

The Extrapolated Taylor Model

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Overview

- Introduction
- Extrapolation Algorithm
- Asymptotic Sequences and Expansion
- The Richardson Extrapolation Process (REP) and Algorithm
- Other Extrapolation Algorithm
- Error control in Convergence Acceleration Processes
- Brezinski's Interval
- Application of Extrapolation to Taylor Model
- Asymptotic Expansion of Taylor Model
- REP for Taylor Model
- Procedure for Extrapolated Taylor Model with example
- Numerical Examples (for 1, 2, 3, 6 and 10 Dimensions)
- Conclusions
- Future Scope

Introduction

- We present a new method ‘**Extrapolated Taylor Model Method**’ to compute the range enclosure of the functions using **Taylor Model and Extrapolation Algorithms**, such as **Richardson Extrapolation Algorithm (REP)**.
- Aim is to ‘*accelerate the order of convergence of Taylor Model*’ by extrapolation, without going for higher order Taylor Model.

Extrapolation Algorithm

- Extrapolation Algorithms can be used to ‘**accelerate**’ the convergence of a given sequence.
- Let (S_n) be a sequence of (real or complex) numbers converging to S . Extrapolation method *transforms* the sequence (S_n) into another sequence (T_n) which converges to the same limit S , faster than (S_n) .
- This is possible *only if* there exists an ‘**Asymptotic Expansion**’ for the sequence to be accelerated.

Asymptotic Sequence

Definition:

The sequence of functions $\{\phi_k(x)\}_{k=0}^{\infty}$ is called an *asymptotic sequence* as $x \rightarrow x_0$ if

$$\frac{\phi_{k+1}(x)}{\phi_k(x)} = o(1) \text{ as } x \rightarrow x_0, \quad k = 0, 1, \dots$$

means
$$\frac{\phi_{k+1}(x)}{\phi_k(x)} \rightarrow 0 \text{ as } x \rightarrow x_0$$

Asymptotic Expansion

- Consider a given sequence of real or complex numbers $\{A(h)\}$ and natural numbers s and N . The sequence $\{A(h)\}$ is said to possess an Asymptotic expansion of order s , if each $A(h)$ as $h \rightarrow 0+$ can be written in the form

$$A(h) = A + \sum_{k=1}^s \alpha_k h^k + O(h^{s+1}) \text{ as } h \rightarrow 0+ \dots(1)$$

Where - α_k are independent of h .

Richardson Extrapolation Algorithm (REP)

- The 'A' in expression (1) can be approximated by 'A(h)' with sufficiently small values of 'h', the error in this approximation being

$$A(h) - A = O(h) \text{ as } h \rightarrow 0+$$

- The deeper idea of REP is to eliminate the 'h' term from the expression (1) and to obtain a new approximation $A_1(h)$ to A whose error is

$$A_1(h) - A = O(h^2) \text{ as } h \rightarrow 0+$$

Recursive Algorithm for REP

- Let us take a constant $\omega \in (0,1)$, $h_0 \in (0,b]$ and let $h_m = h_0 \omega^m$, $m=1,2,\dots$, then $\{h_m\}$ is a decreasing sequence in $(0,b]$ and as $\lim_{m \rightarrow \infty} h_m = 0$.

Algorithm:

- Set $A_0^{(j)} = A(h_j)$, $j=0,1,2,\dots$
- Set $c_n = \omega^n$, $C_n = 1/c_n$ and compute $A_n^{(j)}$ by the recursion

$$A_n^{(j)} = \frac{C_n A_{n-1}^{(j+1)} - A_{n-1}^{(j)}}{(C_n - 1)} \quad j = 0,1,\dots, \quad n = 1,2,\dots$$

$A_n^{(j)}$ are approximations to A produced by Richardson Extrapolation process

Romberg Table by REP

i	$k=0$	$k=1$	$k=2, \dots$
0	$A_0^{(0)}$	$A_1^{(0)} = \frac{2 A_0^{(1)} - A_0^{(0)}}{(2-1)}$	
1	$A_0^{(1)}$	$A_1^{(1)} = \frac{2 A_0^{(2)} - A_0^{(1)}}{(2-1)}$	$A_2^{(0)} = \frac{2^2 A_1^{(1)} - A_1^{(0)}}{(2^2-1)}$
2	$A_0^{(2)}$	$A_1^{(2)} = \frac{2 A_0^{(3)} - A_0^{(2)}}{(2-1)}$	$A_2^{(1)} = \frac{2^2 A_1^{(2)} - A_1^{(1)}}{(2^2-1)}$
3	$A_0^{(3)}$		
	$O(h)$	$O(h^2)$	$O(h^3)$

Other Extrapolation Algorithms

1. Aitkin's Δ^2 iterated process
2. Rational Extrapolation
3. The ε - algorithm
4. The E – algorithm
5. The G - transformation
6. Levin's transform
7. Overholt's process

Error control in Convergence Acceleration Processes

- From user's point of view it is not sufficient to know that, for a given sequence (S_n) , the extrapolated sequence (T_n) will converge faster.
- One should have *an estimate* of the error $(T_n - S)$ or, still better, to know a *sequence of intervals* containing the unknown limit S of the sequence (S_n) .
- Brezenski's method can be used for estimating the error on the transformed sequences (T_n) obtained through Extrapolation process.

Brezenski's Interval

- Let (T_n) and (V_n) be the transformed sequences of (S_n) converging to the same limit S , and let $V_n(b)$ and $J_n(b)$ be defined as

$$V_n(b) = V_n - b * (V_n - T_n), n \in \mathbb{N}, b \in (0,1)$$

$$J_n(b) = [\min (V_n(b), V_n(-b)), \max (V_n(b), V_n(-b))]]$$

- Theorem :**

If the sequence (T_n) converges to S faster than the original sequence (S_n) with limit S and if the sequence (V_n) converges to S faster than the (T_n) , then for all $b \neq 0$, $\exists N: \forall n \geq N$, the true value S will be contained in an interval $J_n(b)$. Moreover, the sequence $V_n(\pm b)$ will converge faster than the (S_n) .

Application of Extrapolation Method to “Taylor Model”

Asymptotic Expansion of Taylor Series

- When x_0 is finite, the sequence $\{(x - x_0)^k\}_{k=0}^{\infty}$ is asymptotic as $x \rightarrow x_0$.
- If $f \in C^{\infty}[x_0, x_0 + \delta]$, $\delta > 0$, then its Taylor series about x_0 , namely,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \quad \text{as } x \rightarrow x_0 +$$

represents $f(x)$ as Asymptotically as $x \rightarrow x_0 +$.

The Taylor Form

- Let $f: X \rightarrow \mathfrak{R}$ be a function that is $m+1$ times differentiable on X . Then, the Taylor expansion of f of order m is given as

$$f(x) = f(c) + \sum_{|\lambda|=1}^m D^\lambda f(c) / \lambda! (x-c)^\lambda + \sum_{|\lambda|=m+1} f^{(\lambda)}(\xi) / \lambda! (x-c)^\lambda$$

$$\longleftarrow \quad p(x) \quad \longrightarrow \quad \longleftarrow \quad r(x) \quad \longrightarrow$$

Where

$l =$ the number of variables, $x = (x_1, \dots, x_l) \in \mathfrak{R}^l$,

$p(x)$ = the polynomial part; $r(x)$ = the remainder part;

$c = \text{mid}(x)$, $x \in X$, $\xi \in X$

$$\lambda = \{\lambda_1, \dots, \lambda_l\}, \quad |\lambda| = \lambda_1 + \dots + \lambda_l, \quad \lambda! = \lambda_1! \dots \lambda_l!, \quad D^\lambda f(x) = \frac{\partial^{\lambda_1 + \dots + \lambda_l} f(x)}{\partial x_1^{\lambda_1} \dots \partial x_l^{\lambda_l}}$$

Existence of Asymptotic Expansion for Taylor Form

- The sequence $\{A(h)\}$ is said to possess an Asymptotic expansion of order s , if each $A(h)$ as $h \rightarrow 0+$ can be written in the form

$$A(h) = A + \sum_{k=1}^s \alpha_k h^k + O(h^{s+1}) \text{ as } h \rightarrow 0+ \dots(1)$$

Where - α_k are independent of h .

continued....

Continued...

