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The method of topological sections in the rigorous numerics of dynamical systems

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A sample equation

Consider the following differential equation in the complex plane

$$z' = (1 + e^{i\varphi t}|z|^2)\bar{z}.$$

Theorem. (Srzednicki, Wójcik 1997)
For $\varphi \in (0, 1/288]$ the Poincaré map of this equation admits a chaotic invariant set, which is semiconjugate to symbolic dynamics on two symbols.

Theorem. (Wójcik, Zgliczyński, 2000)
For $\varphi \in (0, 495/1000]$ the Poincaré map of this equation admits a chaotic invariant set, which is semiconjugate to symbolic dynamics on three symbols.

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The idea of the analytic proof [3]

- Adding the equation
$$t' = 1$$
we obtain an ODE which induces a flow on $\mathbb{R} \times \mathbb{C}$
- The right-hand-side of the equation is T periodic in t variable with $T = 2\pi/\varphi$.
- Therefore there is an induced flow on $S^1_T \times \mathbb{C}$, where $S^1_T = [0, T]/\sim$ with $\sim$ the relation identifying 0 and T.
- One studies the Poincaré map P on the Poincaré section $X := \{0\} \times \mathbb{C}$.
- The dynamical features of this Poincaré map may be captured by means of so called isolating segments.

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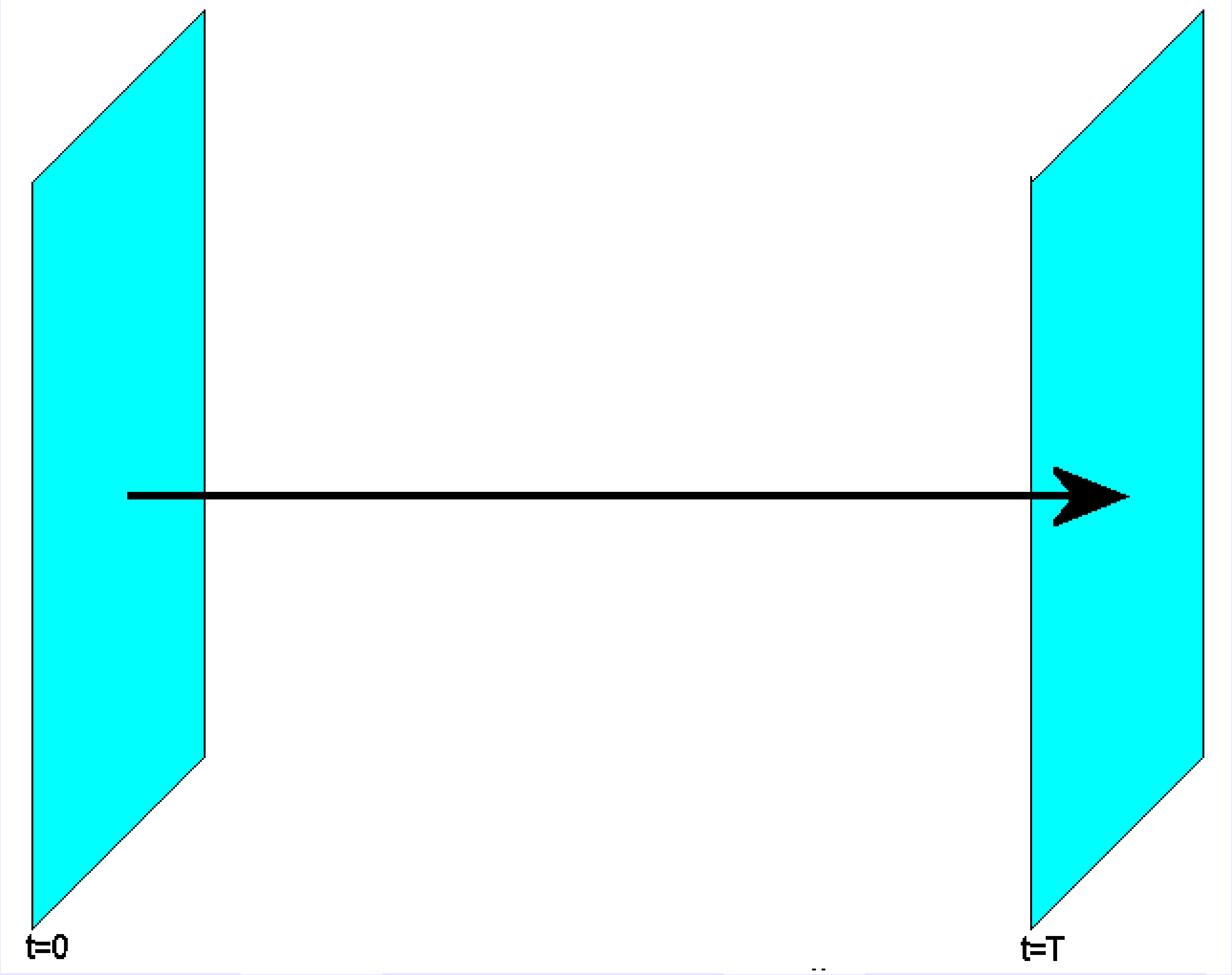
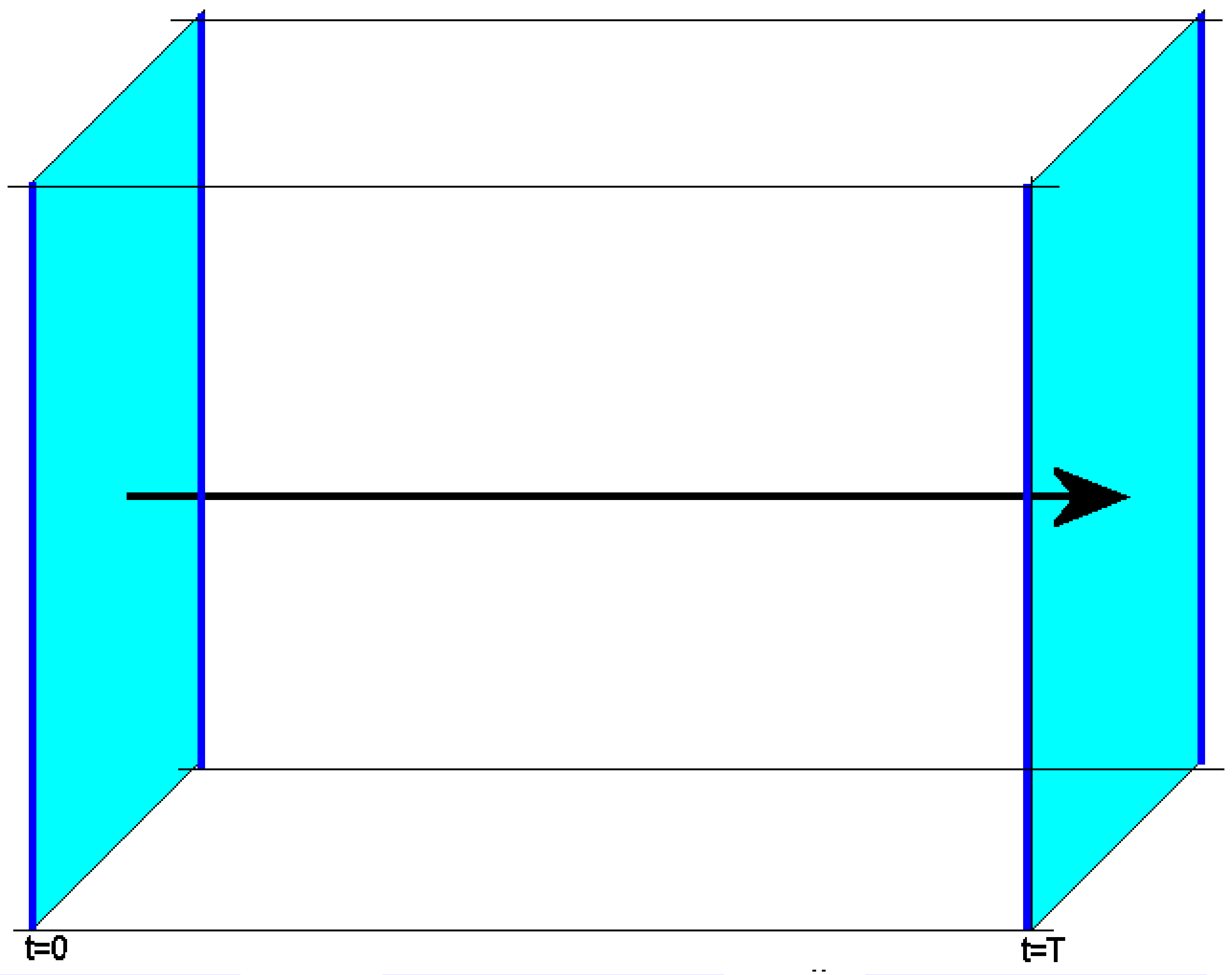
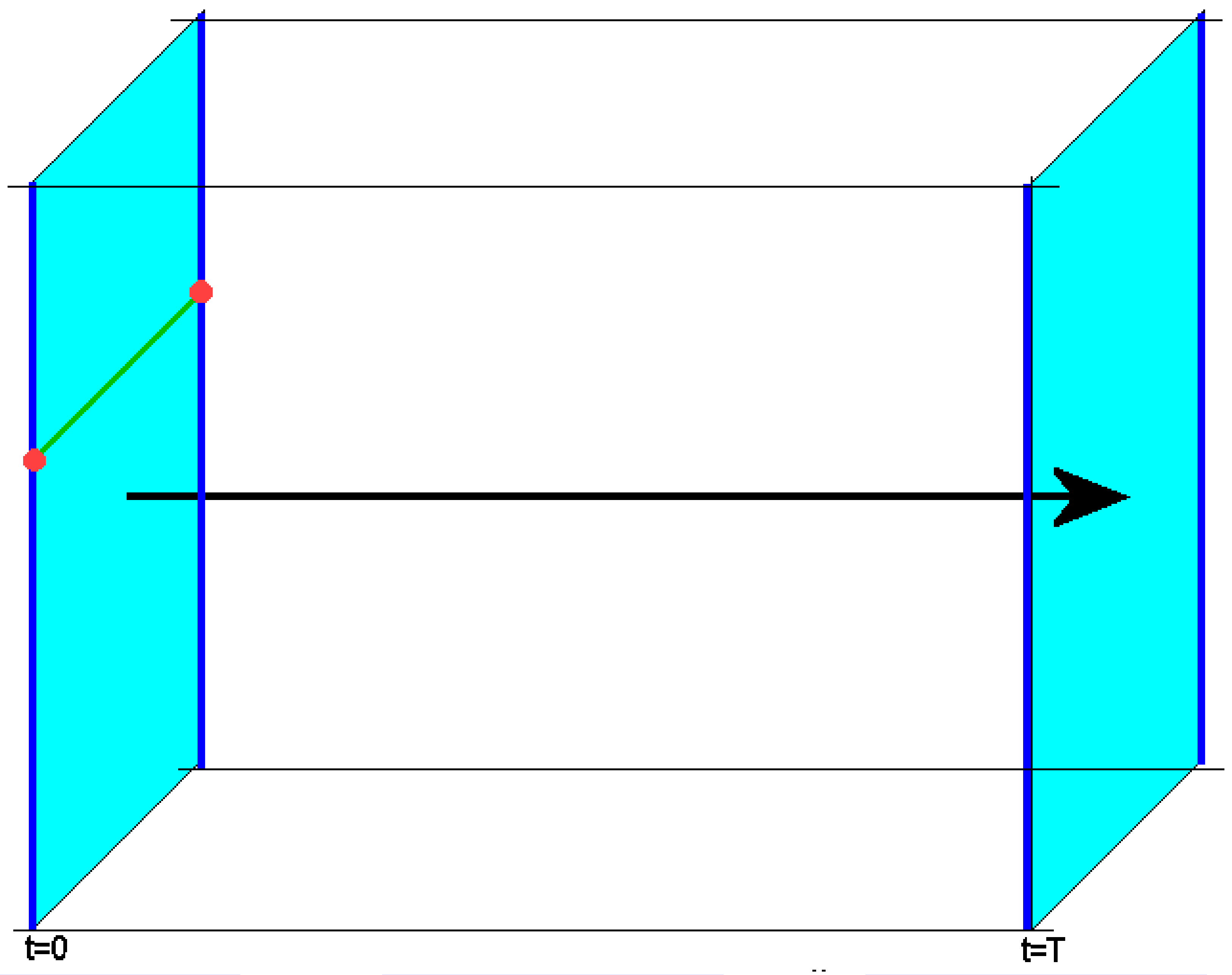
Poincaré map

