Study of Validated Inclusion Functions For a Test Example with Variable Dependency

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In this note we show results of utilizing various validated inclusion functions for the case of a function that can easily be studied by hand for the various methods under consideration, yet can illustrate many of the relevant points. We use an approximation of the function $\cos(x)$ by its power series of order 60; so

$$f(x) = \sum_{i=0}^{30} (-1)^i \frac{x^{2i}}{(2i)!}.$$

For the domain $[0, 4\pi]$, this power series represents the cos function to an accuracy of better than 10^{-15}, which is sufficient for work in conventional double precision. Although of course this is one of the worst ways to obtain validated bounds for the cos function, this function is useful for comparisons of bounding methods because it has the following useful features:

(1) Properties of the function are well known
(2) Dependency increases with x from very small to very large
(3) Periodicity allows the study of the same functional behavior with various amounts of dependency
(4) Study at points with both non-stationary and stationary points is possible

The results are shown for the expansion points $x_0 = n \cdot \pi/4$ for $n = 0, ..., 16$. For each of these points, domains are chosen as $x_0 + [-2^{-j}, 2^{-j}]$ for $j = 1, ..., 8$. For each of the values of j, we show relative overestimation $q =$ (computed range - true range) /(true range) as well as the empirical approximation order (EAO) calculated as $1 + \log_2(q_j/q_{j+1})$ for the various methods. EAOs are only calculated until the floor of machine precision is used. We also compute the average empirical approximation order (AEAO) of each method, calculated as the average over j of the EAOs. In all figures, the left shows the results obtained by TM without LDB, i.e. only with direct interval evaluation of the Taylor polynomial, and the right shows the results using LDB.

As expected, the advantage of TM increases with both x_0 and j and reaches up to 16 orders of magnitude for the non-stationary cases in which the linear dominated bounder LDB is applicable, and 5 orders of magnitude for the stationary cases where it is not applicable. In general, the behavior of the high order TM depends mostly on the Taylor polynomial, which is the same except for the signs of coefficients for the families $P_1 = \{0, \pi, 2\pi, 3\pi, 4\pi\}$, $P_2 = \{\pi/2, 3\pi/2, 5\pi/2, 7\pi/2\}$ and $P_3 = \{\pi/4, 3\pi/4, 5\pi/4, ..., 15\pi/4\}$. Interval, MF, and CF as well as low order TM eventually suffer from the dependency problem. High order TM significantly alleviate the dependency problem. For elements of P_1, the collection of all stationary points, the LDB bounder does not offer any benefit over direct interval evaluation of the Taylor polynomial; still the higher order TM methods provide significantly sharper bounds than CF and MF because of the ability to suppress the dependency problem. For elements of P_2 and P_3, LDB provides additional significant improvements in sharpness.

There are a few interesting observations:

(1) The TM convergence order without LDB for elements of P_2 exceed that of CF, but it does not do this for elements of P_3
(2) The TM convergence order for the elements in P_1 exceed that of CF and MF
(3) It exceeds it by 2
(4) Some of the behavior of TMs at stationary points generalizes to various obvious multidimensional extensions of the example

An account for the first observation can be given based on a property of the Taylor polynomial of the cosine function at points $(2n-1)\pi/2$ for integers n, such as the elements of P_2. There are no even order terms in the Taylor polynomial of *cos* at those points. Because the default tightening method for the Taylor model bounds the linear term exactly, using interval arithmetic, the TM range enclosures around the points in group P_2 have a minimum convergence order of 3 without LDB, due to the missing second order term in the Taylor polynomial. However, at the points in P_3, the Taylor polynomial has a nonvanishing second order term, amounting to the quadratically decreasing overestimation produced by the TM range enclosures without LDB.

Similarly, the property of the Taylor polynomial of *cos* at the stationary points $n\pi$ provides an explanation for the second observation. Contrary to the points in P_2, the Taylor polynomial at the stationary points in P_1 has no odd order terms. The leading term in the Taylor polynomial, which is of the second order, is bounded exactly by the default tightening method for the TM, utilizing the square operation for interval objects. Hence, the TM range enclosures around these stationary points produce absolute convergence order of 4 without LDB, because the second lowest term in the Taylor polynomial is of the fourth order.

On the figures included, however, one observes the empirical approximation order of 3 for the TM range enclosures around the stationary points in P_1. The reason for this is that the *relative* overestimation values have been used in calculating the EAO for the range bounding methods tested, instead of the absolute overestimation. As stated above, the relative overestimation values are calculated as $q =$(computed range - true range)/(true range), and in the formula for the EAO, $1 + log_2(q_j/q_{j+1})$, the constant 1 has been added based on the assumption that the true range scales linearly with the width of the domain interval. This formula, therefore, results in the same value of EAO as that based on the absolute overestimation, if the actual range does scale linearly with the domain width. Around the stationary points in P_1, however, the actual range scales quadratically with the size of the domain. This entails that the EAO formula used in this study gives an approximation order less than the order of decrease of the absolute overestimation values by 1, for all the methods tested around the stationary points. Hence, according to the current formula for the EAO, the TM range enclosures have a convergence order of 3 around the stationary points, while the range bounds produced by the CF and MF have EAO of 1, because these methods usually yield quadratically decreasing absolute overestimation values. This explanation is also consistent with the third observation posed above.

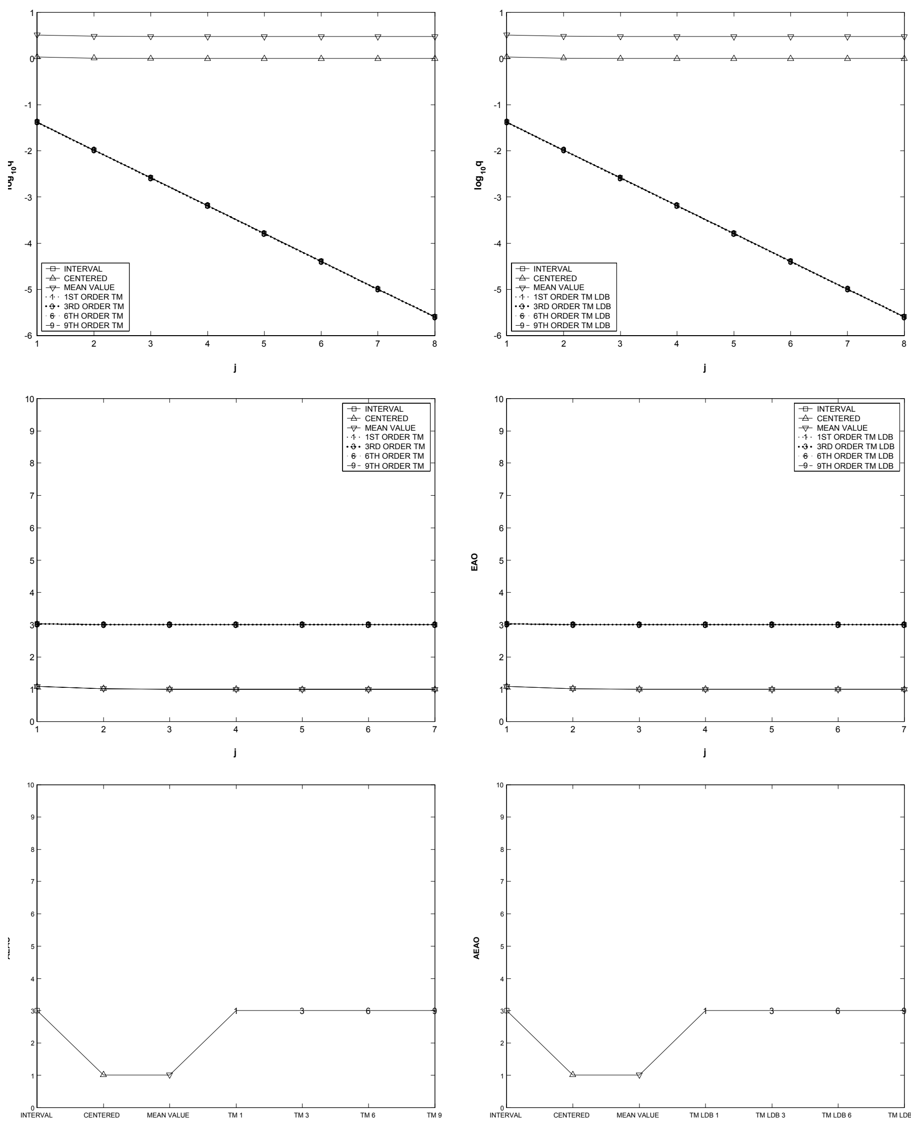

FIGURE 1. At the expansion point $x_0 = 0$.