- The Taylor form given below is analogous to asymptotic form given in expression (1)

$$f(x) = f(c) + \sum_{|\lambda|=1}^m D^\lambda f(c) / \lambda! (x-c)^\lambda + \sum_{|\lambda|=m+1} f^{(\lambda)}(\xi) / \lambda! (x-c)^\lambda$$

Where

$$A(h) \leftrightarrow f(x) \qquad A \leftrightarrow f(c) \qquad h \leftrightarrow (x-c)$$

$$\alpha_k \leftrightarrow \sum_{|\lambda|=1}^m D^\lambda f(c) / \lambda!$$

$$O(h^{s+1}) \leftrightarrow \sum_{|\lambda|=m+1} f^{(\lambda)}(\xi) / \lambda! (x-c)^\lambda$$

REP for Taylor Model

- Let $f \in C^{s+1}[a,b]$, let $\{n_0=n < n_1 < n_2 < \dots\}$ be an increasing sequence of positive integers and let $h_i = n_i^{-1}$, a step size and $N_i = n_i$, a number of sub boxes, $\{A_k^{(i)}\}_{inf}$ a sequence of infimum of range enclosure $f(x)$ obtained through Taylor Model of order m .

$$A_k^{(i)} = A_{k-1}^{(i)} + \frac{A_{k-1}^{(i)} - A_{k-1}^{(i-1)}}{(2^{k+1} - 1)} \quad (\text{stable formula})$$

(correction factor)

with $n_i = \rho^i$; $\rho = 2$, $i=0,1,2..$ (called Geometric progression)
 $k=1,2,\dots,s$; $i=0,1,\dots$

Procedure to construct the sequences

Do $i = 1, 2, \dots$

1. Subdivide the box X uniformly into $N=2^i$ subboxes

$$\text{Let } X = \bigcup_{j_i=1}^N (x_{1,j1}, x_{2,j2}, \dots, x_{n,jn})$$

2. Expand the given function with Taylor Model order '1' over all subboxes.
3. For each subbox, compute the exact range of the polynomial part $p(x)$ of Taylor Model using NIE and add the corresponding Taylor Model remainder interval $R(X)$.
4. Compute the range enclosure $F_N(X)$

$$F_N(X) = \bigcup_{j_i=1}^N F(x_{1,j1}, x_{1,j2}, \dots, x_{1,jn})$$

5. Set $A_0^{(i)}|_{inf} = \inf(F_N(X))$ and $A_0^{(i)}|_{sup} = \sup(F_N(X))$

End do

Romberg Table for Infimum

$k=0$	$k=1$	$k=2, \dots$
$A_0^{(0)} = \inf(F_2(X)),$	$A_1^{(0)} = A_0^{(1)} + \frac{A_0^{(1)} - A_0^{(0)}}{(2^{k+1} - 1)}$	
$A_0^{(1)} = \inf(F_4(X)),$	$A_1^{(1)} = A_0^{(2)} + \frac{A_0^{(2)} - A_0^{(1)}}{(2^{k+1} - 1)}$	$A_2^{(0)} = A_1^{(1)} + \frac{A_1^{(1)} - A_1^{(0)}}{(2^{k+1} - 1)}$
$A_0^{(2)} = \inf(F_8(X)),$	$A_1^{(2)} = A_0^{(3)} + \frac{A_0^{(3)} - A_0^{(2)}}{(2^{k+1} - 1)}$	$A_2^{(1)} = A_1^{(2)} + \frac{A_1^{(2)} - A_1^{(1)}}{(2^{k+1} - 1)}$
$A_0^{(3)} = \inf(F_{16}(X))$	$A_1^{(3)} = A_0^{(4)} + \frac{A_0^{(4)} - A_0^{(3)}}{(2^{k+1} - 1)}$	$A_2^{(2)} = A_1^{(3)} + \frac{A_1^{(3)} - A_1^{(2)}}{(2^{k+1} - 1)}$
$A_0^{(4)} = \inf(F_{32}(X))$		

Romberg Table for Supremum

$k=0$	$k=1$	$k=2, \dots$
$A_0^{(0)} = \sup(F_2(X)),$	$A_1^{(0)} = A_0^{(1)} + \frac{A_0^{(1)} - A_0^{(0)}}{(2^{k+1} - 1)}$	
$A_0^{(1)} = \sup(F_4(X)),$	$A_1^{(1)} = A_0^{(2)} + \frac{A_0^{(2)} - A_0^{(1)}}{(2^{k+1} - 1)}$	$A_2^{(0)} = A_1^{(1)} + \frac{A_1^{(1)} - A_1^{(0)}}{(2^{k+1} - 1)}$
$A_0^{(2)} = \sup(F_8(X)),$	$A_1^{(2)} = A_0^{(3)} + \frac{A_0^{(3)} - A_0^{(2)}}{(2^{k+1} - 1)}$	$A_2^{(1)} = A_1^{(2)} + \frac{A_1^{(2)} - A_1^{(1)}}{(2^{k+1} - 1)}$
$A_0^{(3)} = \sup(F_{16}(X))$	$A_1^{(3)} = A_0^{(4)} + \frac{A_0^{(4)} - A_0^{(3)}}{(2^{k+1} - 1)}$	$A_2^{(2)} = A_1^{(3)} + \frac{A_1^{(3)} - A_1^{(2)}}{(2^{k+1} - 1)}$
$A_0^{(4)} = \sup(F_{32}(X))$		<i>continued</i>

Continued...

i	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
1	$A_{N=2}$	A_2-A_4				
2	$A_{N=4}$	A_4-A_8	A_2-A_8	A_2-A_{16}		
3	$A_{N=8}$	A_8-A_{16}	A_4-A_{16}	A_4-A_{32}	A_2-A_{32}	
4	$A_{N=16}$	$A_{16}-A_{32}$	A_8-A_{32}	A_8-A_{64}	A_4-A_{64}	A_2-A_{64}
5	$A_{N=32}$	$A_{32}-A_{64}$	$A_{16}-A_{64}$			
6	$A_{N=64}$					

Procedure to construct the Brezenski's Interval

1. Let $S_n = A_k^{(i)}$, $T_n = A_{k+1}^{(i)}$, and $V_n = A_{k+2}^{(i)}$, $k=0,1,2,\dots$ and $i = 0,1,2,\dots$
2. Construct Brezenski's Interval $J_n(b)$ for the entries in the Romberg Table for Infimum and Supremum.
3. Call the resulting table as Brezenski's Table of Intervals for Infimum (BTII) and Brezenski's Table of Intervals for Supremum (BTIS)
4. Construct the range enclosure by $F(X) = [\inf(\text{BTII}), \sup(\text{BTIS})]$

Example:

$$1/(1+X) + 1/(1-X) - 2/(1 - X^2)$$

Domain $[-0.5, 0.5]$

- Range enclosure obtained with Taylor Model Order '1'

$F_2(X)$	$[-3.0888888888888908e-001, 5.6250000000000037e-001]$
$F_4(X)$	$[-9.902701254974052e-002, 1.3300000000000010e-001]$
$F_8(X)$	$[-2.822497863135687e-002, 3.275262960180347e-002]$
$F_{16}(X)$	$[-7.552813925502175e-003, 8.141258116216093e-003]$
$F_{32}(X)$	$[-1.955038747264521e-003, 2.030179217273669e-003]$
$F_{64}(X)$	$[-4.974450742398358e-004, 5.069428624799704e-004]$
$F_{128}(X)$	$[-1.254688165082416e-004, 1.266628200249031e-004]$
$F_{256}(X)$	$[-3.150705459406010e-005, 3.165673530568016e-005]$

Romberg Table for Infimum

i	k=0	k=1	k=2
1.	-3.088888888888907e-001		
2.	-9.902701254974052e-002	-2.907305377002378e-002	
3.	-2.822497863135686e-002	-4.624300658562307e-003	-1.131621642639239e-003
4.	-7.552813925502173e-003	-6.620923568839460e-004	-9.606259950132305e-005
5.	-1.955038747264521e-003	-8.911368785196998e-005	-7.259592275973409e-006
6.	-4.974450742398356e-004	-1.158051656494061e-005	-5.043492382221307e-007
7.	-1.254688165082416e-004	-1.476730597710230e-006	-3.333260239160436e-008
8.	-3.150705459406010e-005	-1.864672893329348e-007	-2.143959564749814e-009
	k=3	k=4	k=5
4.	-2.702532995879532e-005		
5.	-1.339391794283434e-006	-5.108131438153084e-007	
6.	-5.399970237204542e-008	-1.253544134264580e-008	-4.626271462127342e-009
7.	-1.931493336235940e-009	-2.518736899195051e-010	-5.689642559051633e-011
8.	-6.471670962617721e-011	-4.498108767797779e-012	-5.715122415802028e-013
	k=6	k=7	
7.	-2.091709459416338e-011		
8.	-1.280089868641703e-013	-4.648316095319303e-014	

BTII (True infimum = 0)

i	k=1	k=2	k=3
1.	[-2.35e-002, 1.42e-002]		
2.	[-6.14e-003, 4.85e-003] [-5.48e-004, 3.56e-004]		
3.	[-1.58e-003, 1.40e-003] [-7.27e-005, 5.82e-005] [-6.07e-006, 3.39e-006]		
4.	[-4.00e-004, 3.77e-004] [-9.36e-006, 8.35e-006] [-4.14e-007, 3.06e-007]		
5.	[-1.00e-004, 9.77e-005] [-1.18e-006, 1.12e-006] [-2.70e-008, 2.31e-008]		
6.	[-2.52e-005, 2.48e-005] [-1.49e-007, 1.45e-007] [-1.72e-009, 1.59e-009]		
7.	[-6.32e-006, 6.27e-006] [-1.87e-008, 1.84e-008] [-1.09e-010, 1.05e-010]		
	k=4	k=5	k=6
4.	[-4.57e-008, 2.06e-008]		
5.	[-1.59e-009, 1.09e-009] [-2.12e-010, 9.90e-011]		
6.	[-5.26e-011, 4.36e-011] [-3.71e-012, 2.56e-012] [-4.82e-013, 2.26e-013]		
7.	[-1.69e-012, 1.54e-012] [-6.12e-014, 5.10e-014] [-4.21e-015, 2.92e-015]		
	k=7		
7.	[-5.42e-016, 2.56e-016]		

Overestimation and Ratio for BTII

Overestimation

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1.	3.7761e-002						
2.	1.1025e-002	9.0565e-004					
3.	2.9855e-003	1.3097e-004	9.4723e-006				
4.	7.7738e-004	1.7722e-005	7.2056e-007	6.6343e-008			
5.	1.9839e-004	2.3094e-006	5.0242e-008	2.6874e-009	3.1196e-010		
6.	5.0113e-005	2.9492e-007	3.3268e-009	9.6350e-011	6.2826e-012	7.0961e-013	
7.	1.2593e-005	3.7266e-008	2.1419e-010	3.2321e-012	1.1233e-013	7.1359e-015	7.9918e-016
8.	3.1566e-006	4.6838e-009	1.3590e-011	1.0471e-013	1.8816e-015	6.3738e-017	4.0121e-018

