Computer Assisted Proofs for the Restricted 3-Body Problem

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2-body problem with mass 1 and period 2π

clockwise circular orbit

$$x_1(t) = 2^{-2/3}(\cos t, -\sin t)$$
$$x_2(t) = 2^{-2/3}(-\cos t, \sin t)$$

The Lagrangian for the R3BP in sidereal coordinates (fixed) is

$$L(w) = \frac{|\dot{w}|^2}{2} - V(w)$$

with the Keplerian potential

$$V(w) = -\frac{1}{|w - x_1|} - \frac{1}{|w - x_2|}$$

$$R_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$w = R_t v \qquad c = (2^{-2/3}, 0)$$

$$\dot{w} = R_t(\dot{v} - Jv)$$

Lagrangian in synodical coordinates

$$L(v) = \frac{1}{2}|\dot{v} - Jv|^2 - V(v)$$

or

$$L(v) = \frac{1}{2}|\dot{v}|^2 - (\dot{v}, Jv) + \Omega(v)$$

$$\Omega(v) = \frac{|v|^2}{2} + \frac{1}{|v-c|} + \frac{1}{|v+c|} - \frac{2}{|c|}$$

This system admits an invariant called the Jacobi integral:

$$H(v) = |\dot{v}|^2 - 2\Omega(v)$$

Equations of motion:

$$\begin{cases} \ddot{x} + 2\dot{y} = \Omega_x \\ \ddot{y} - 2\dot{x} = \Omega_y \end{cases}$$

Poincaré section

$H(x, y) = h$ intersected with $y = 0$ ($\dot{y} > 0$)

The phase space can be described by the plane $(x, \dot{x})$

Rigorous numerics for ODE's

The main problem one faces when trying to make a rigorous integration of an ordinary differential equation is the wrapping effect.

Let R_α be the rotation matrix and let $[x] = [-\delta, \delta]^2$.

Consider now the evaluation in interval arithmetic of $R_\alpha[x]$. If $|\alpha| < \pi/2$ the result is

$$R_\alpha[x] = (\cos(\alpha) + |\sin(\alpha)|)[-\delta, \delta]^2$$

Hence the computer realization of an isometry has a growth factor

$$\cos(\alpha) + |\sin(\alpha)| > 1 \text{ (when } \alpha \neq 0).$$

The wrapping effect was controlled by a version of the Lohner algorithm.

Consider the ordinary differential equation

$$x' = f(x), \quad x \in \mathbb{R}^n$$

inducing a dynamical system which we denote by φ.

Let $\Phi(h, x)$ be a Taylor method for solving the above equation ($h \in \mathbb{R}$ and $x \in \mathbb{R}^n$). We have

$$\varphi(h, x) \in \Phi(h, x) + [w]$$

where $[w]$ is the remainder term computed over the set containing $\varphi([0, h], x)$.

Our aim is to estimate $[\varphi(h, x_k + [r_k])]$, where x_k is a vector and $[r_k]$ is a set. One of the basic ideas of Lohner is to use the Jacobian matrix for the explicitly known function $\Phi(h, \cdot)$ instead of trying to estimate the Jacobian matrix for $\varphi(h, \cdot)$.

We proceed as follows:

Step 1 We find a rough enclosure D for $[\varphi([0,h], x_k + [r_k])]$.

Step 2 We compute the error term $[w] := [w](D)$ on D, by evaluating $x^{(r+1)}$ on D using an algorithm for the automatic computation of the time derivatives.

Step 3 We have

$$[\varphi(h, x_k + [r_k])] \subset [\Phi(h, x_k + [r_k])] + [w]\,.$$

Consider $\Delta x \in [r_k]$. We apply the Lagrange mean value theorem to $\Phi_i(h, x_k + \Delta x)$ to obtain

$$\Phi_i(h, x_k + \Delta x) = \Phi_i(h, x_k) + \sum_{i=1}^{n} \frac{\partial \Phi_i(h, x_k + \theta \Delta x)}{\partial x_j} \Delta x_j$$

for every $i = 1, \ldots, n$, $\Delta x \in [r_k]$ and $\theta = \theta(i, \Delta x) \in [0, 1]$.

Let us denote by $[A_k]$ an array whose entries are given by

$$[A_{k,ij}] := \left[\frac{\partial \Phi_i(h, x_k + [r_k])}{\partial x_j} \right]$$

Since $\Delta x \in [r_k]$ we obtain

$$[\varphi(h, x_k + [r_k])] \subset \Phi(h, x_k) + [A_k] \cdot [r_k] + [w]$$

Step 4 Let

$$\begin{aligned} x_{k+1} &= \text{the middle point of } (\langle \Phi(h, x_k) \rangle + [w]) \\ [z_{k+1}] &= (\langle \Phi(h, x_k) \rangle + [w]) - x_{k+1} \\ [r_{k+1}] &= [A_k][r_k] + [z_{k+1}]. \end{aligned}$$

It follows easily that

$$[\varphi(h, x_k + [r_k])] \subset x_{k+1} + [r_{k+1}].$$

Iterating this procedure we obtain

$$[\varphi(nh, x_0 + [r_0])] \subset x_n + [r_n]$$

where

$$[r_{k+1}] = [A_k][r_k] + [z_{k+1}].$$

Let us remark that in the above discussion we used ideal mathematical arithmetic operations, but in numerical calculations we have to prevent the wrapping effect, therefore we have to find an efficient algorithm to evaluate the recursively defined sequence of $[r_k]$. More precisely we need to avoid the multiplication $[A_k][r_k]$, which is the source of the wrapping effect.

We used the method which Lohner calls inner enclosure, which is very efficient in the cases when the initial size of $[r_0]$ is large when compared with other errors such as round–off errors and the rest term in the Taylor method.

In this approach we never compute $[r_{k+1}]$ explicitly, but we use an additional variable $[\tilde{r}_{k+1}]$ recursively defined by setting

$$[\tilde{r}_0] = \{0.0\}^n$$

and

$$[\tilde{r}_{k+1}] = [A_k][\tilde{r}_k] + [z_{k+1}] + ([A_k]C_k - C_{k+1})[r_0]\,,$$

where $C_0 := I$ and C_{k+1} is the middle point of $[A_k]C_k$.

By these definitions we have

$$[r_{k+1}] = C_{k+1}[r_0] + [\tilde{r}_{k+1}]\,,$$

and due to the splitting of $[r_k]$ into $C_k[r_0]$ and $[\tilde{r}_k]$, the wrapping effect is present only in the term $[\tilde{r}_k]$, which is in our computations much smaller than the other one arising from $[r_0]$.

The Lohner method and the analytic expansion method

It is quite hard to compare these method. The only point in common is that they yield approximate solutions of differential equations with rigorous bounds on the errors.

The Lohner method applies to initial value problems, and it is particularly suitable for searching both periodic orbits and symbolic dynamics.

The analytic expansion method in Fourier polynomials is extremely efficient when looking for periodic solutions, it is very fast and provides more information on the solutions.