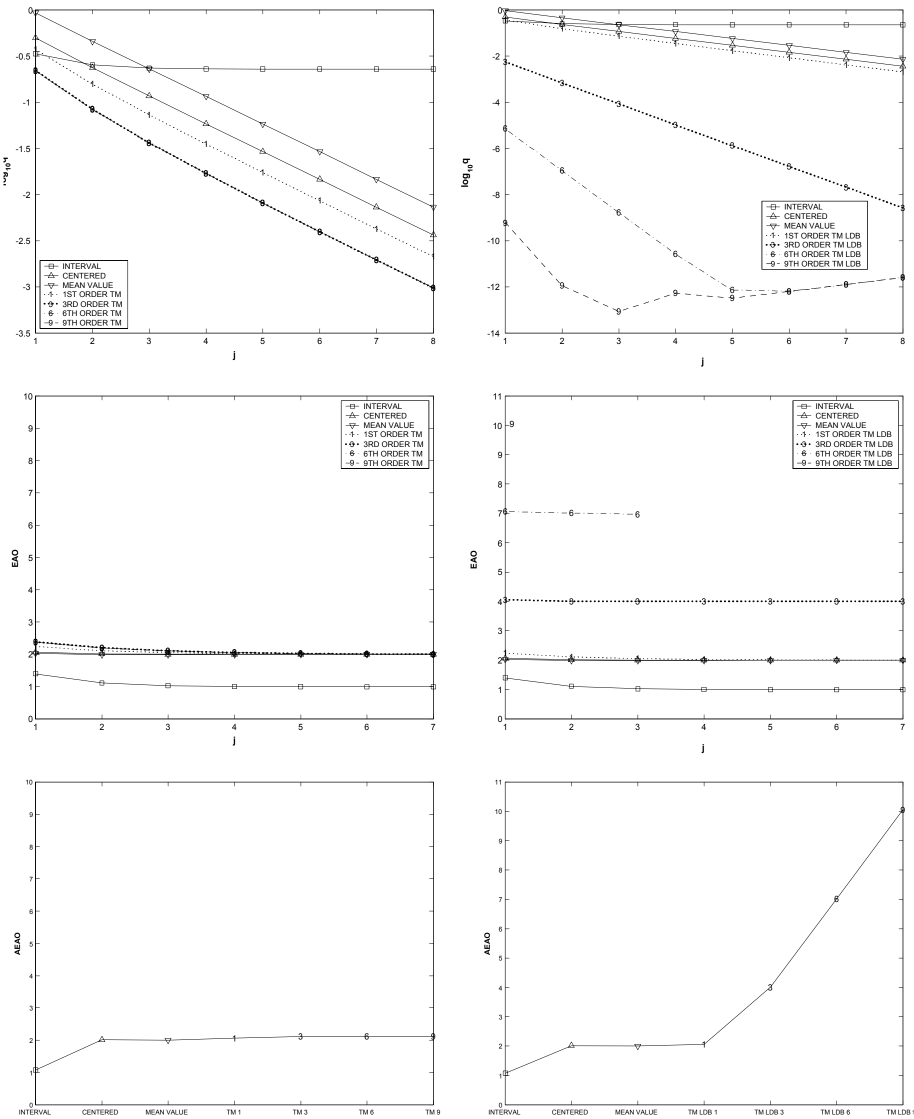

FIGURE 2. At the expansion point $x_0 = \pi/4$.

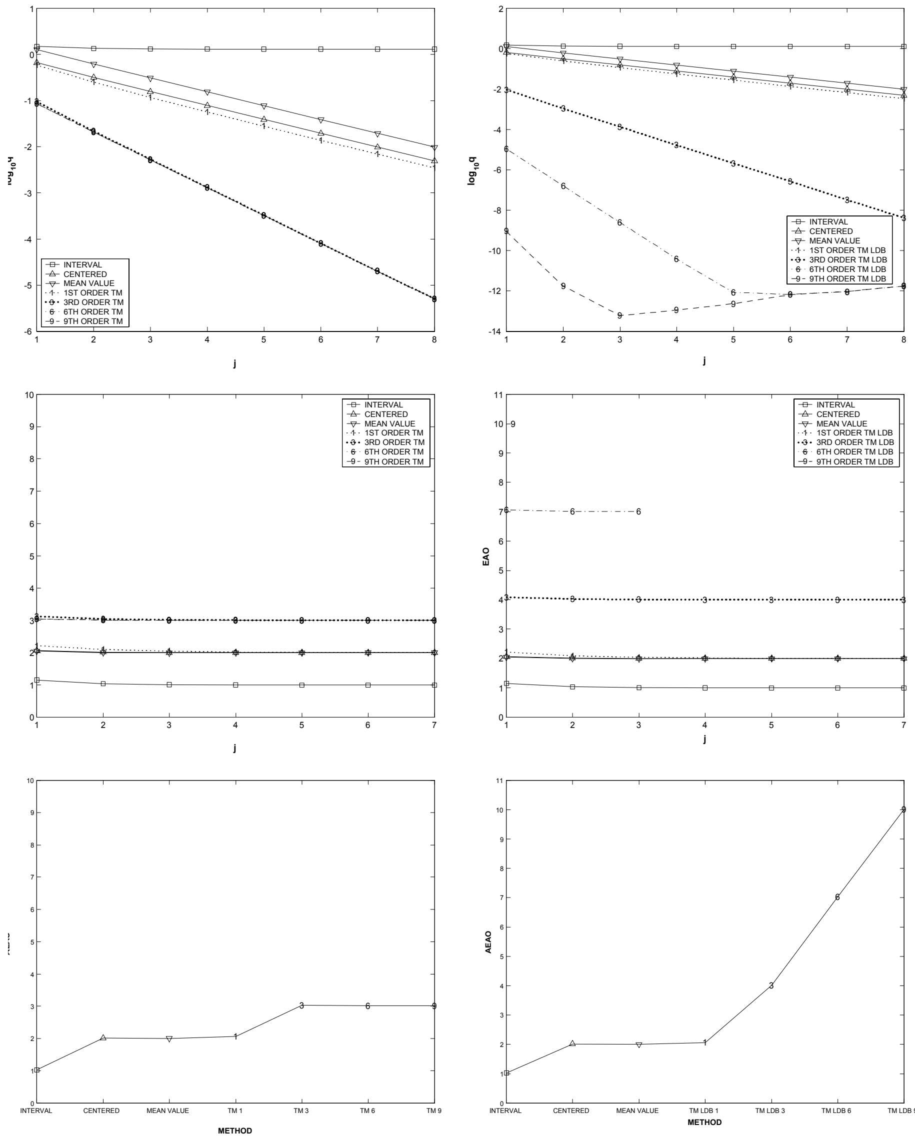

FIGURE 3. At the expansion point $x_0 = \pi/2$.