Overestimation Ratio

	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1.	3.4250e+000						
2.	3.6929e+000	6.9151e+000					
3.	3.8404e+000	7.3901e+000	1.3146e+001				
4.	3.9185e+000	7.6737e+000	1.4342e+001	2.4687e+001			
5.	3.9588e+000	7.8308e+000	1.5102e+001	2.7892e+001	4.9656e+001		
6.	3.9793e+000	7.9138e+000	1.5532e+001	2.9811e+001	5.5932e+001	9.9442e+001	
7.	3.9896e+000	7.9565e+000	1.5761e+001	3.0868e+001	5.9698e+001	1.1196e+002	1.9919e+002

$O(h)$ $O(2^{k+1})=4$ $O(2^{k+1})=8$ $O(2^{k+1})=16$ $O(2^{k+1})=32$ $O(2^{k+1})=64$ $O(2^{k+1})=128$ $O(2^{k+1})=256$

Romberg Table for Supremum

i	k=0	k=1	k=2
1.	5.625000000000037e-001		
2.	1.330000000000010e-001	-1.016666666666657e-002	
3.	3.275262960180345e-002	-6.631605309290536e-004	6.944832027477349e-004
4.	8.141258116216091e-003	-6.253237897969628e-005	2.327164272735477e-005
5.	2.030179217273668e-003	-6.847082373806273e-006	1.107959998463727e-006
6.	5.069428624799703e-004	-8.025891179289156e-007	6.090991862499258e-008
7.	1.266628200249031e-004	-9.719412678597180e-008	3.576586234448741e-009
8.	3.165673530568015e-005	-1.195960072749292e-008	2.167601380040589e-010
	k=3	k=4	k=5
4.	-2.147579460733724e-005		
5.	-3.696188501290093e-007	3.112255291357722e-007	
6.	-8.893420030923049e-009	2.742884165789409e-009	-2.153665754369047e-009
7.	-2.456359249208487e-010	3.332485269212552e-011	-9.684025293545645e-012
8.	-7.228268425586576e-012	4.623011387767161e-013	-5.932666365739183e-014
	k=6	k=7	
7.	7.197720604686650e-012		
8.	1.645836492440543e-014	-1.170344778054454e-014	

BTIS (True supremum = 0)

i	k=1	k=2	k=3
1.	[-2.73e-002, 2.60e-002]		
2.	[-6.62e-003, 6.50e-003] [-4.53e-005, 9.19e-005]		
3.	[-1.63e-003, 1.62e-003] [-5.25e-006, 7.47e-006] [-1.55e-006, 8.12e-007]		
4.	[-4.07e-004, 4.05e-004] [-6.29e-007, 7.51e-007] [-6.47e-008, 4.69e-008]		
5.	[-1.01e-004, 1.01e-004] [-7.70e-008, 8.41e-008] [-3.30e-009, 2.81e-009]		
6.	[-2.53e-005, 2.53e-005] [-9.52e-009, 9.95e-009] [-1.86e-010, 1.71e-010]		
7.	[-6.33e-006, 6.33e-006] [-1.18e-009, 1.21e-009] [-1.10e-011, 1.06e-011]		
	k=4	k=5	k=6
4.	[-6.56e-009, 1.20e-008]		
5.	[-1.89e-010, 2.56e-010] [-4.40e-011, 2.47e-011]		
6.	[-5.69e-012, 6.61e-012] [-4.76e-013, 3.57e-013] [-4.41e-014, 7.70e-014]		
7.	[-1.74e-013, 1.87e-013] [-6.19e-015, 5.37e-015] [-3.19e-016, 4.22e-016]		
	k=7		
7.	[-6.41e-017, 3.87e-017]		

Overestimation and Ratio for BTIS

Overestimation

i	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1.	5.3465e-002						
2.	1.3126e-002	1.3729e-004					
3.	3.2592e-003	1.2728e-005	2.3641e-006				
4.	8.1239e-004	1.3816e-006	1.1169e-007	1.8618e-008			
5.	2.0282e-004	1.6123e-007	6.1156e-009	4.4634e-010	6.8814e-011		
6.	5.0670e-005	1.9482e-008	3.5838e-010	1.2305e-011	8.3460e-013	1.2126e-013	
7.	1.2663e-005	2.3946e-009	2.1698e-011	3.6175e-013	1.1568e-014	7.4224e-016	1.0296e-016
8.	3.1653e-006	2.9682e-010	1.3349e-012	1.0969e-014	1.7051e-016	5.1641e-018	3.2354e-019

Overestimation Ratio

	k=1	k=2	k=3	k=4	k=5	k=6	k=7
1.	4.0732e+000						
2.	4.0273e+000	1.0786e+001					
3.	4.0119e+000	9.2126e+000	2.1168e+001				
4.	4.0056e+000	8.5689e+000	1.8263e+001	4.1713e+001			
5.	4.0027e+000	8.2759e+000	1.7064e+001	3.6273e+001	8.2451e+001		
6.	4.0013e+000	8.1359e+000	1.6517e+001	3.4015e+001	7.2149e+001	1.6336e+002	
7.	4.0007e+000	8.0675e+000	1.6255e+001	3.2979e+001	6.7843e+001	1.4373e+002	3.1823e+002

$O(h)$ $O(2^{k+1})=4$ $O(2^{k+1})=8$ $O(2^{k+1})=16$ $O(2^{k+1})=32$ $O(2^{k+1})=64$ $O(2^{k+1})=128$ $O(2^{k+1})=256$

Range Enclosure

i	k=1	k=2	k=3
1.	[-2.35e-002, 2.60e-002]		
2.	[-6.17e-003, 6.50e-003] [-5.48e-004, 9.19e-005]		
3.	[-1.58e-003, 1.62e-003] [-7.27e-005, 7.47e-006] [-6.07e-006, 8.12e-007]		
4.	[-4.00e-004, 4.05e-004] [-9.36e-006, 7.51e-007] [-4.14e-007, 4.69e-008]		
5.	[-1.00e-004, 1.01e-004] [-1.18e-006, 8.41e-008] [-2.70e-008, 2.81e-009]		
6.	[-2.52e-005, 2.53e-005] [-1.49e-007, 9.95e-009] [-1.72e-009, 1.71e-010]		
7.	[-6.32e-006, 6.33e-006] [-1.87e-008, 1.21e-009] [-1.09e-010, 1.06e-011]		
	k=4	k=5	k=6
4.	[-4.57e-008, 1.20e-008]		
5.	[-1.59e-009, 2.56e-010] [-2.12e-010, 2.47e-011]		
6.	[-5.26e-011, 6.61e-012] [-3.71e-012, 3.57e-013] [-4.82e-013, 7.70e-014]		
7.	[-1.69e-012, 1.87e-013] [-6.12e-014, 5.37e-015] [-4.21e-015, 4.22e-016]		
	k=7		
7.	[-5.42e-016, 3.87e-017]		

Overestimation Ratio

i	k=1	k=2	k=3	k=4	k=5	k=6
1.	3.9111					
2.	3.9553	7.9886				
3.	3.9776	7.9287	14.9340			
4.	3.9888	7.9521	15.4440	31.1862		
5.	3.9944	7.9734	15.7176	31.2385	58.3677	
6.	3.9972	7.9861	15.8580	31.5522	61.0893	120.8535
$O(h)$	$O(4)$	$O(8)$	$O(16)$	$O(32)$	$O(64)$	$O(128)$

Numerical Examples (1 Dim)

Examples from ' *Taylor forms – use and limits* ' by Arnold Neumaier, 2000 MSC Classification, Oct. 2002.

1. **Example 1** : $1/(1+X) + 1/(1-X) - 2/(1 - X^2)$
Domain $[-0.5, 0.5]$
2. **Example 2** : $1/(1+X)^4$
Domain $[-0.5, 0.5]$
3. **Example 3** : $(1.0/5.0) * (\sin(x) + \sin(2.0*x) + \sin(3.0*x) + \sin(4.0*x) + \sin(5.0*x))$
Domain $[-0.25, 0.25]$
4. **Example 4** : $(X - 1/4)^2$
Domain $[-0.5, 0.5]$
5. **Example 5** : $1/(1-X) - 1/(2-X)$
Domain $[-0.5, 0.5]$
6. **Example 6** : Gritton's
Domain $[0.4, 2.4]$ (from K. Makino & M. Berz – IJPAM, 2003)

Numerical Examples: 3 Dim

Examples from the paper by K. Makino & M. Berz – IJPAM, 2003.

