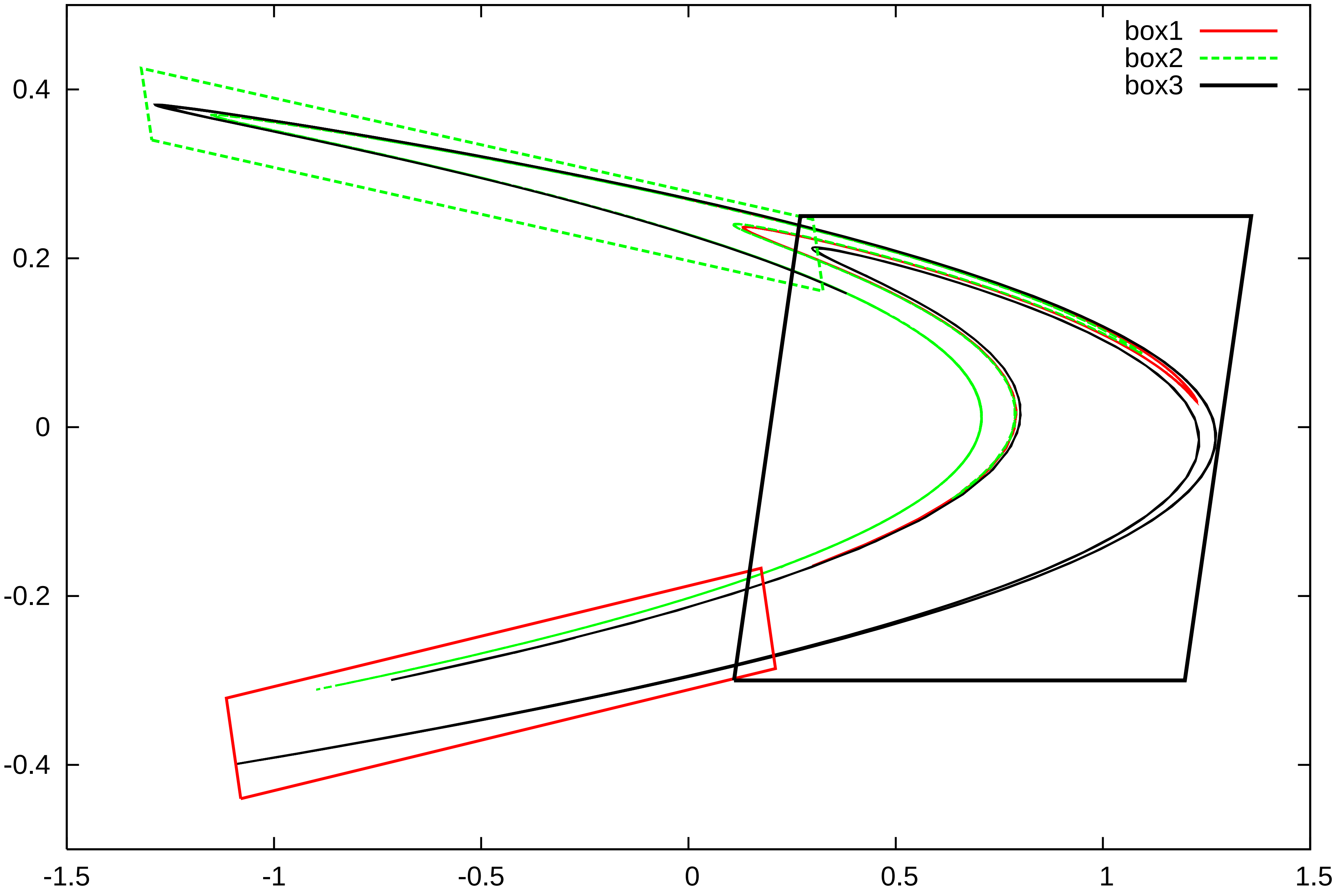
rhenon. step 1. 3/3/08



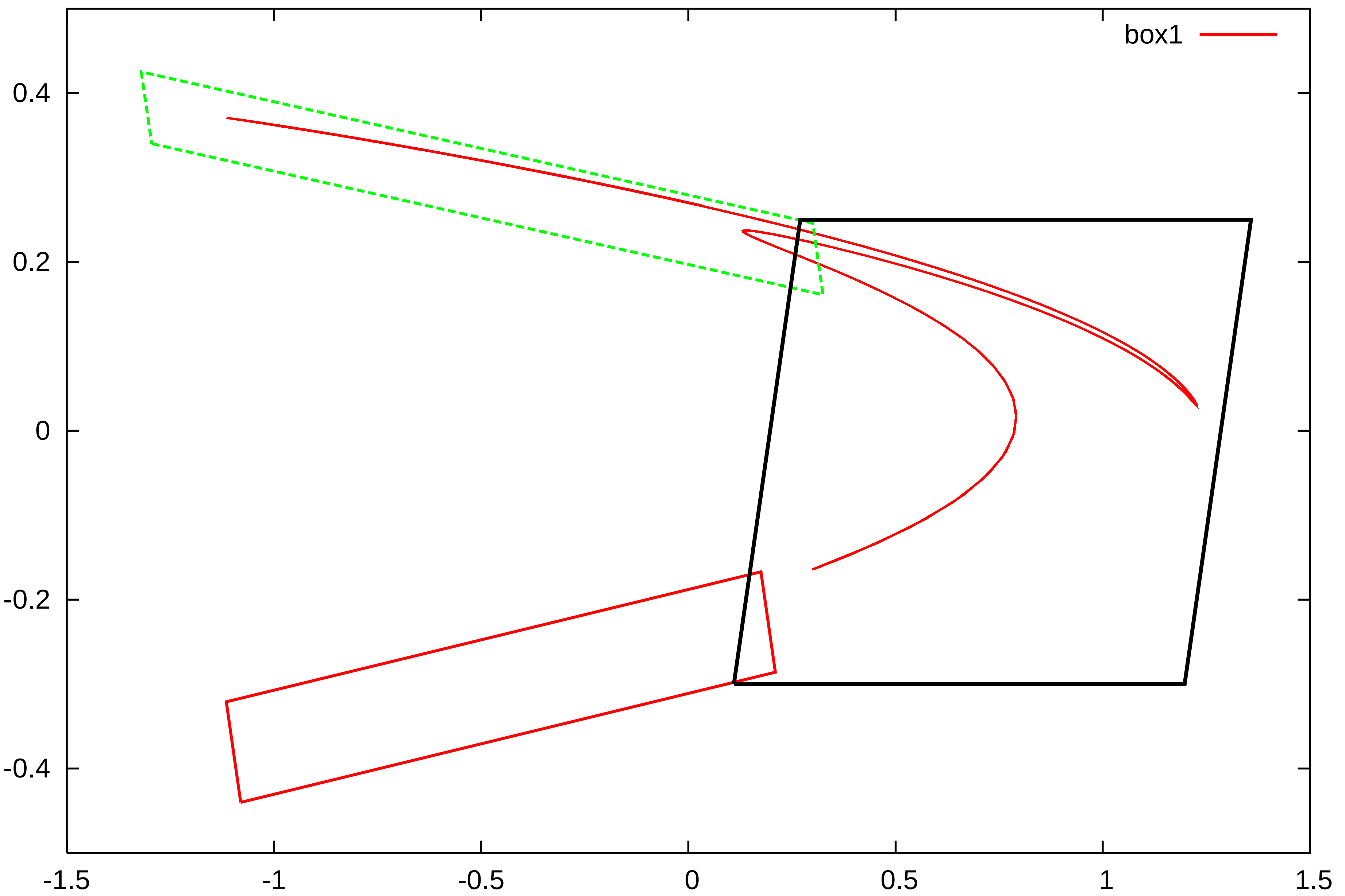

rhenon. step 2. 3/3/08
box1
box2
box3
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5


rhenon. step 3. 3/3/08
box1
box2
box3
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5