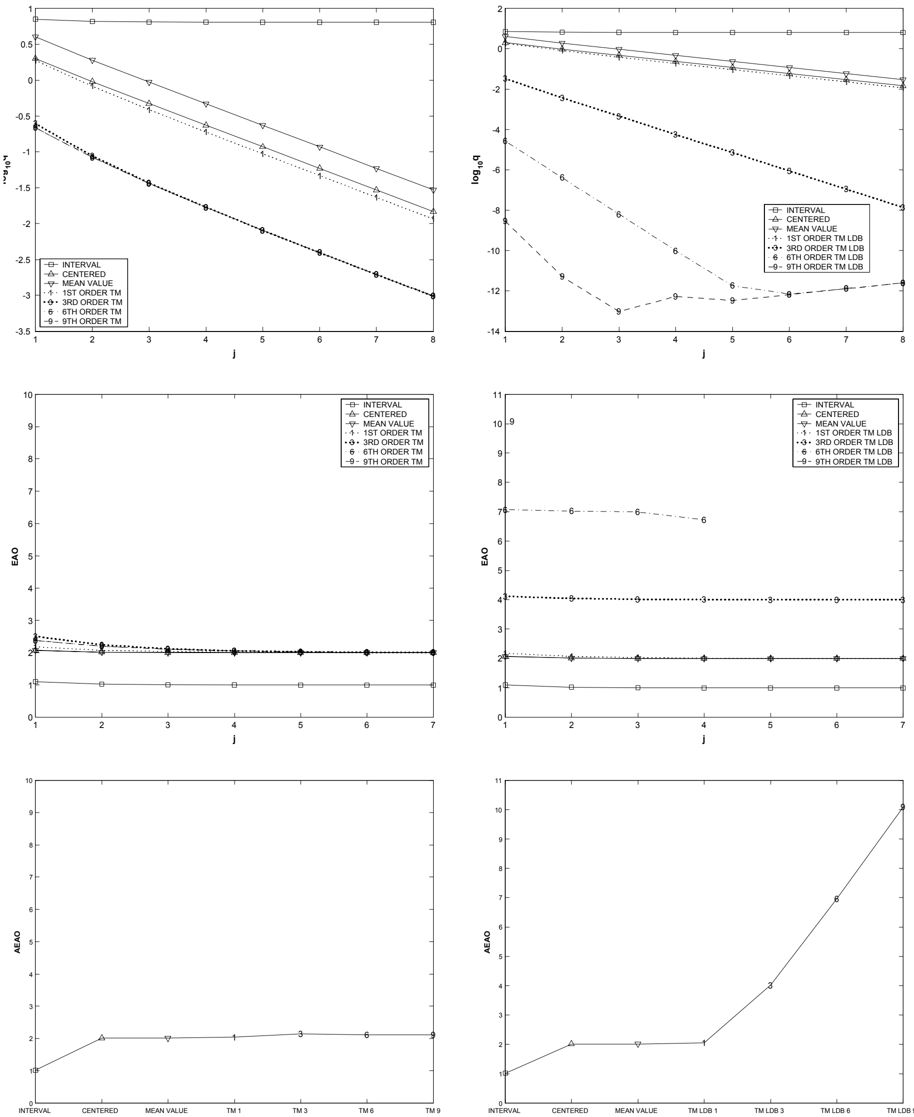

Figure 4. At the expansion point $x_0 = 3\pi/4$.

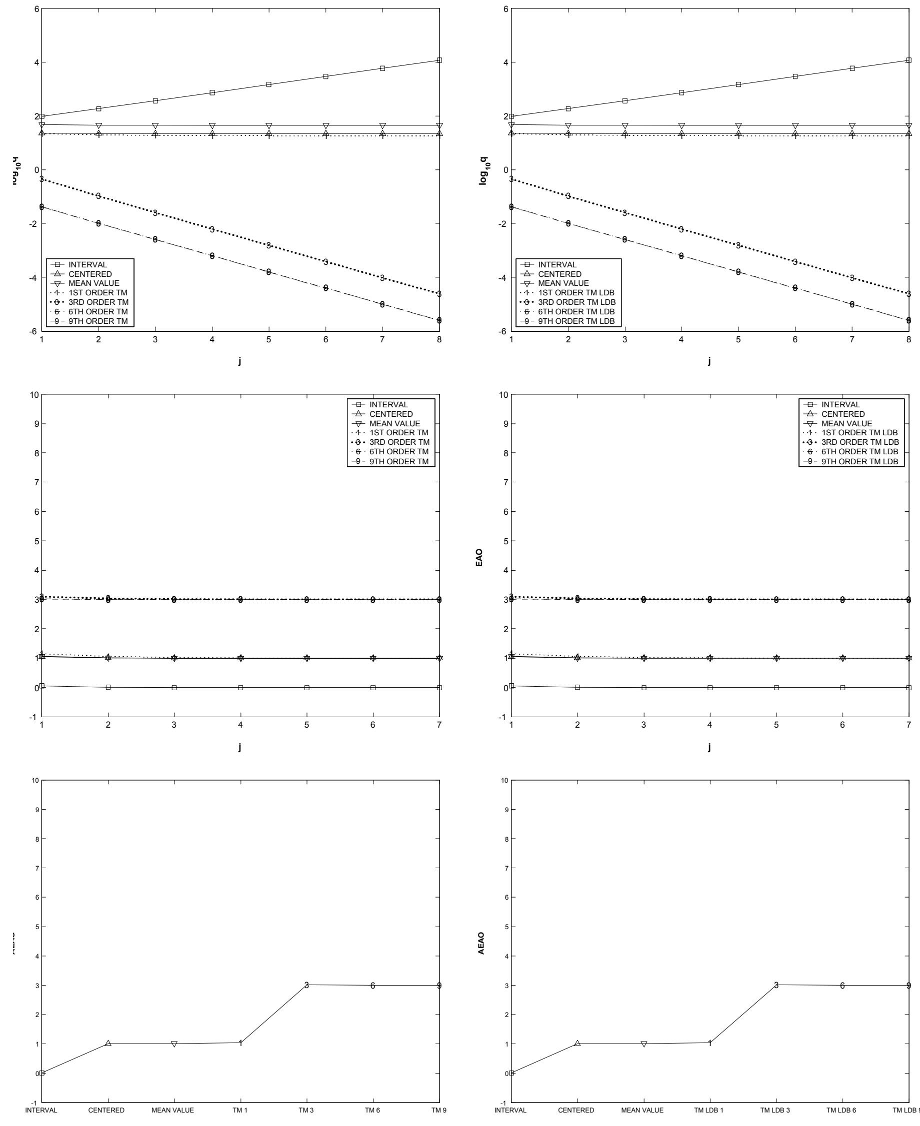

FIGURE 5. At the expansion point $x_0 = \pi$.