7. Example 7 :

$$f = ((4.0d0 * \tan(3.0d0 * x_2)) / (3.0d0 * x_1 + x_1 * \sqrt{6.0d0 * x_1 / (-7.0d0 * (x_1 - 8.0d0))})) \\ - 120.0d0 - 2.0d0 * x_1 - 7.0d0 * x_3 * (1.0d0 + 2.0d0 * x_2) - \sinh(0.5d0 + \\ 6.0d0 * x_2 / (8.0d0 * x_2 + 7.0d0)) + (3.0d0 * x_2 + 13.0d0)^2 / (3.0d0 * x_3) - \\ 20.0d0 * x_3 * (2.0d0 * x_3 - 5.0d0) + (5.0d0 * x_1 * \tanh(0.9d0 * x_3)) / \sqrt{5.0d0 * x_2} - \\ 20.0d0 * x_2 * \sin(3.0d0 * x_3);$$

Domain : [1.75, 2.25], [0.75, 1.25], [0.75, 1.25]

8. Example 8 :

$$f(x_1, x_2, x_3) = f(x_1, x_2, x_3) + \sum_{i=1}^{10} f(x_1, x_2, x_3) - f(x_1, x_2, x_3)$$

where $f(x_1, x_2, x_3)$ is the function specified in Example 7.

Domain : [1.75, 2.25], [0.75, 1.25], [0.75, 1.25]

Numerical Examples: 6 Dim

Example from the paper by K. Makino & M. Berz
IJPAM, 2003.

9. Example 9: Trigonometric function

$$f(x) = \sum_{i=1}^6 f_i(x)^2, \quad f_i(x) = 6 - \sum_{j=1}^6 \cos x_j + i(1 - \cos x_j) - \sin x_j$$

Domain $[0.75, 2.75]^6$

Numerical Examples: 10 Dim

Example from the paper by K. Makino & M. Berz –
IJPAM, 2003.

10. Example 10: Trigonometric function

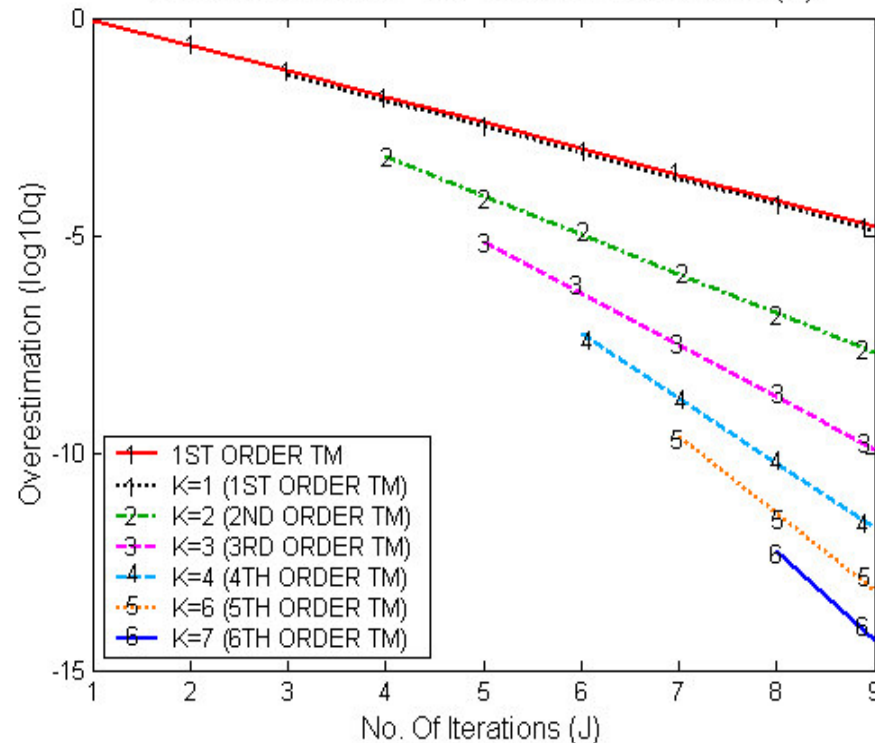
$$f(x) = \sum_{i=1}^{10} f_i(x)^2, \quad f_i(x) = 10 - \sum_{j=1}^{10} \cos x_j + i(1 - \cos x_j) - \sin x_j$$

Domain $[0.75, 2.75]^{10}$

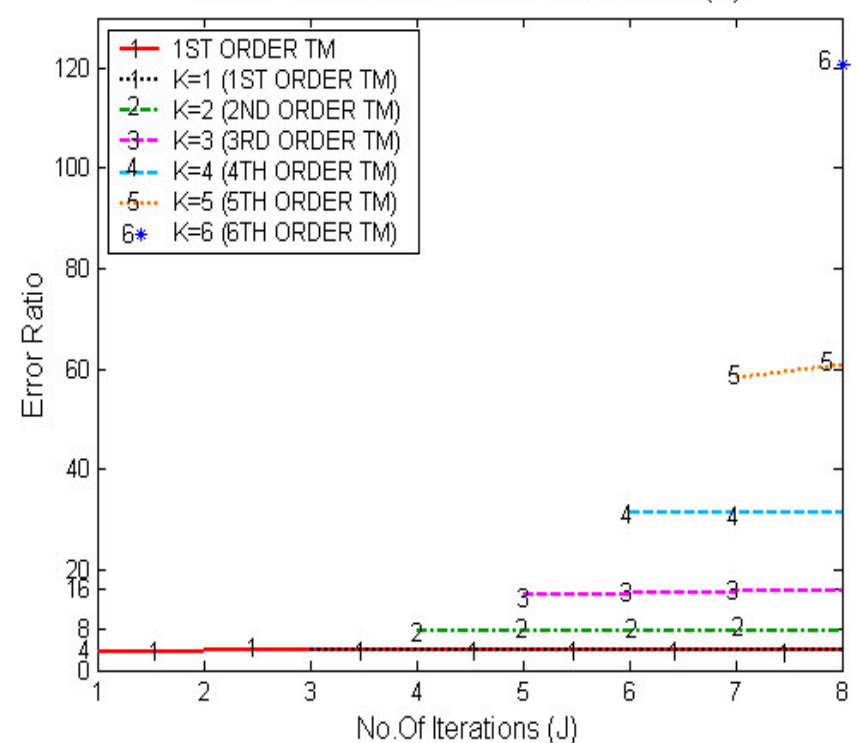
Example 1: $1/(1+X) + 1/(1-X) - 2/(1 - X^2)$ Domain $[-0.5, 0.5]$

(From '*Taylor forms – use and limits*' by Arnold Neumaier, 2002)

Overestimation vs. No. of Iterations (J)



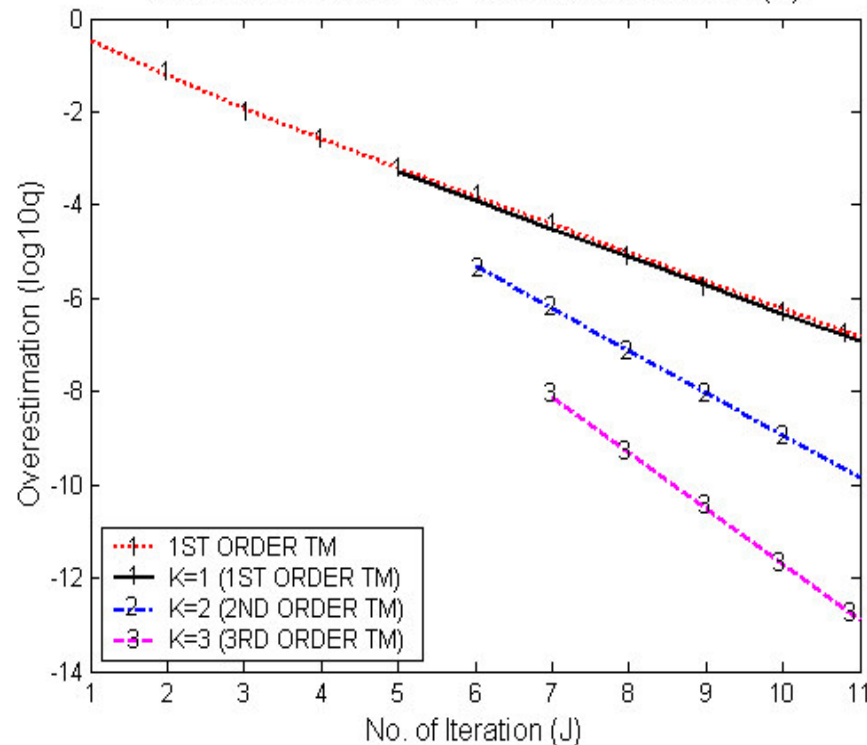
Error Ratio vs. No. of Iterations(J)



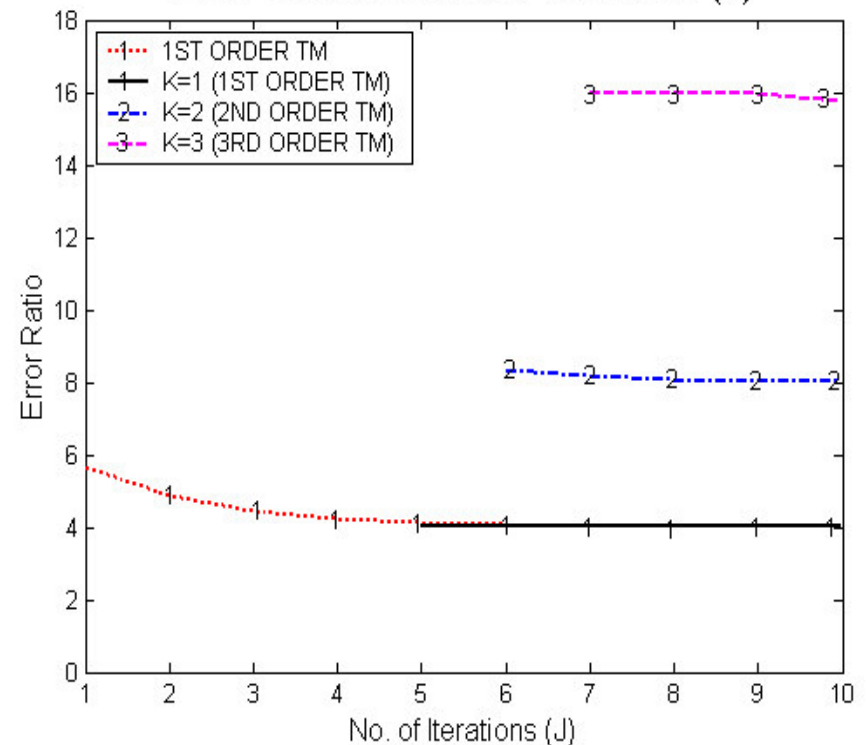
Example 2 : $1/(1+X)^4$ Domain $[-0.5, 0.5]$

