A functional analytic approach to computer assisted proofs based on Taylor expansions.

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Taylor Model Methods V
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1st example: the Brezis-Nirenberg eigenvalue problem

In a joint work with Filippo Gazzola, Hans-Christoph Grunau and Edoardo Sassone (The second bifurcation branch for radial solutions of the Brezis-Nirenberg problem in dimension four, NoDEA OnLine First), we study the equation

$$-\Delta u = \lambda u + u^3$$

in the unit ball of $\mathbb{R}^4$ under homogeneous Dirichlet boundary conditions. We consider the branch of radial solutions bifurcating from the second (radial) eigenvalue of $-\Delta$.

1st example: the Brezis-Nirenberg eigenvalue problem

The problem becomes an ordinary differential equation. More precisely, we set $r := |x|$ (so that $0 < r < 1$) and assuming that $u = u(r)$, the equation reads

$$u''(r) + \frac{3}{r}u'(r) + \lambda u(r) + u^3(r) = 0$$

with boundary conditions $u'(0) = u(1) = 0$.

1st example: the Brezis-Nirenberg eigenvalue problem

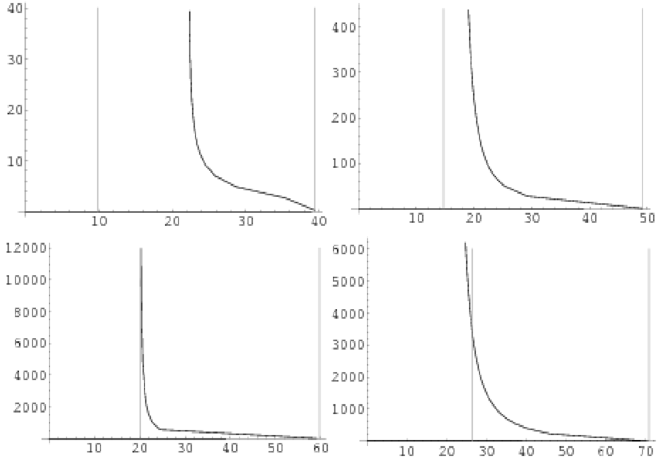
We overdetermine the problem by adding the "shooting condition" $u(0) = \omega$. In general, the equation admits no solution since it involves 3 boundary conditions. However, for any $\omega > 0$ and for a suitable $\lambda = \lambda(\omega)$, there exists a solution u_ω with precisely one zero in $[0, 1)$, the second zero being at $r = 1$. We are interested in studying the behaviour of the map $\omega \mapsto \lambda(\omega)$.

Theorem

Let $\lambda(\omega)$ be defined as above. Then, for all $\omega > 0$ we have $\lambda(\omega) > \mu_1$ (the first eigenvalue of $-\Delta$).

1st example: the Brezis-Nirenberg eigenvalue problem

Second bifurcation branch for $n = 3, 4, 5, 6$.



1st example: the Brezis-Nirenberg eigenvalue problem

Sketch of the sketch of the proof.

- Use analytical estimates for $\omega > 359$ and $\omega \in (0, 5.87\dots)$.
- Partition the interval $[5, \dots, 359]$ in small intervals.
- Use a computer assisted proof to prove the statement in the small intervals.

2nd example: the biharmonic equation

In a joint work with Filippo Gazzola and Hans-Christoph Grunau (Entire solutions for a semilinear fourth order elliptic problem with exponential nonlinearity, J. Diff. Eq. 230 (2006) 743-770) we studied entire *regular* radial solutions of the semilinear supercritical biharmonic equation

$$\Delta^2 u = \lambda e^u \quad \text{in} \quad \mathbb{R}^n, \qquad n \geq 5, \qquad \lambda > 0\,,$$

i.e. in solutions $u(x) = \tilde{u}(|x|)$, which exist for all $x \in \mathbb{R}^n$.

2nd example: the biharmonic equation

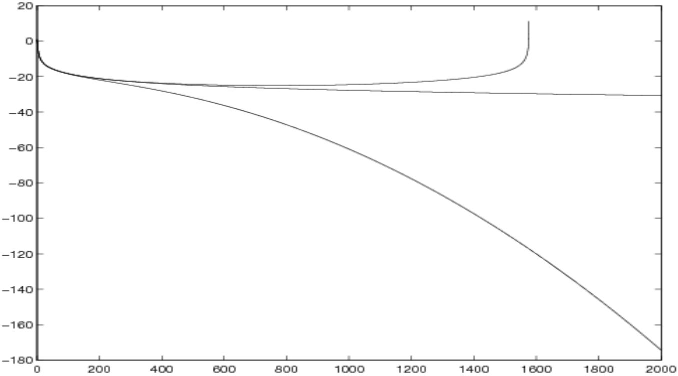
The function $\tilde{u}(r)$ satisfies the ordinary differential equation

$$u^{(4)}(r) + \frac{2(n-1)}{r} u'''(r) + \frac{(n-1)(n-3)}{r^2} u''(r)$$

$$- \frac{(n-1)(n-3)}{r^3} u'(r) = \lambda e^{u(r)}$$

when $r > 0$ and with initial conditions $u(0) = 1$, $u''(0) = \beta$ and $u'(0) = u'''(0) = 0$.