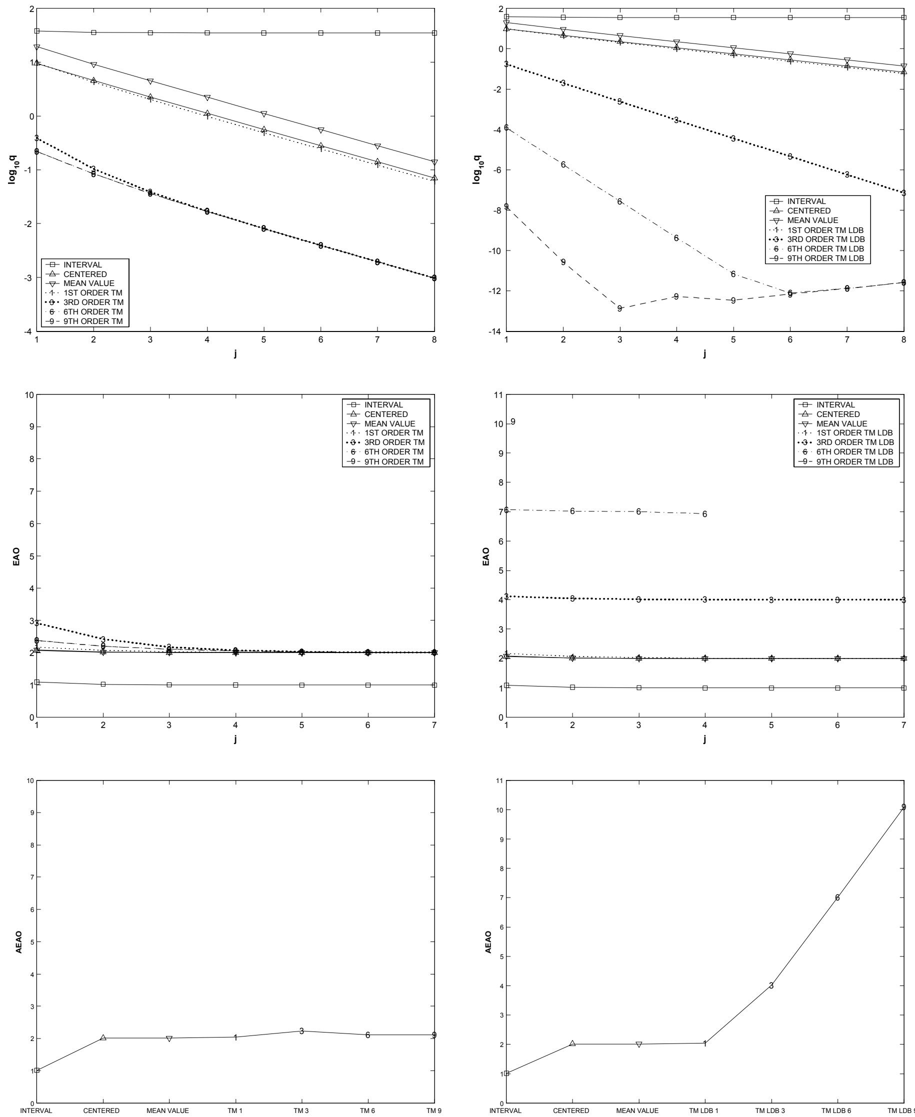

FIGURE 6. At the expansion point $x_0 = 5\pi/4$.

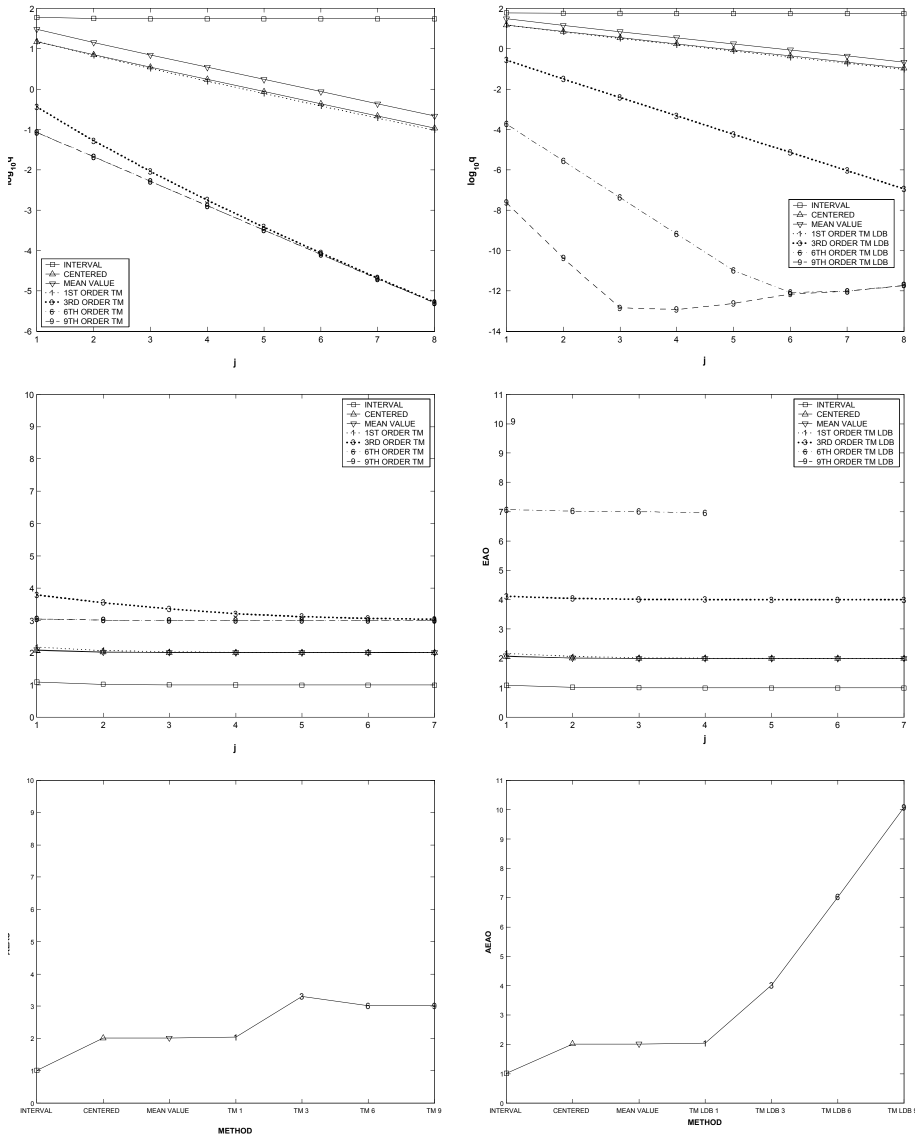

FIGURE 7. At the expansion point $x_0 = 3\pi/2$.

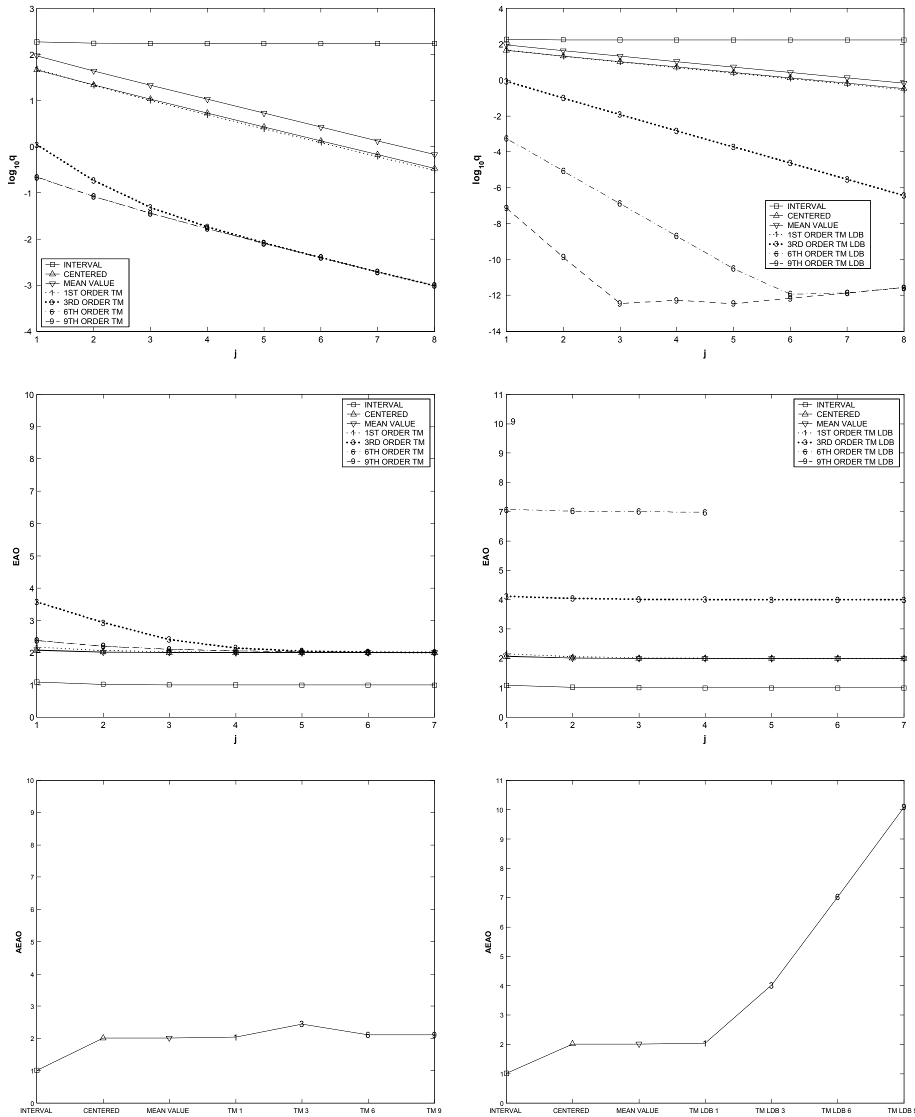

FIGURE 8. At the expansion point $x_0 = 7\pi/4$.