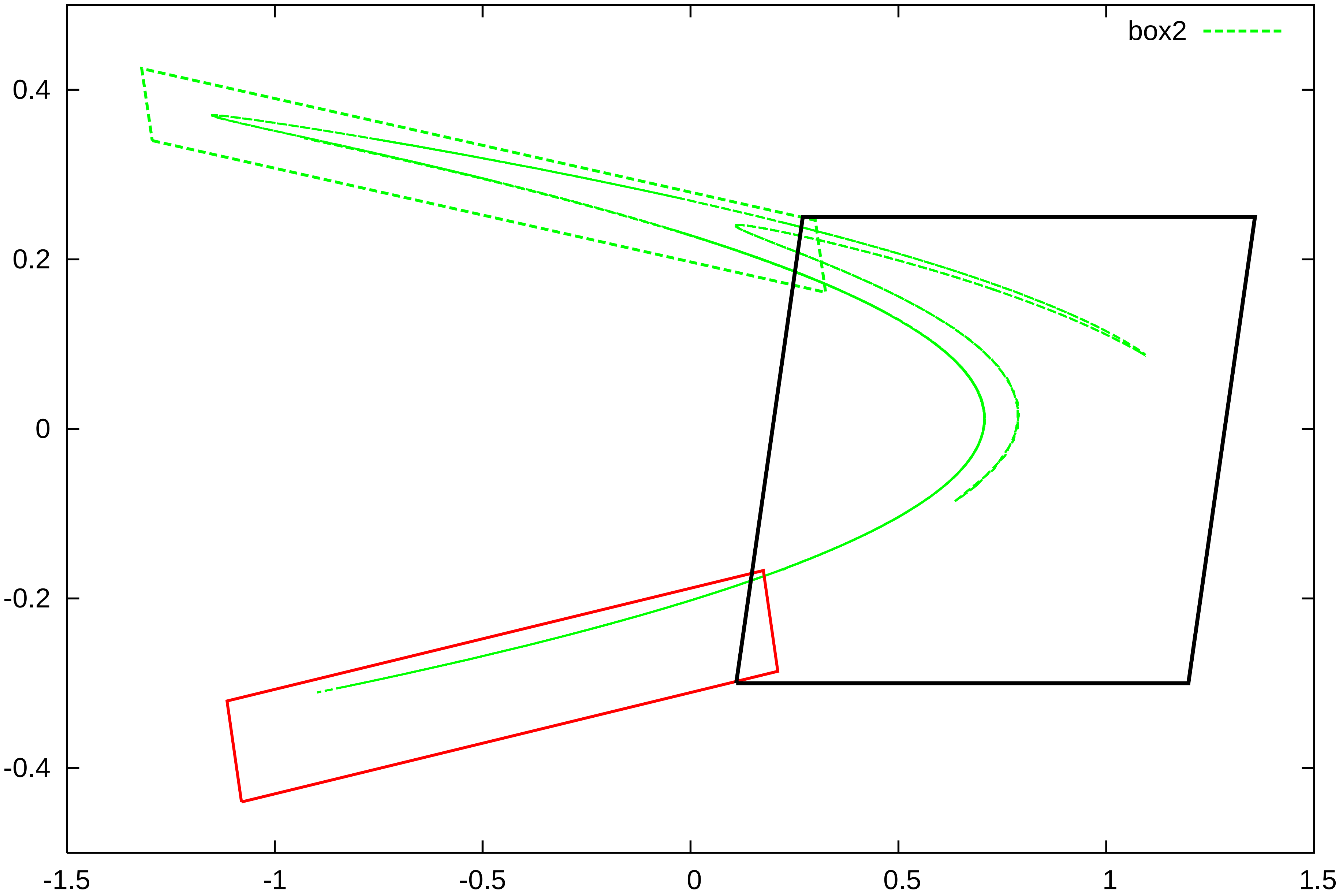
rhenon. step 4. 3/3/08



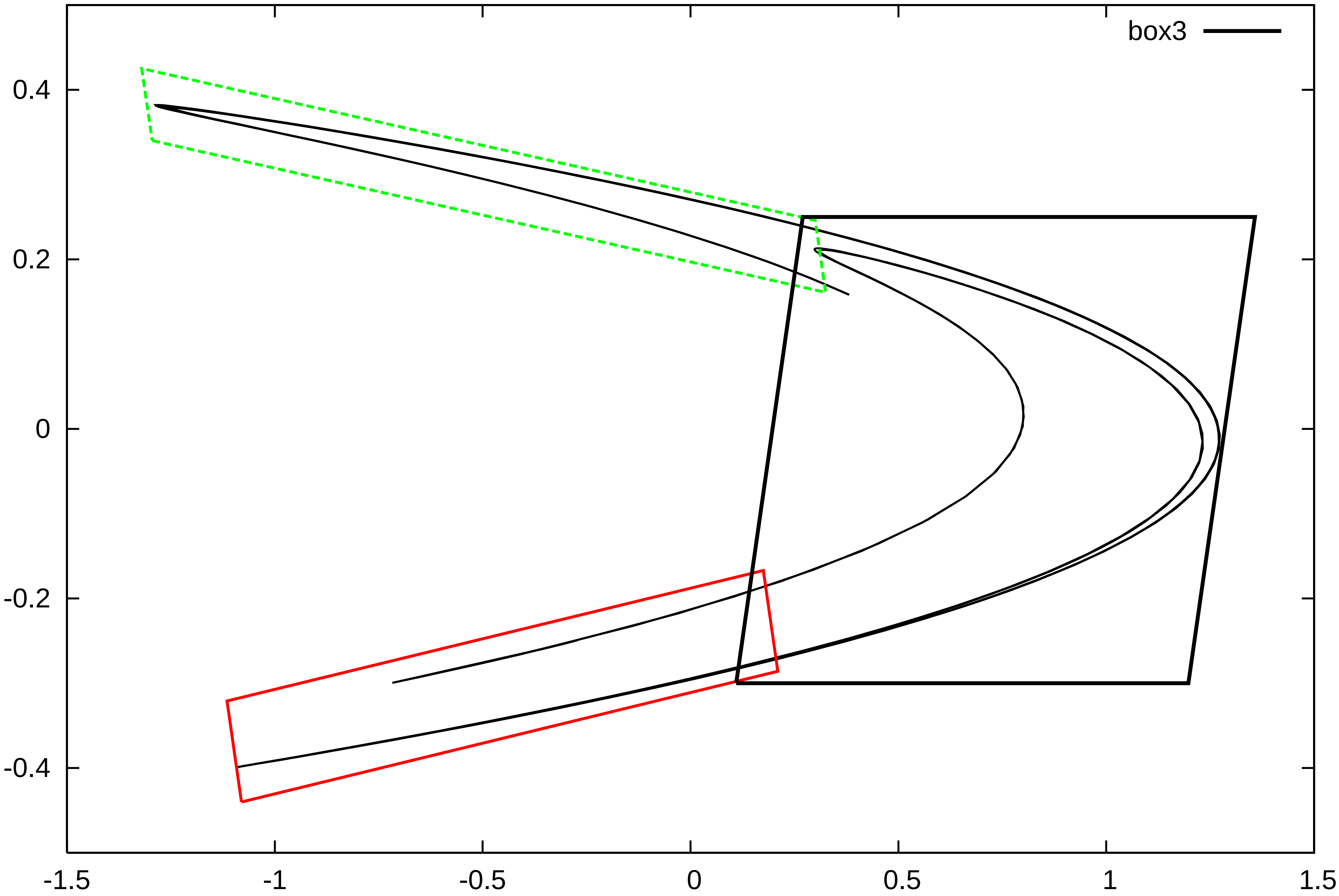
rhenon. step 4. box1. 3/3/08

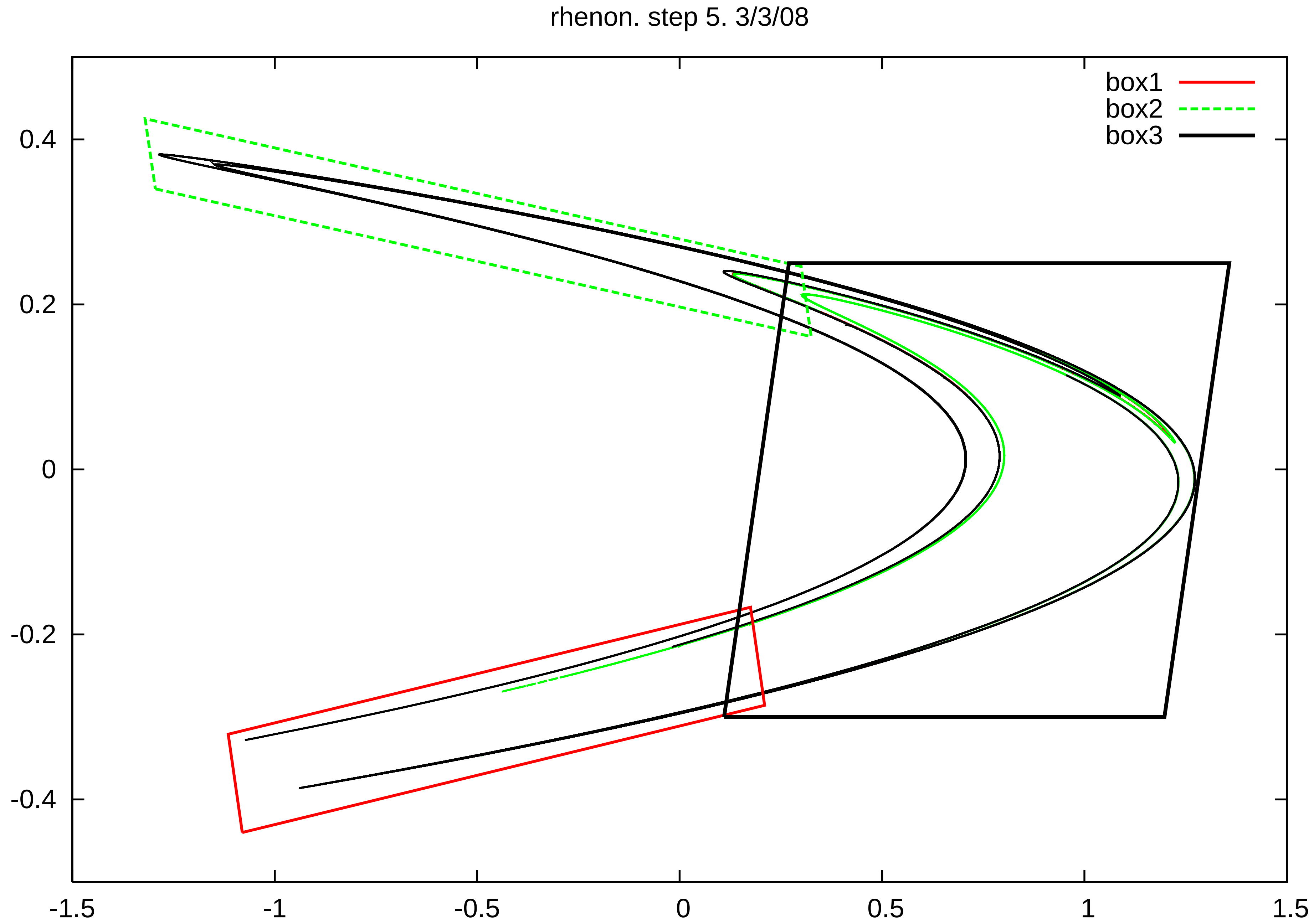