(From '*Taylor forms – use and limits*' by Arnold Neumaier, 2002)

Overestimation vs. No. of Iterations (J)



Error Ratio vs. No. of Iterations (J)



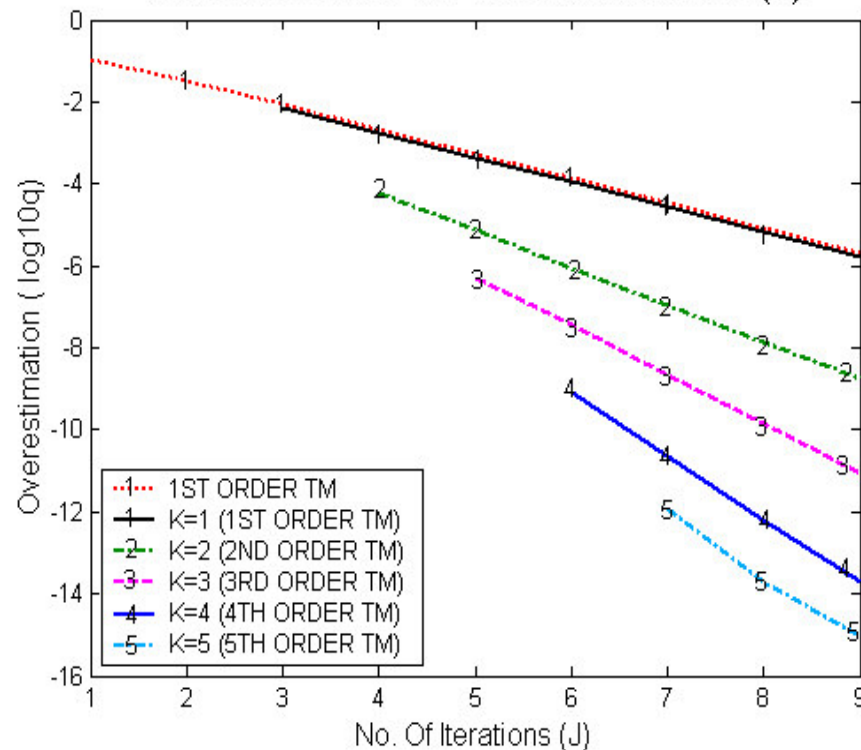
Example 3:

$$F(x) = (1.0/5.0) * (\sin(x) + \sin(2.0*x) + \sin(3.0*x) + \sin(4.0*x) + \sin(5.0*x))$$

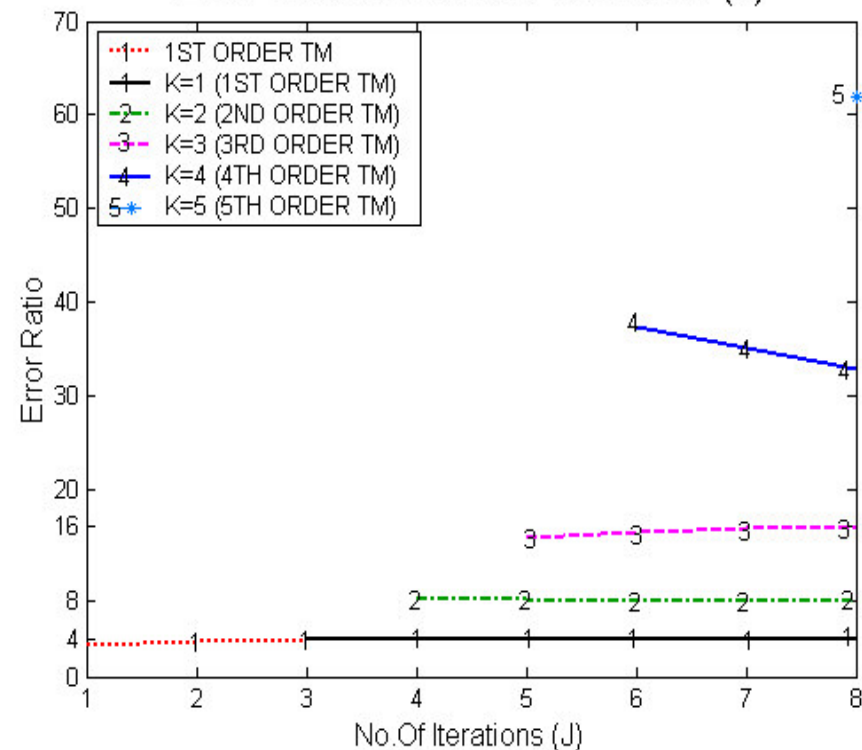
Domain [-0.25, 0.25]

(From '*Taylor forms – use and limits*' by Arnold Neumaier, 2002)

Overestimation vs. No. of Iterations (J)



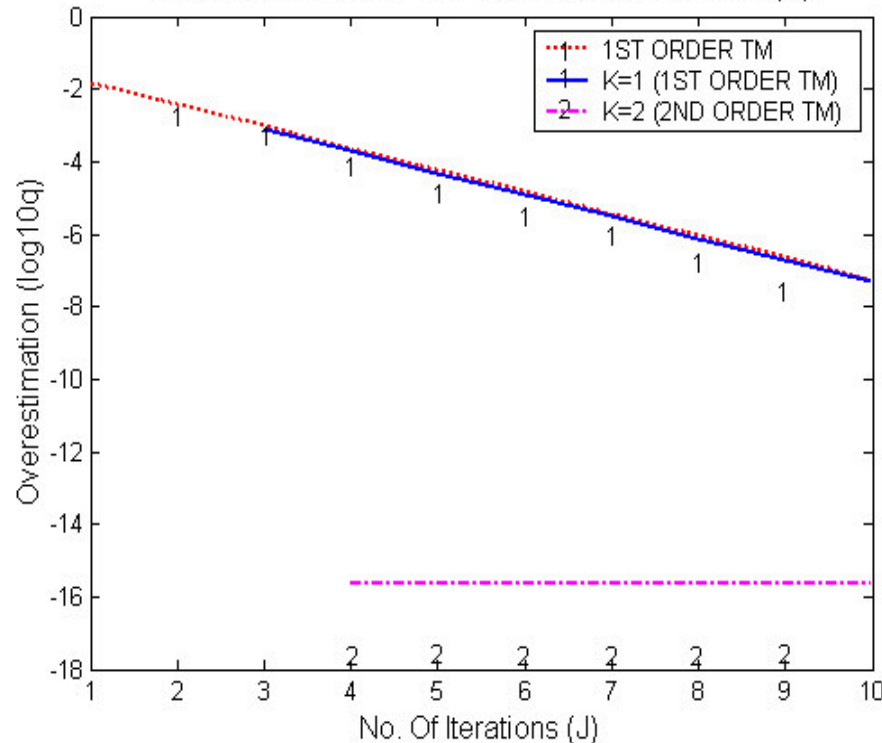
Error Ratio vs. No. of Iterations (J)



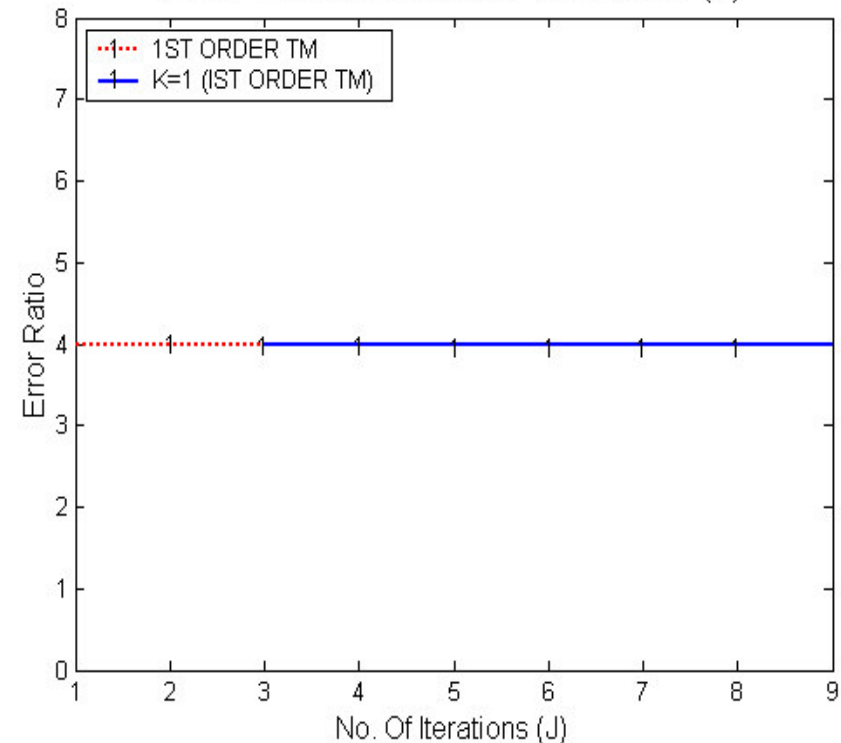
Example 4 : $(X - 1/4)^2$ Domain $[-0.5, 0.5]$