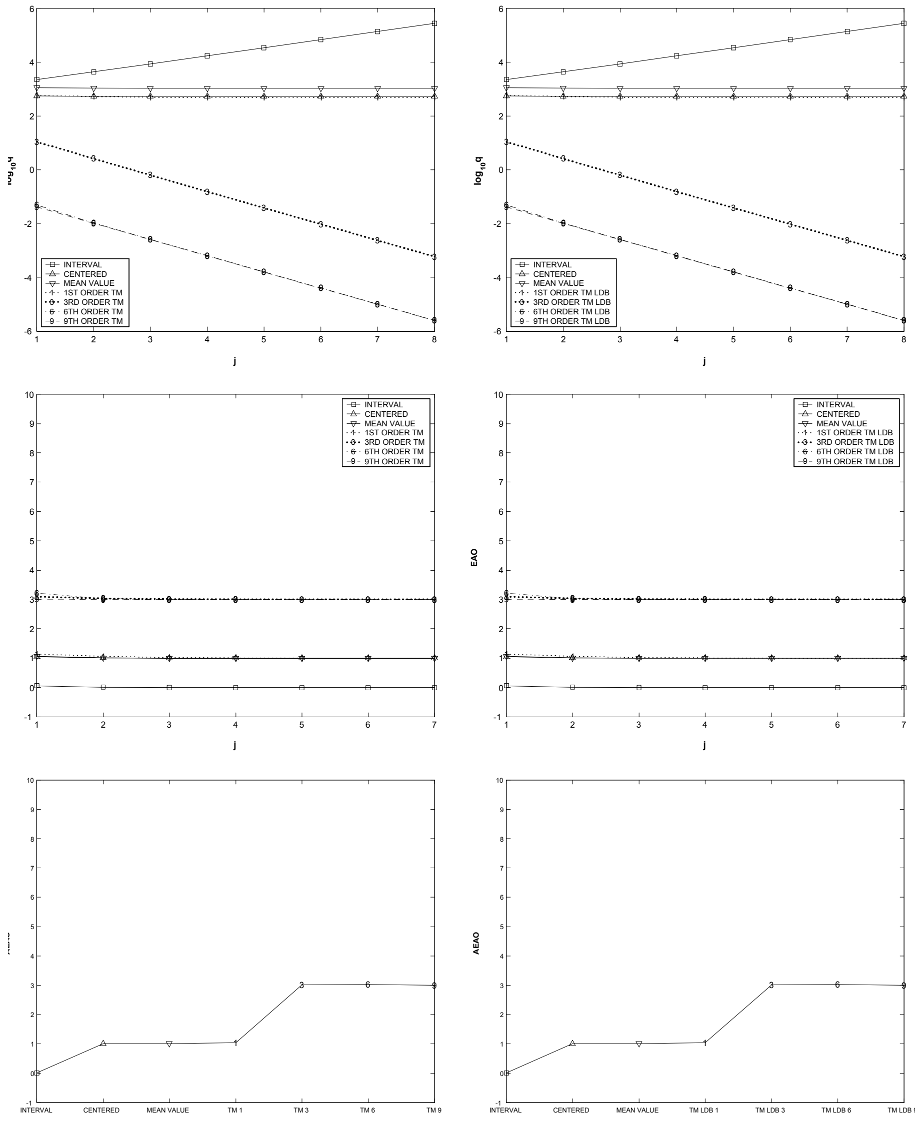

FIGURE 9. At the expansion point $x_0 = 2\pi$.

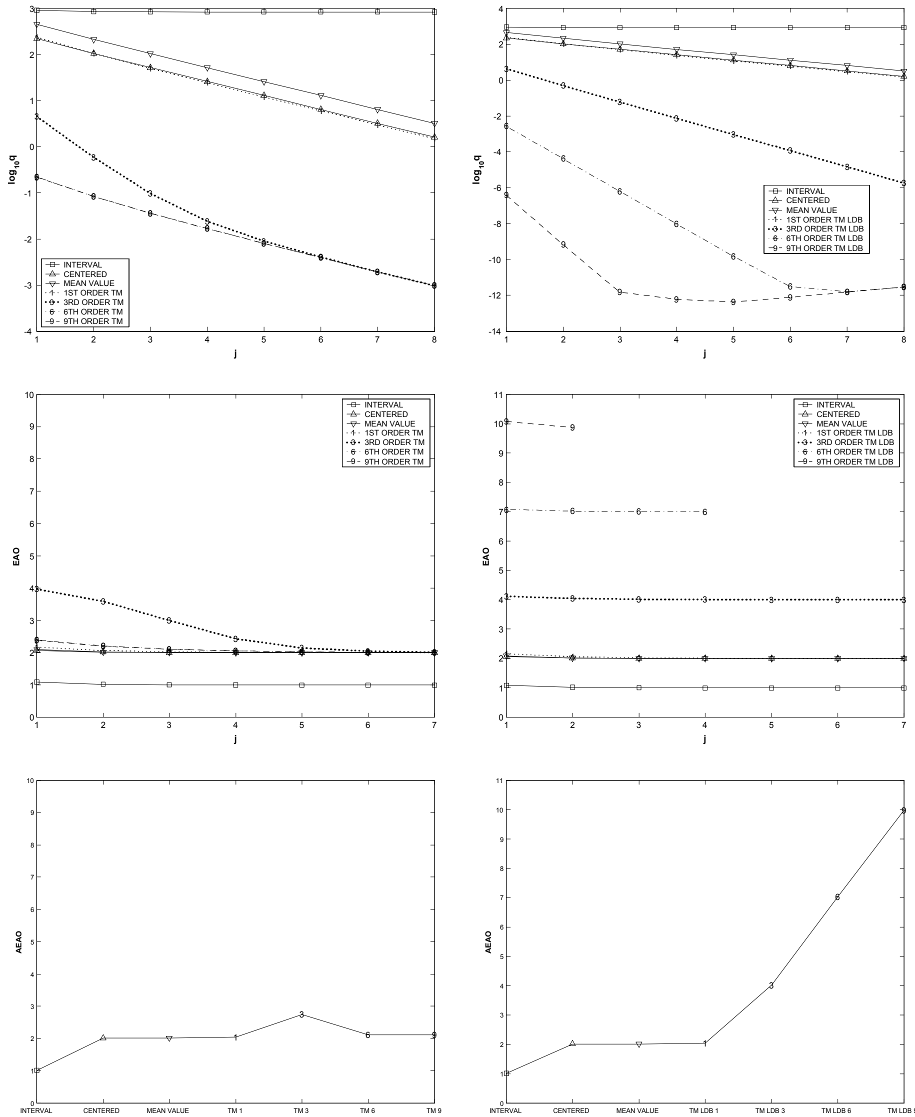

FIGURE 10. At the expansion point $x_0 = 9\pi/4$.