rhenon. step 4. box2. 3/3/08

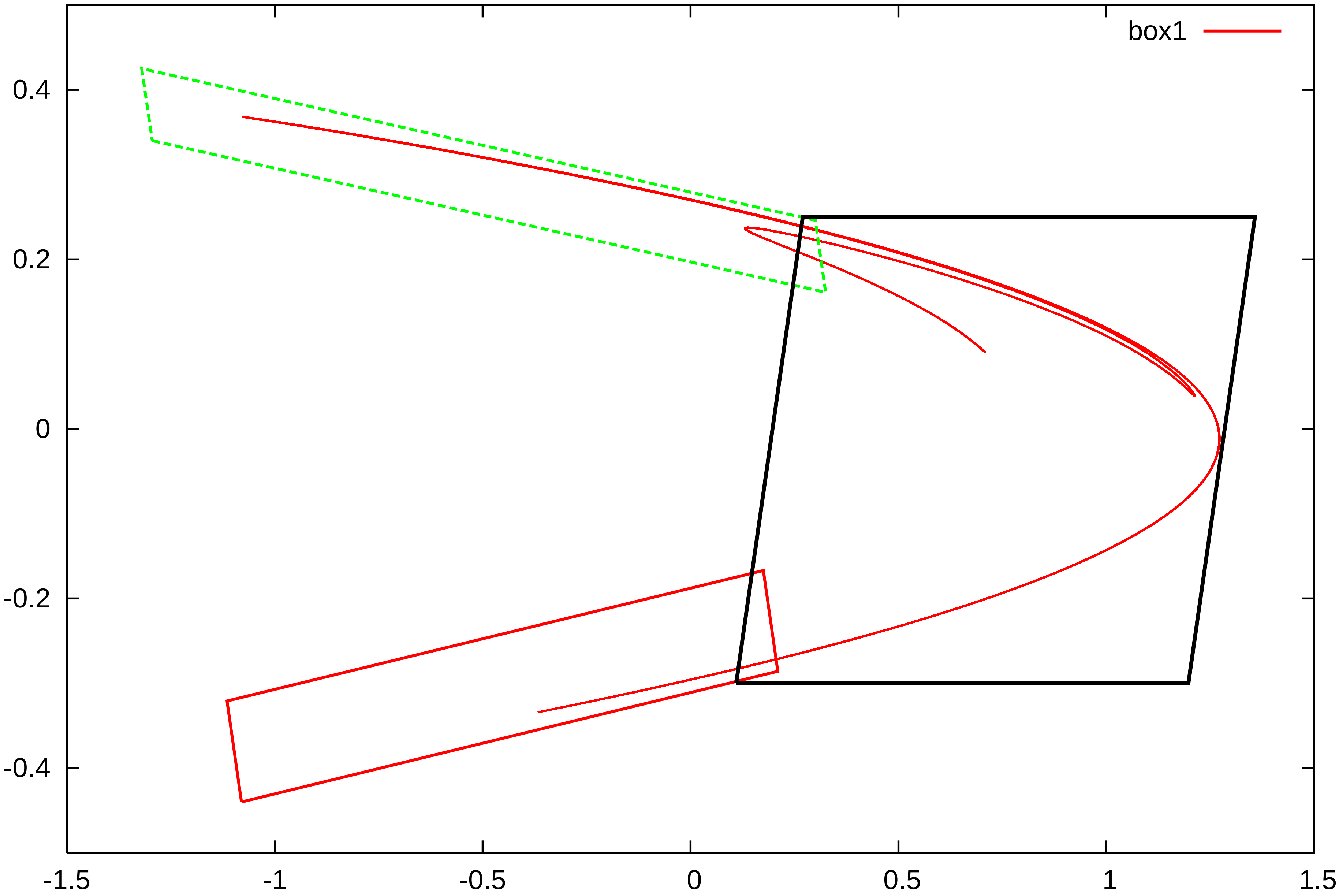


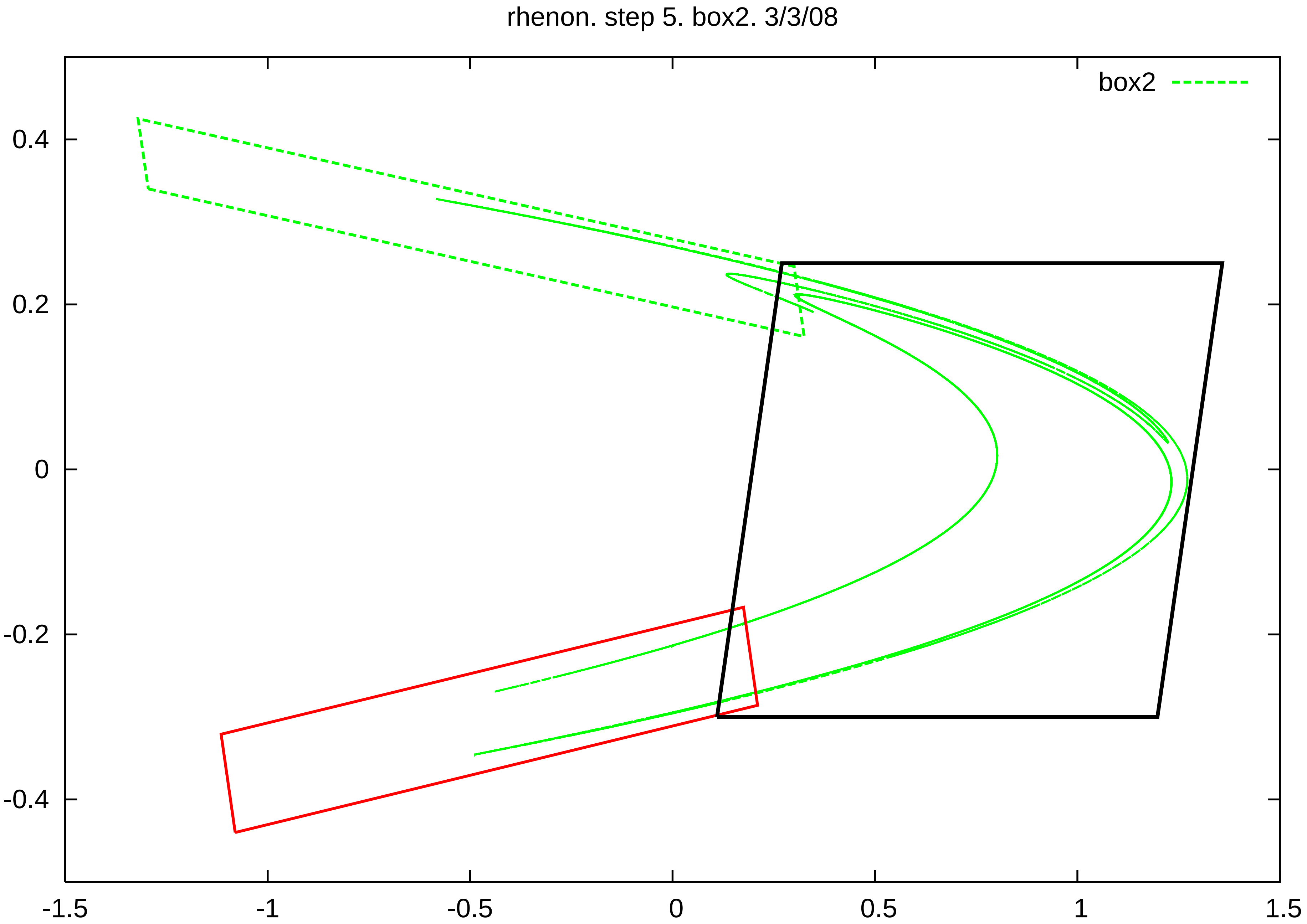
rhenon. step 4. box3. 3/3/08