(From '*Taylor forms – use and limits*' by Arnold Neumaier, 2002)

Overestimation vs. No. of Iterations (J)



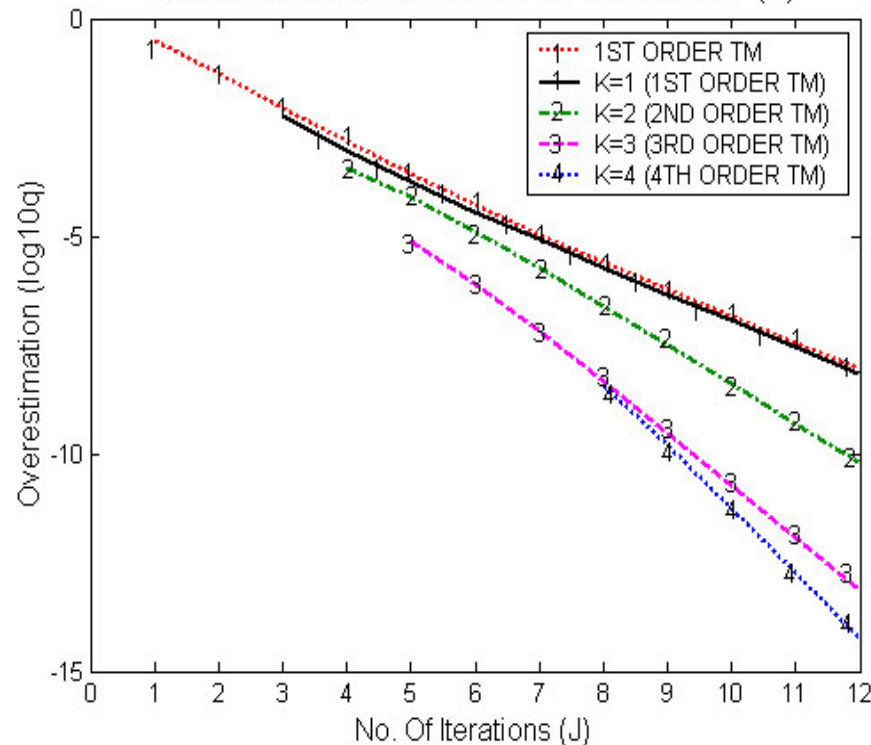
Error Ratio vs. No. of Iterations (J)



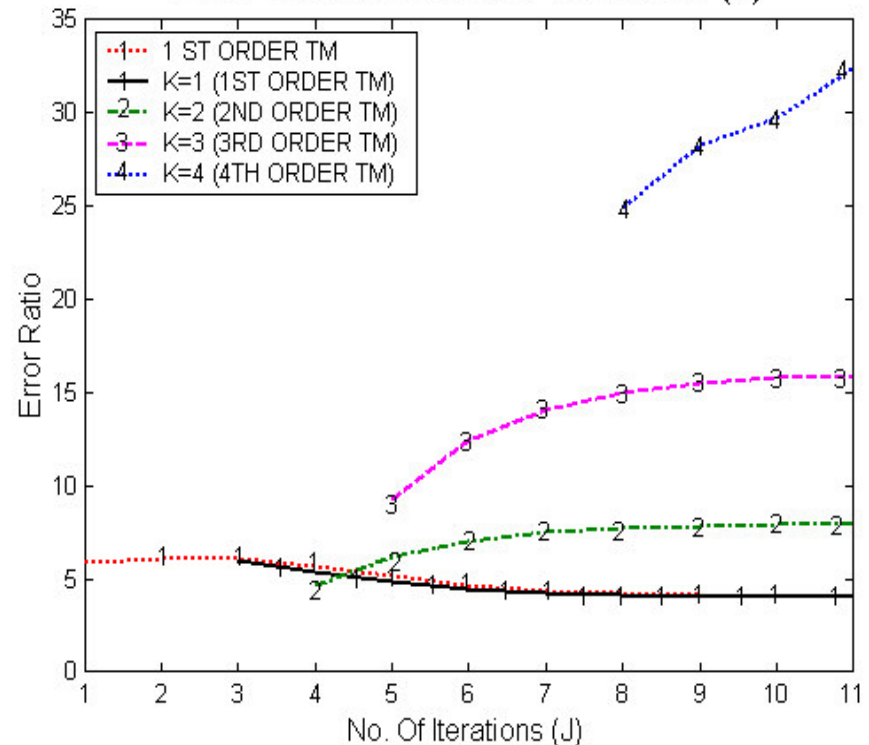
Example 5 : $1/(1-X) - 1/(2-X)$ Domain $[-0.5, 0.5]$

(From '*Taylor forms – use and limits*' by Arnold Neumaier, 2002)

Overestimation vs. No. of Iterations (J)



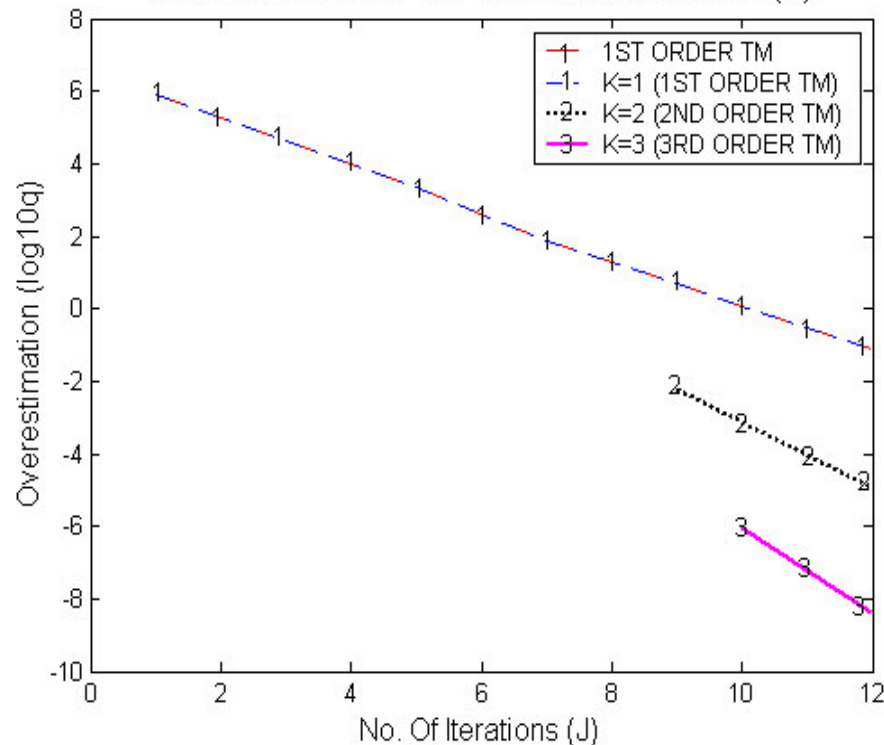
Error Ratio vs. No. of Iterations (J)



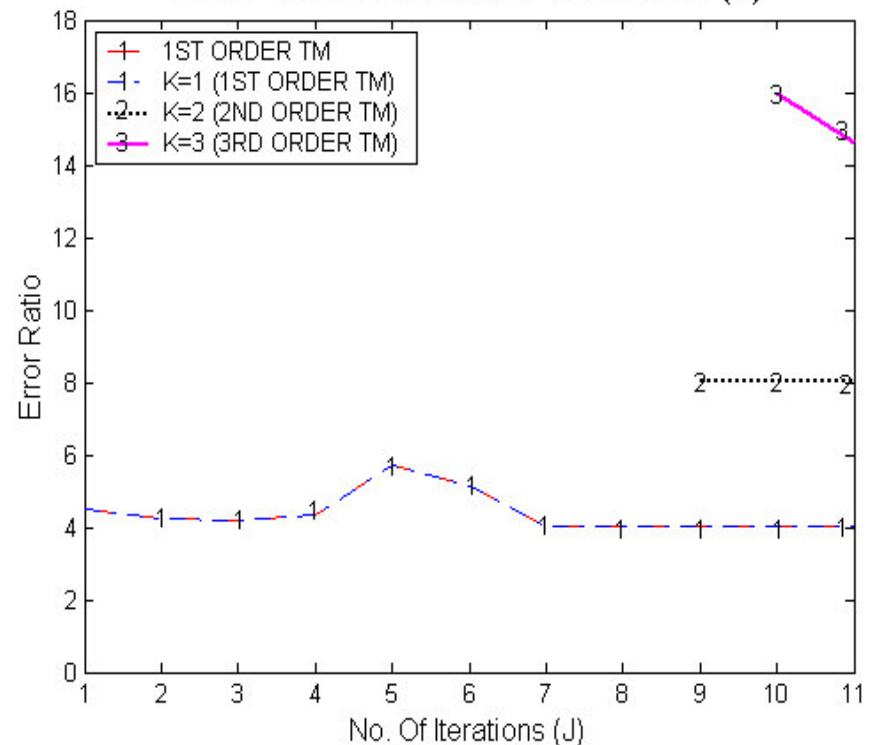
Example 6 : Gritton Domain [0.4, 2.4]