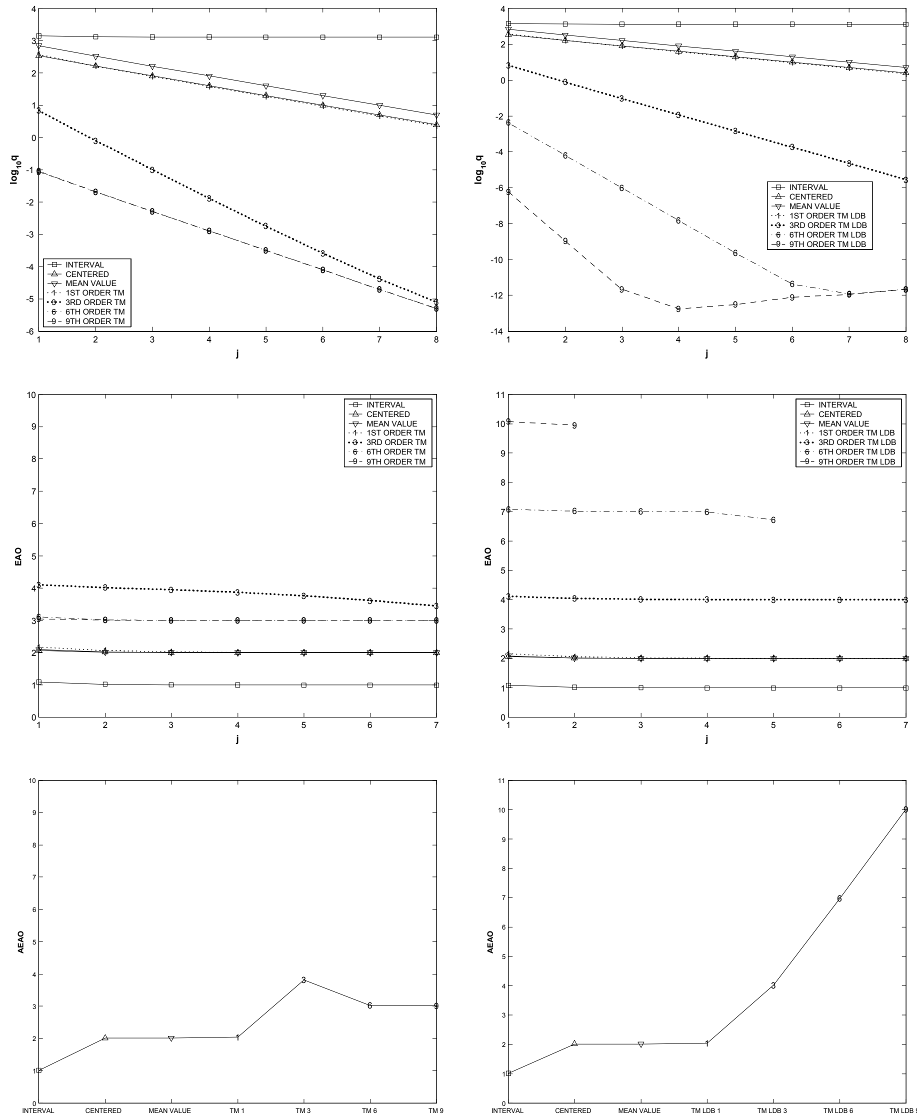

Figure 11. At the expansion point $x_0 = 5\pi/2$.

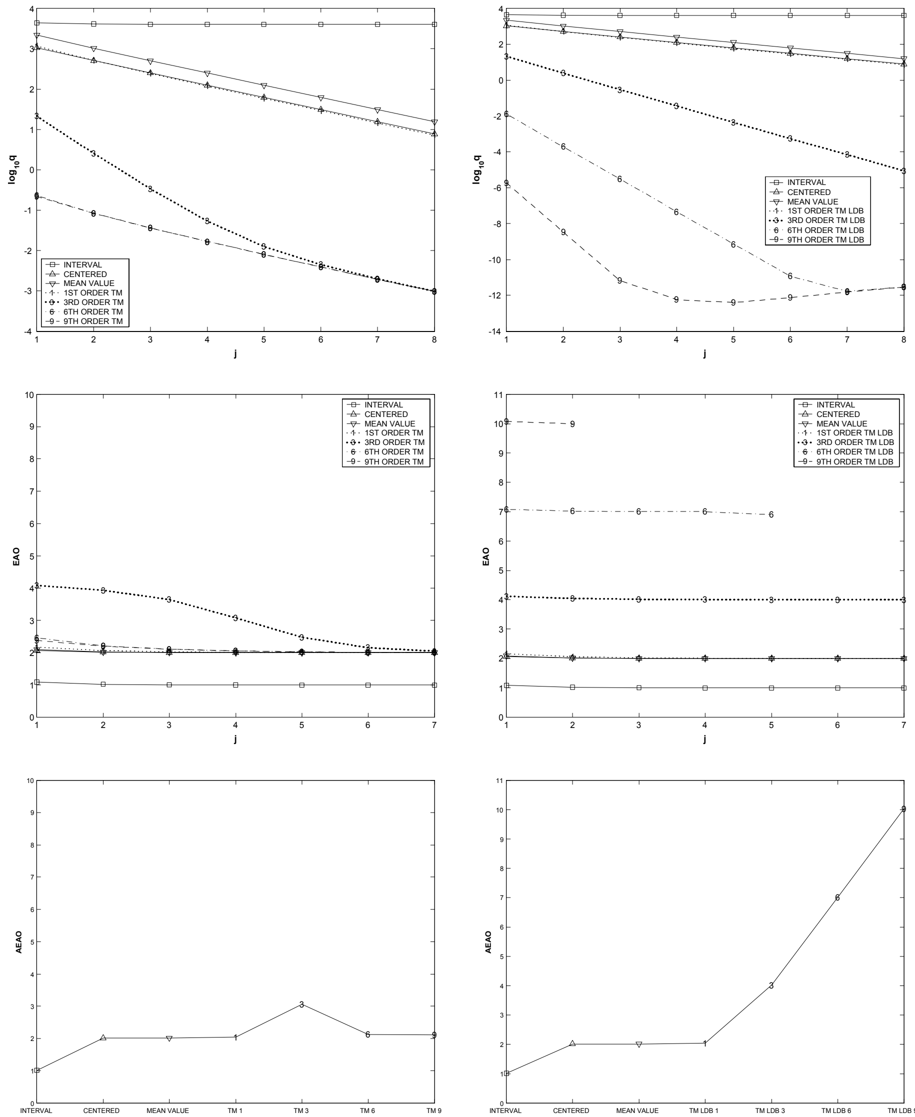

FIGURE 12. At the expansion point $x_0 = 11\pi/4$.