2nd example: the biharmonic equation



Finite time blow up, entire and infinite time blow down solution with $n = 5$.

2nd example: the biharmonic equation

Thanks to scaling it is enough to consider just one value of the parameter λ. It turns out that

$$\lambda = 8(n-2)(n-4).$$

is particularly convenient. For this value of λ the equation admits the singular solution $x \mapsto -4\log|x|$.

2nd example: the biharmonic equation

If $s = \log r$ and $w(s) := u(e^s) + 4s$, then the equation becomes

$$\frac{d^4 w}{ds^4} + 2(n-4)\frac{d^3 w}{ds^3} + (n^2 - 10n + 20)\frac{d^2 w}{ds^2} - 2(n-2)(n-4)\frac{dw}{ds} =$$

$$\lambda\left(e^{w(s)} - 1\right)$$

Set $\mathbf{w} = (w, w', w'', w''')$; then the singular solution of corresponds to the stationary solution $\mathbf{w}_0(s) \equiv 0$.

Representation of functions analytic in a disk

Let $R > 0$, let $\mathcal{H}_R$ be the space of analytic functions in the open disk $D_R = \{z \in \mathbb{C} : |z| < R\}$ and let $\mathcal{X}_R$ be the subspaces of $\mathcal{H}_R$ with finite norm

$$\|u\|_{\mathcal{X}_R} = \sum_{k=0}^{\infty} |u_k| \, R^k$$

where

$$u(t) = \sum_{k=0}^{\infty} u_k t^k \qquad (1)$$

and $u_k \in \mathbb{R}$.
The space $\mathcal{X}_R$ is a Banach algebra.

Representation of functions analytic in a disk

We wish to make the computer handle functions in $\mathcal{X}_R$ in the most transparent way.
It is well known that a main issue for computer assisted proofs is to have the computer manipulate real numbers. This problem is usually addressed by Interval Arithmetics. The main point of Interval Arithmetics is the representation of real numbers by means of intervals, whose extrema are representable numbers, i.e. number that can be exactly expressed by the arithmetics used by the computer, e.g. according to the IEEE standard.

Representation of functions analytic in a disk

Obviously, it is also essential to teach the computer how to handle these "numbers", how to perform basic arithmetics and how to compute all kind of functions we may need. Nowadays there exists plenty of packages that provide such technique.
We refer to the surrogate of real numbers provided by intervals as $\mathrm{std}(\mathbb{R})$. Our purpose is to create $\mathrm{std}(\mathcal{X}_R)$ with the same properties.

Representation of functions analytic in a disk

We need a (finite) subset of $\mathcal{X}_R$ which can efficiently represent the whole space. We can write functions in $\mathcal{X}_R$ as follows:

$$u(t) = \sum_{k=0}^{N-1} u_k t^k + t^N E_u(t) \,, \qquad (2)$$

where $E_u \in \mathcal{X}_R$. We can store the N (real) coefficients $\{u_k\}$ and a bound for the norm of E_u; more precisely, we store $2N+1$ representable numbers. N pairs represent lower and upper bounds for the value of the coefficients, while the last number is an upper bound of the norm of E_u.

Representation of functions analytic in a disk

Object oriented programming may be very useful to handle this kind of representations. It allows to define an object "Taylor series" with methods corresponding to all the operations we need, and also we can overload some basic function, e.g. '+', '-', '*', '/', and treat functions in a completely transparent way.

Implementation

We wish to solve

$$u^{(4)}(t)+\frac{2(n-1)}{t}u'''(t)+\frac{(n-1)(n-3)}{t^2}u''(t)-\frac{(n-1)(n-3)}{t^3}u'(t)$$
$$=\lambda e^{u(t)}$$

with initial conditions $u(0)=1$, $u''(0)=\beta$ and $u'(0)=u'''(0)=0$. As a first step, we wish to have a rigorous estimate of the solution and its derivatives at a given time $t\in[0,T]$, where $T>0$ is as large as possible.