From paper by K. Makino & M. Berz – IJPAM, 2003

Overestimation vs. No. of Iterations (J)



Error Ratio vs. No. of Iterations (J)

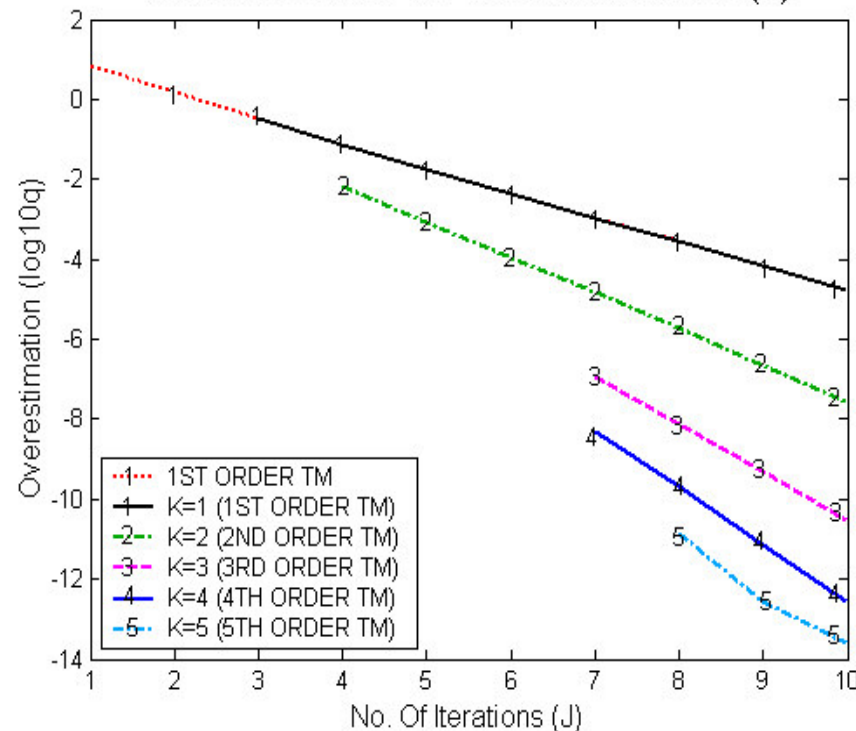


Example 7: Trigonometric 3Dim.

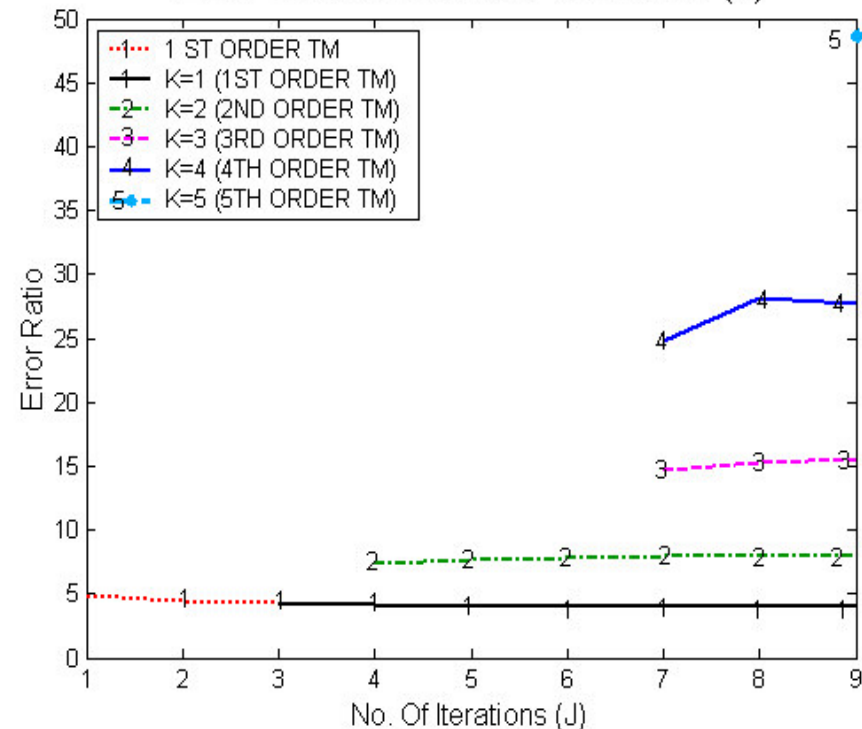
Domain $[1.75, 2.25] [0.75, 1.25][0.75, 1.25]$

From paper by K. Makino & M. Berz – IJPAM, 2003

Overestimation vs. No. of Iterations (J)



Error Ratio vs. No. of Iterations (J)



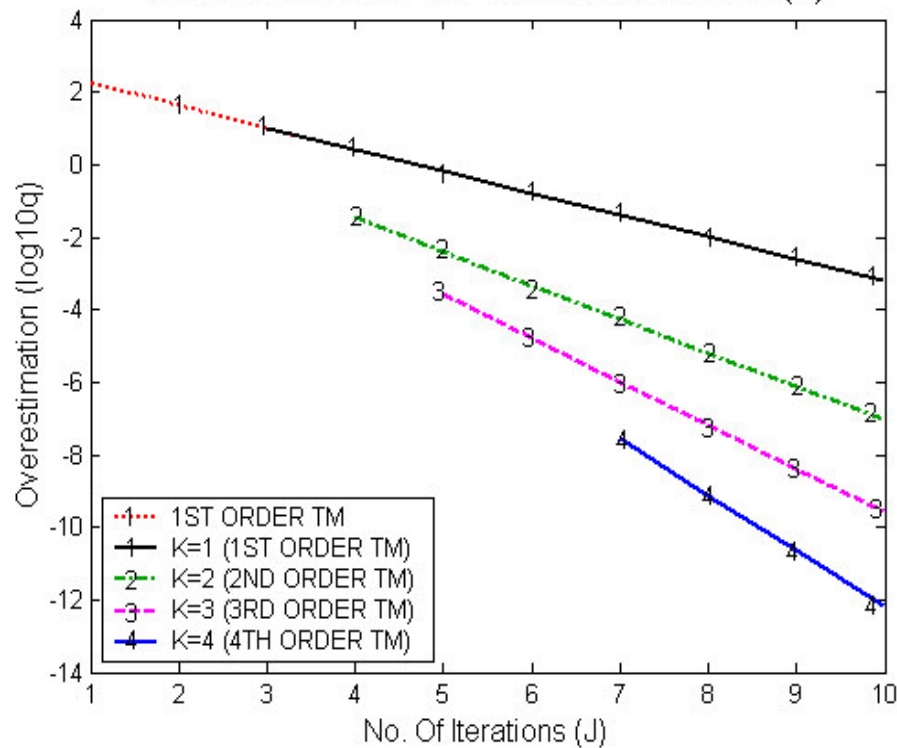
Example 8: Trigonometric 3Dim.

$$f(x_1, x_2, x_3) = f(x_1, x_2, x_3) + \sum_{i=1}^{10} f(x_1, x_2, x_3) - f(x_1, x_2, x_3)$$

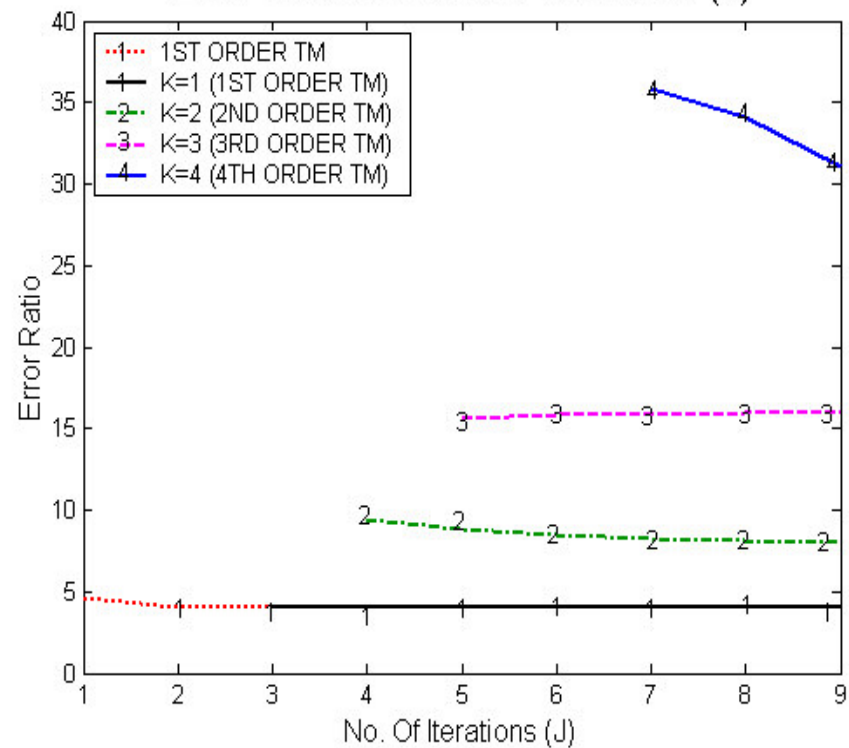
Domain [1.75, 2.25] [0.75, 1.25] [0.75, 1.25]

From paper by K. Makino & M. Berz – IJPAM, 2003

Overestimation vs. No. of Iterations (J)