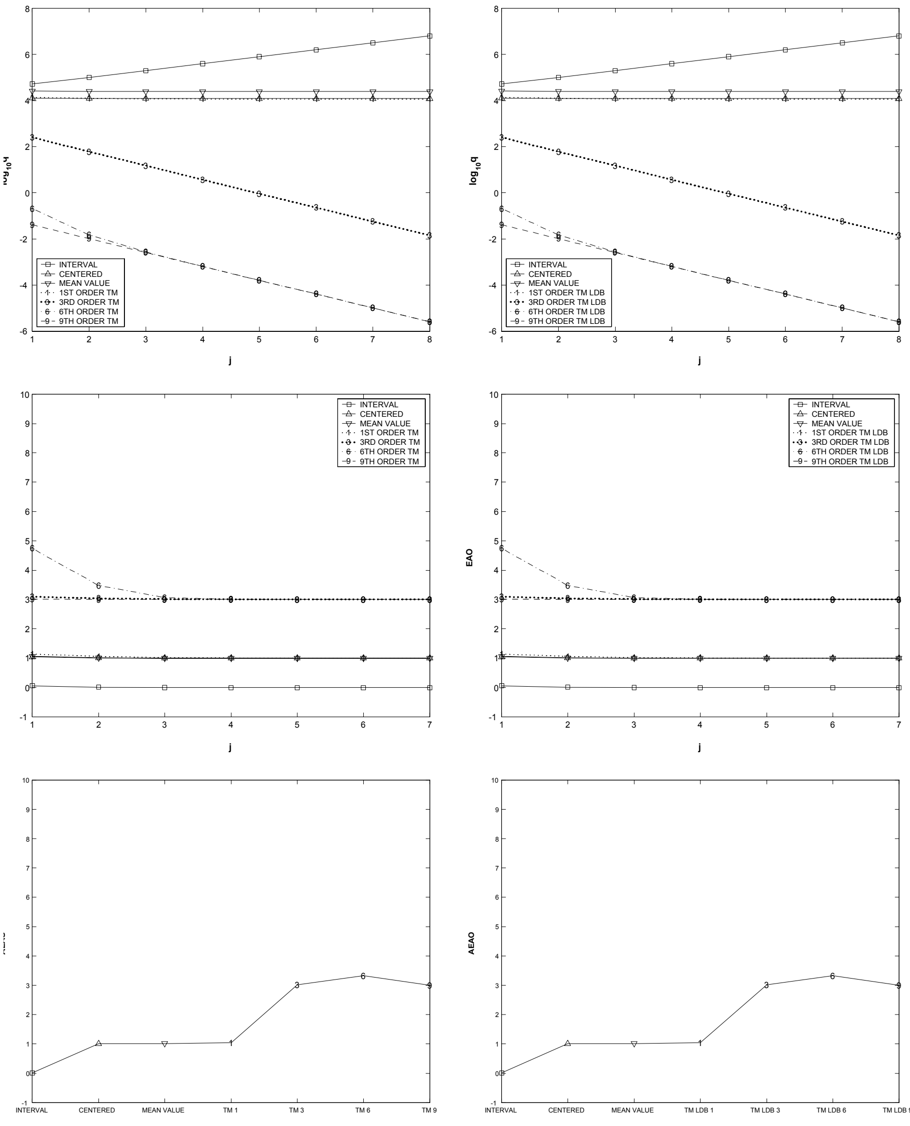

FIGURE 13. At the expansion point $x_0 = 3\pi$.

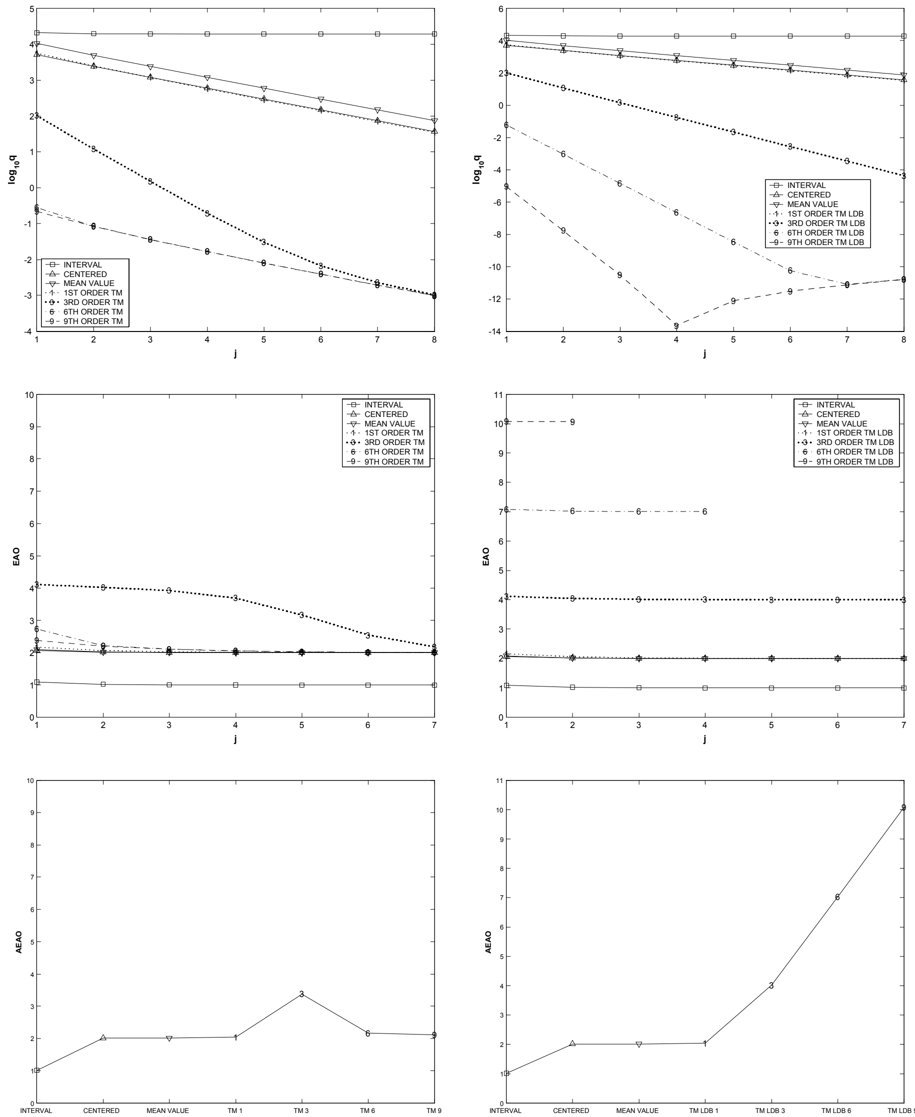

Figure 14. At the expansion point $x_0 = 13\pi/4$.

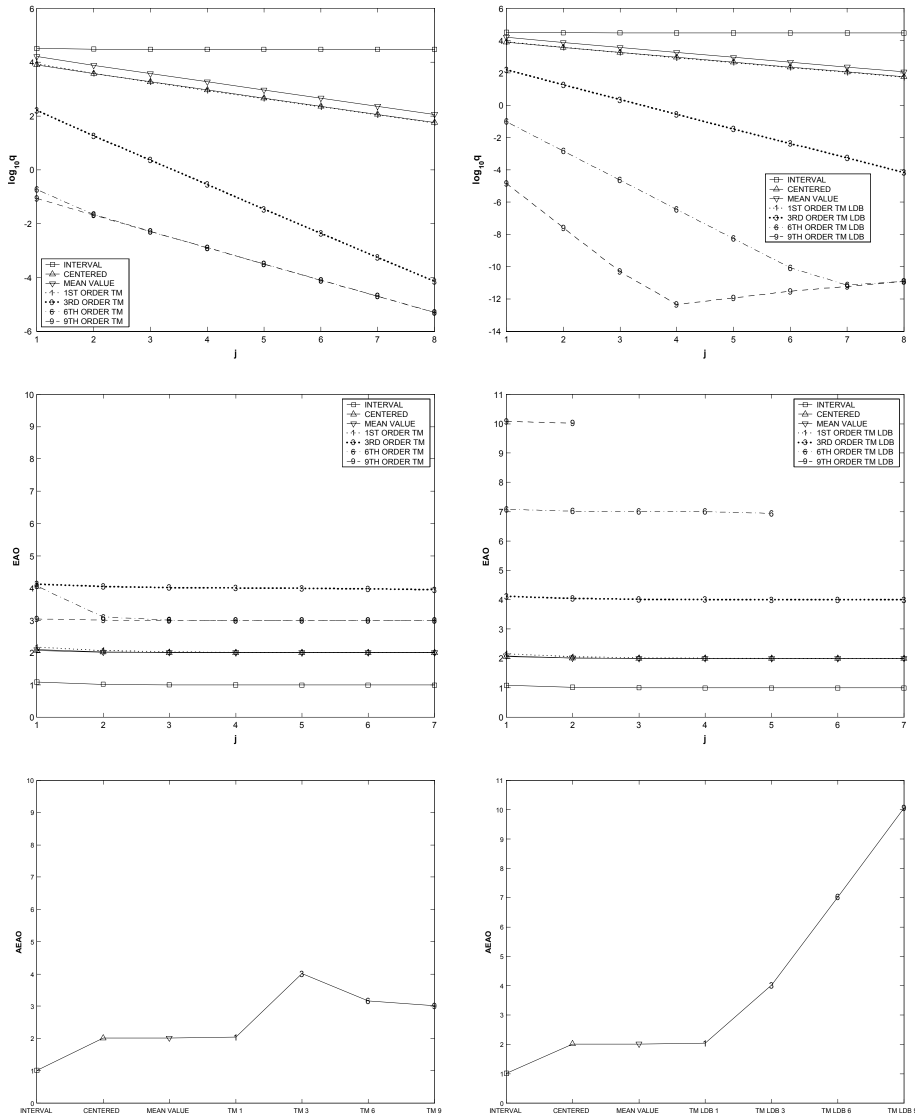

FIGURE 15. At the expansion point $x_0 = 7\pi/2$.