Implementation

Fix $R > 0$ and let

$$\tilde{\mathcal{X}}_R = \{u \in \mathcal{X}_R \;:\; u(0) = 1,\; u''(0) = \beta\}\,.$$

Let $L : \tilde{\mathcal{X}}_R \to \mathcal{H}_R$ be defined by

$$(Lu)(t) = u^{(4)}(t) + \frac{2(n-1)}{t}u'''(t) + \frac{(n-1)(n-3)}{t^2}u''(t)$$
$$-\frac{(n-1)(n-3)}{t^3}u'(t)$$

and $f : \tilde{\mathcal{X}}_R \to \mathcal{X}_R$ be defined by

$$f(u) = \lambda e^u\,.$$

Implementation

The following lemmas are straightforward:

Lemma

The operator L is invertible and solutions of the fourth order differential equation with the assigned initial conditions correspond to fixed points of the operator $F = (L^{-1} f) : \tilde{\mathcal{X}}_R \to \tilde{\mathcal{X}}_R$.

Lemma

Let $B_K = \{u \in \mathcal{X}_R\,,\, ||u||_{\mathcal{X}_R} \leq K\}$. *The Lipschitz constant of the function F restricted to* B_K *is at most* $\frac{\lambda e^K R^4}{C(0,n)}$.

Implementation

Assume that we have an approximate solution $\bar{u}(t) = \sum_{k=0}^{N-1} \bar{u}_k t^k$, where $\{\bar{u}_k\}$ are interval values satisfying $1 \in \bar{u}_0$ and $\beta/2 \in \bar{u}_2$. The following lemma yields a true solution close to $\bar{u}$:

Lemma

Let $\bar{u} \in \mathcal{Z}_R$, $C : \mathcal{Z}_R \to \mathcal{Z}_R$ and $\varepsilon, \rho > 0$. If $||C(\bar{u}) - \bar{u}||_{\mathcal{Z}_R} < \varepsilon$ and the restriction of C to the ball $B(\bar{u}, \rho)$ has Lipschitz constant $L(C) \leq 1 - \varepsilon/\rho$, then there exists a fixed point of C in $B(\bar{u}, \rho)$.

Implementation

By applying the lemmas shown above we can now rigorously compute $u(t)$ and its derivatives for all $t \in [0, T]$, where $0 < T < R$. We remark that, independently of the accuracy of the computations and of the order N, it is clear that we cannot use this approach for computing the solution at values of T larger than the (unknown) radius of analyticity of the solution of the problem. Nonetheless, since we know the solution at some positive time T, we can reiterate the procedure by computing the power expansion centered at $t = T$.

Implementation

This is not very convenient from the numerical point of view, since we would have to compute the power expansion at $t = T$ of the functions t^{-1}, t^{-2} and t^{-3}. It is more convenient to set $w(s) = u(e^s) + 4s$, so that w satisfies the autonomous equation

$$w^{(4)}(s)+2(n-4)w'''(s)+(n^2-10n+20)w''(s)-2(n-2)(n-4)w'(s)$$
$$= \lambda(e^{w(s)} - 1)\,,$$

for which we may always assume that an initial value problem is set at $s = 0$.

A fixed point problem

Our strategy for proving the existence of a fixed point for some operator $F : \mathcal{F} \to \mathcal{F}$ is as follows. First, we use our best numeric algorithm in order to find a good approximation u_0 of the solution of the problem. We choose an “finite” approximation M for the map $\mathrm{Id} + [DF(u_0) - \mathrm{Id}]^{-1}$, and then we define

$$\mathcal{C}(u) = F(u) - M\big[F(u) - u\big].$$

Formally, the map $\mathcal{C}$ is close to the Newton map for F. Thus, our goal is to prove that $\mathcal{C}$ is a contraction.

A fixed point result

The necessary conditions are given in the following version of the Banach theorem:

Lemma

Let u_0 be a function in $\mathcal{F}$. Assume that there exists a bounded linear operator M on $\mathcal{F}$, and constants $\varepsilon, K > 0$, such that $M - \mathrm{Id}$ has a bounded inverse and

$$\|\mathcal{C}(u_0) - u_0\| < \varepsilon \,, \qquad \|D\mathcal{C}(u)\| < K \,, \qquad \varepsilon + Kr < r \,,$$

for all functions u in a closed ball B in $\mathcal{F}$ of radius r, centered at u_0. Then the function F has a unique fixed point u in B.

Representation of other functional spaces

This technique is not restricted to Taylor series, but, at least in principle, can be applied to all functional spaces having a "good" basis. Here the word "basis" is used in a broad sense. Loosely speaking, a good basis is a finite subset of the functional space which spans a large portion of the functional space. The technique has been successfully implemented to Fourier serier and also to Fourier series with coefficients that are Taylor series or again Fourier series.