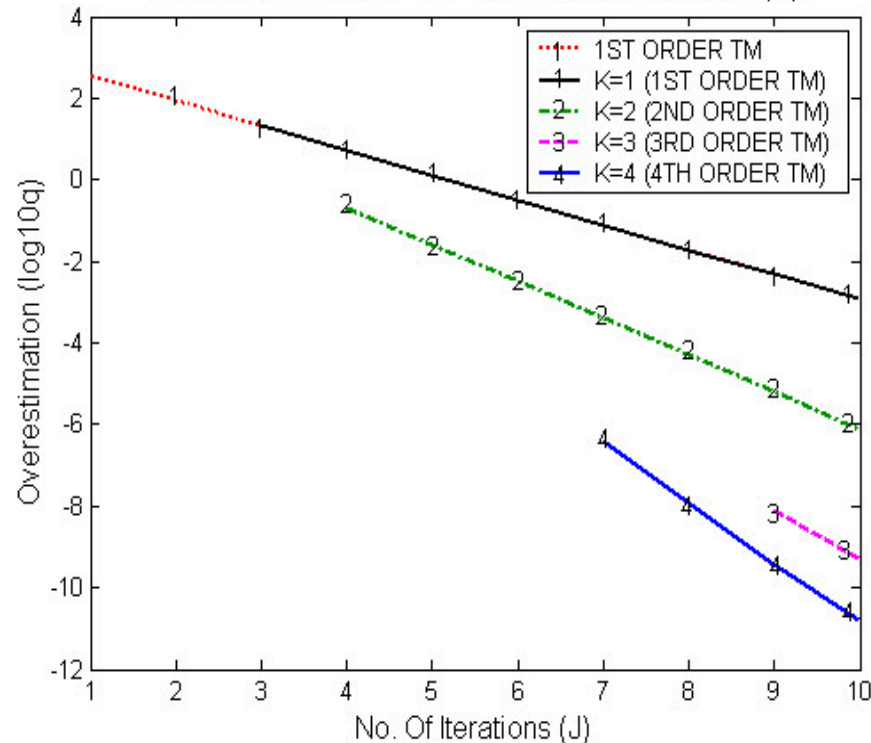
Error Ratio vs. No. of Iterations (J)



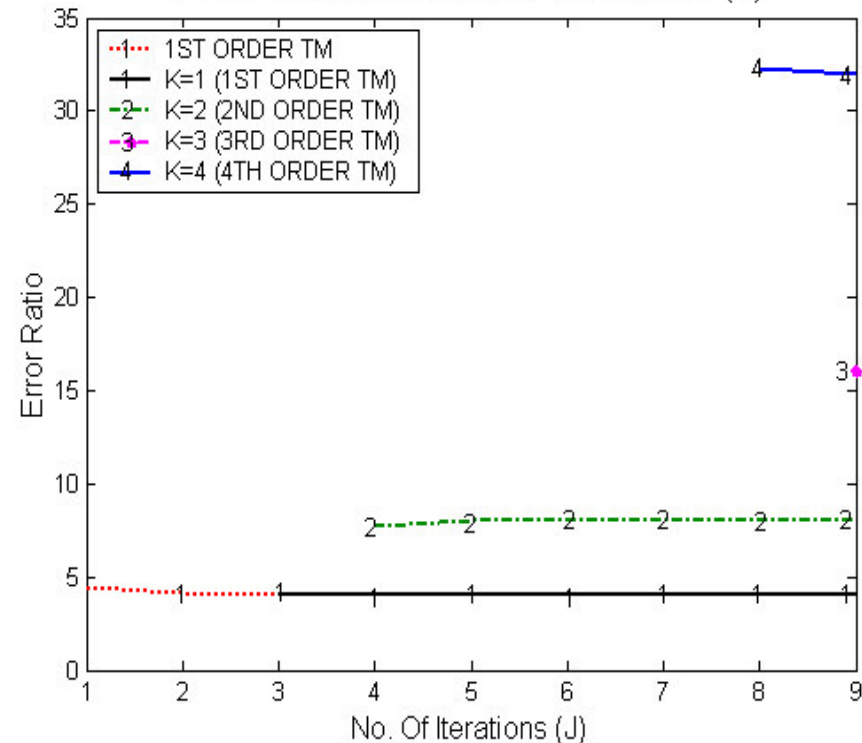
Example 9: Trigonometric 6 Dim. Domain $[0.75, 2.75]^6$

From paper by K. Makino & M. Berz – IJPAM, 2003

Overestimation vs. No. of Iterations (J)



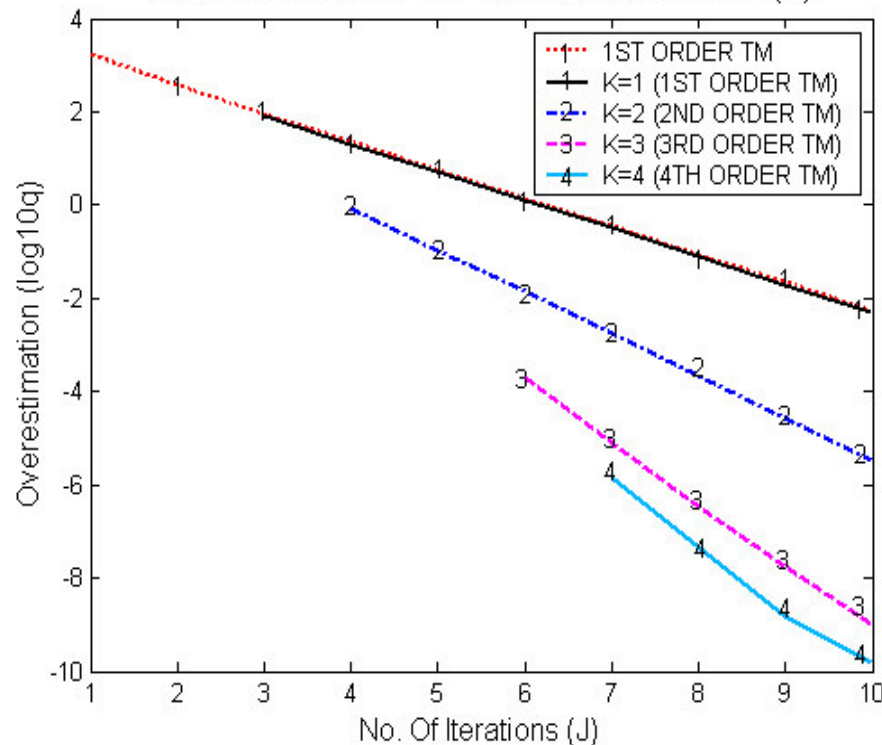
Error Ratio vs. No. of Iterations (J)



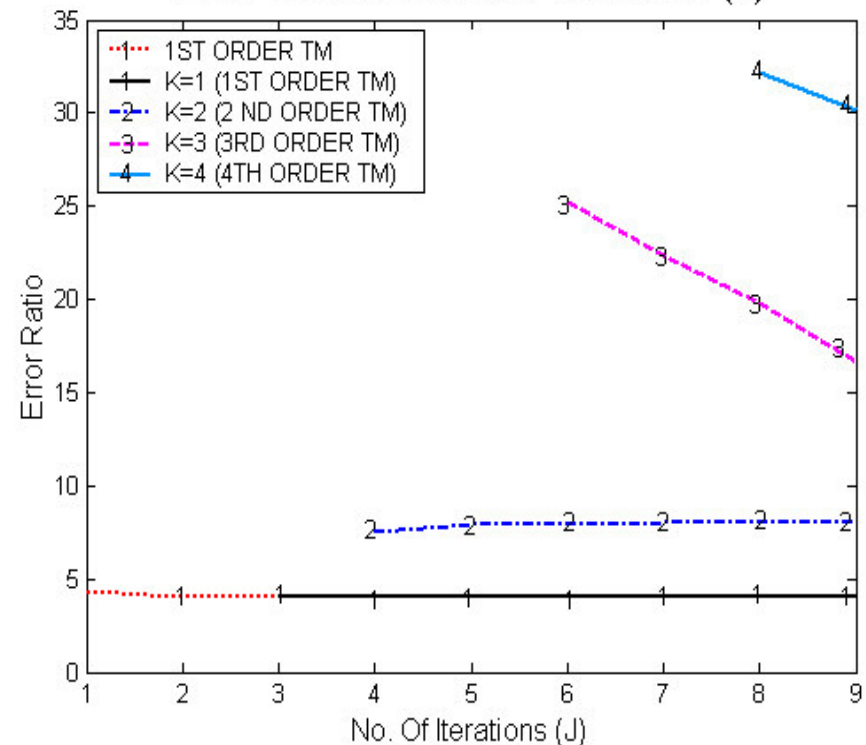
Example 10: Trigonometric 10 Dim. Domain $[0.75, 2.75]^{10}$

From paper by K. Makino & M. Berz – IJPAM, 2003

Overestimation vs. No. of Iterations (J)



Error Ratio vs. No. of Iterations (J)



Conclusions

- We could obtain a *Higher Order Convergence by Extrapolation* without actually going for higher order Taylor Model.
- Exact range of the polynomial is computed using simple *NIE*. No need to go for LDB, QDB or Bernstein method.
- Requires *almost negligible computational effort !*

Future Scope

- The Extrapolated Taylor Model Method can be applied to 2nd and Higher Order Taylor Model to obtain still higher orders of convergence.
- Explore other methods for computation of Rigorous Error Bounds.

The background is a solid blue color with a subtle gradient. A thin, light blue curved line starts from the top left and arcs towards the right. A larger, semi-transparent blue triangular shape is positioned on the right side, pointing towards the center.

THANK YOU !