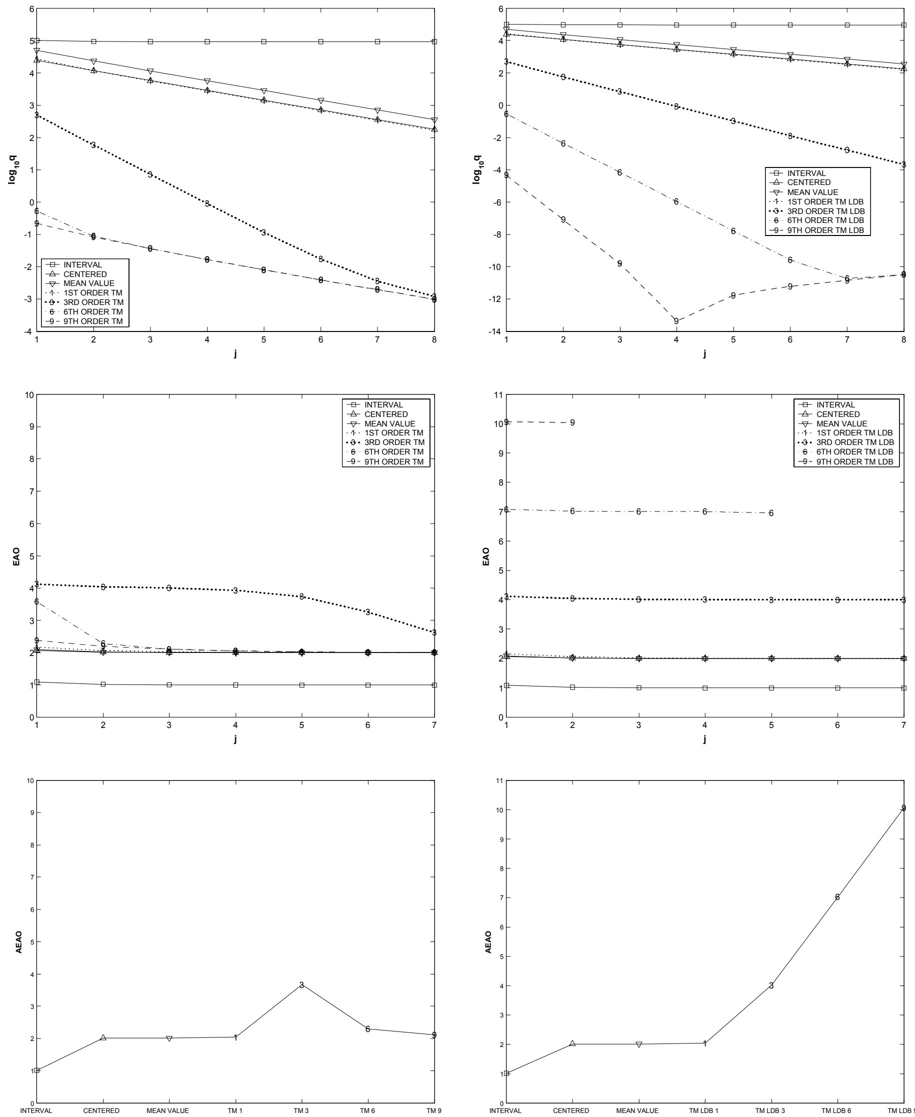

FIGURE 16. At the expansion point $x_0 = 15\pi/4$.

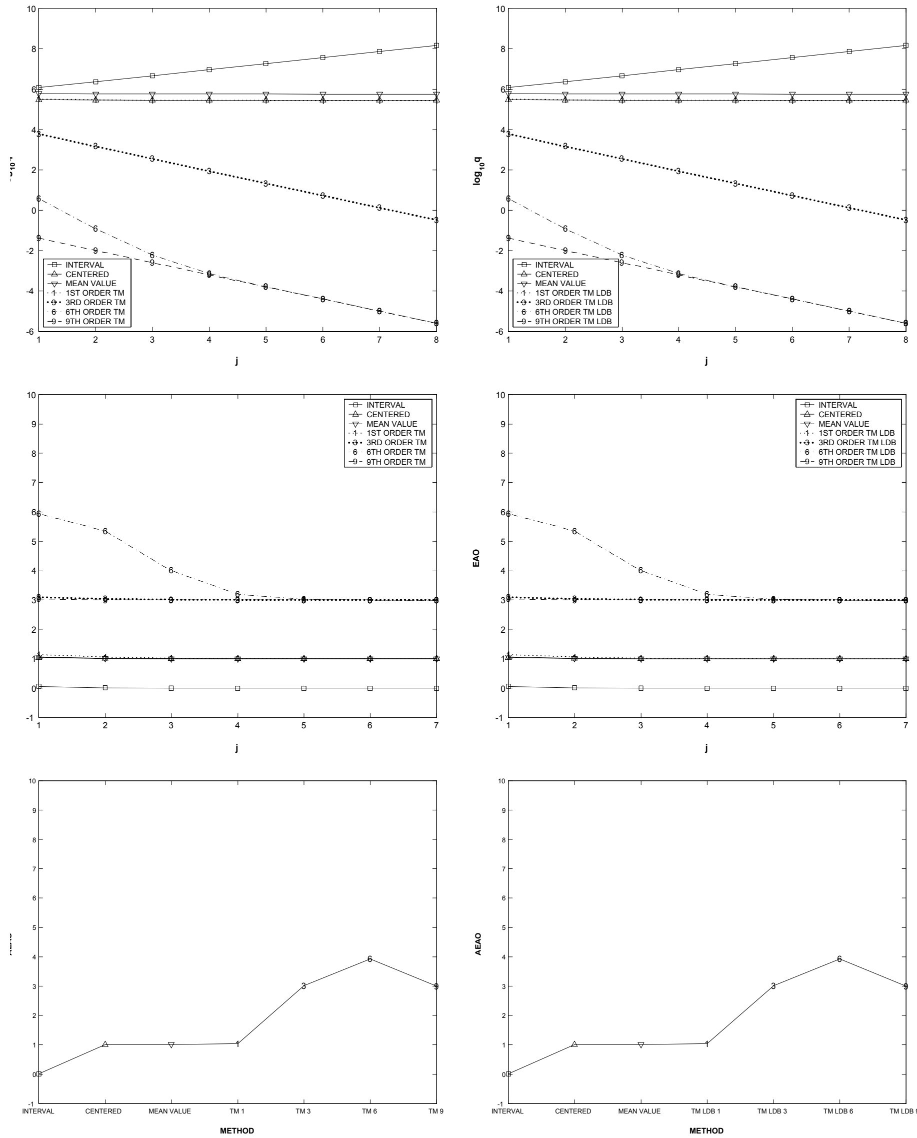

FIGURE 17. At the expansion point $x_0 = 4\pi$.