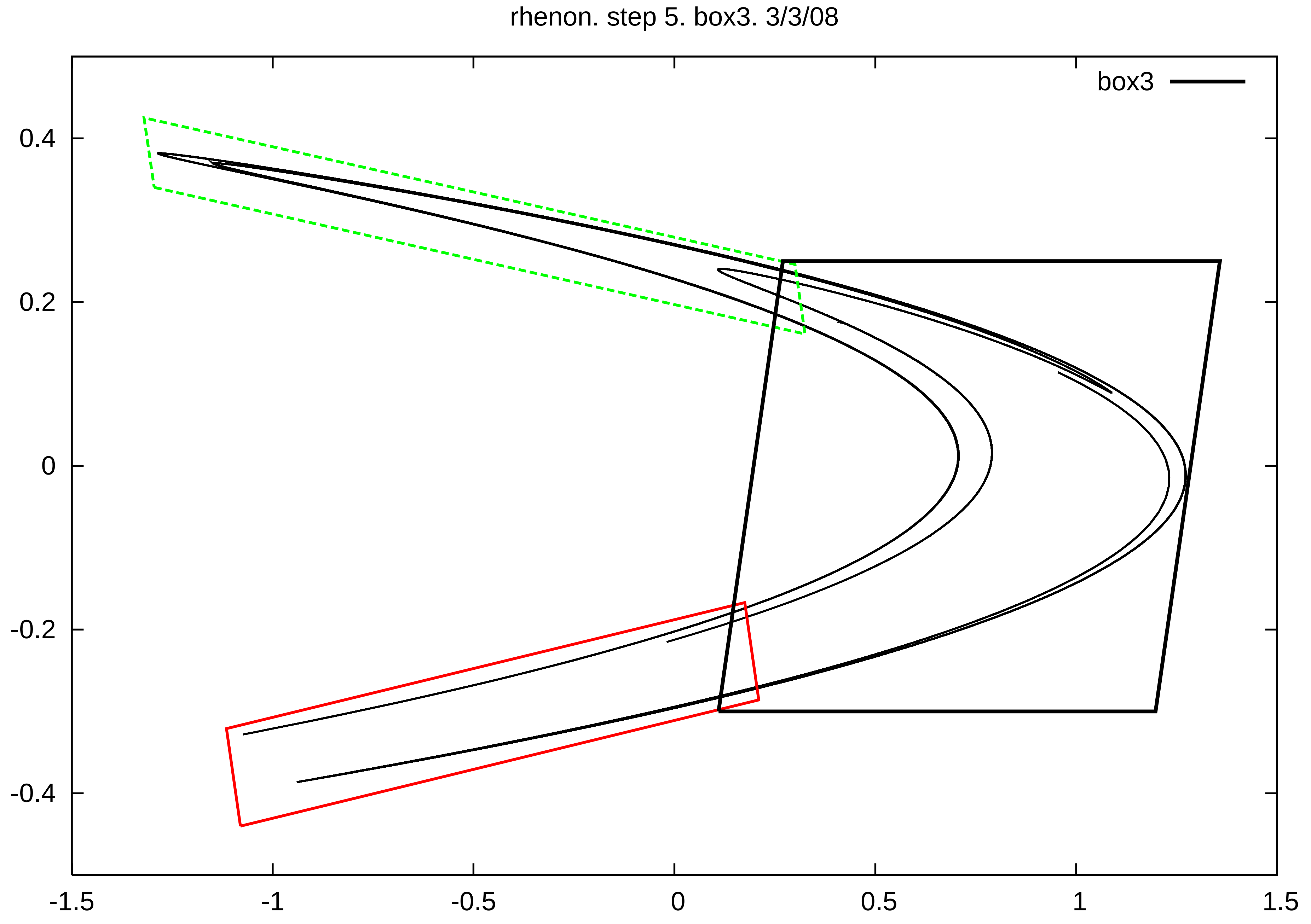



rhenon. step 5. 3/3/08
box1
box2
box3
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5

rhenon. step 5. box1. 3/3/08




rhenon. step 5. box2. 3/3/08
box2
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5


rhenon. step 5. box3. 3/3/08
box3
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5

rhenon: Number of Objects

To carry out multiple mappings of the Henon map, Taylor model objects underwent the domain decomposition.

Number of Taylor model objects used for multiple mappings:

	n	w	for 5 steps	for 7 steps
box1	33	17	3	1386
box2	21	11	148	1691
box3	33	17	8	2839

Coming very soon...

Dynamic Domain Decomposition for the ODE integrator