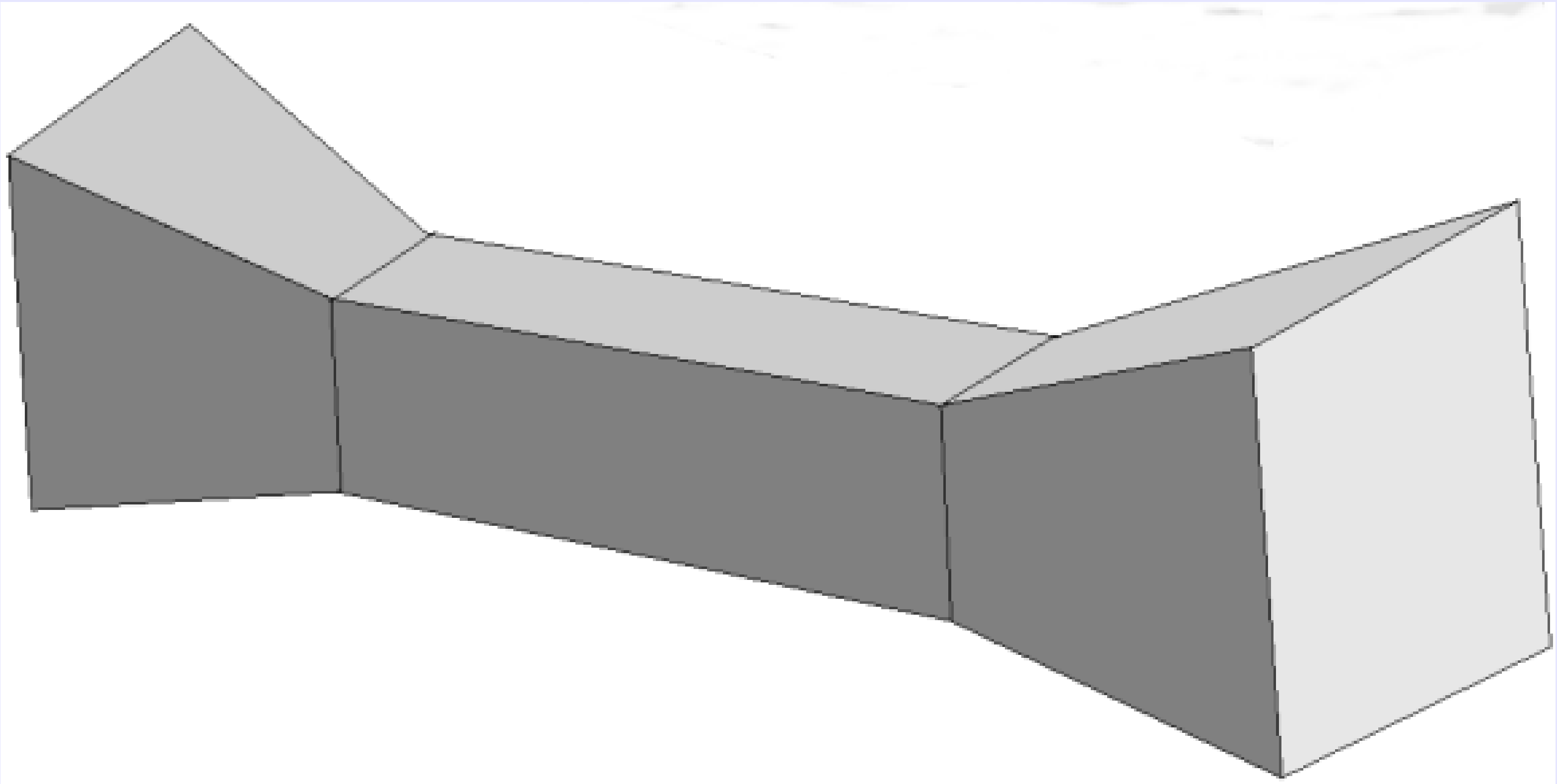


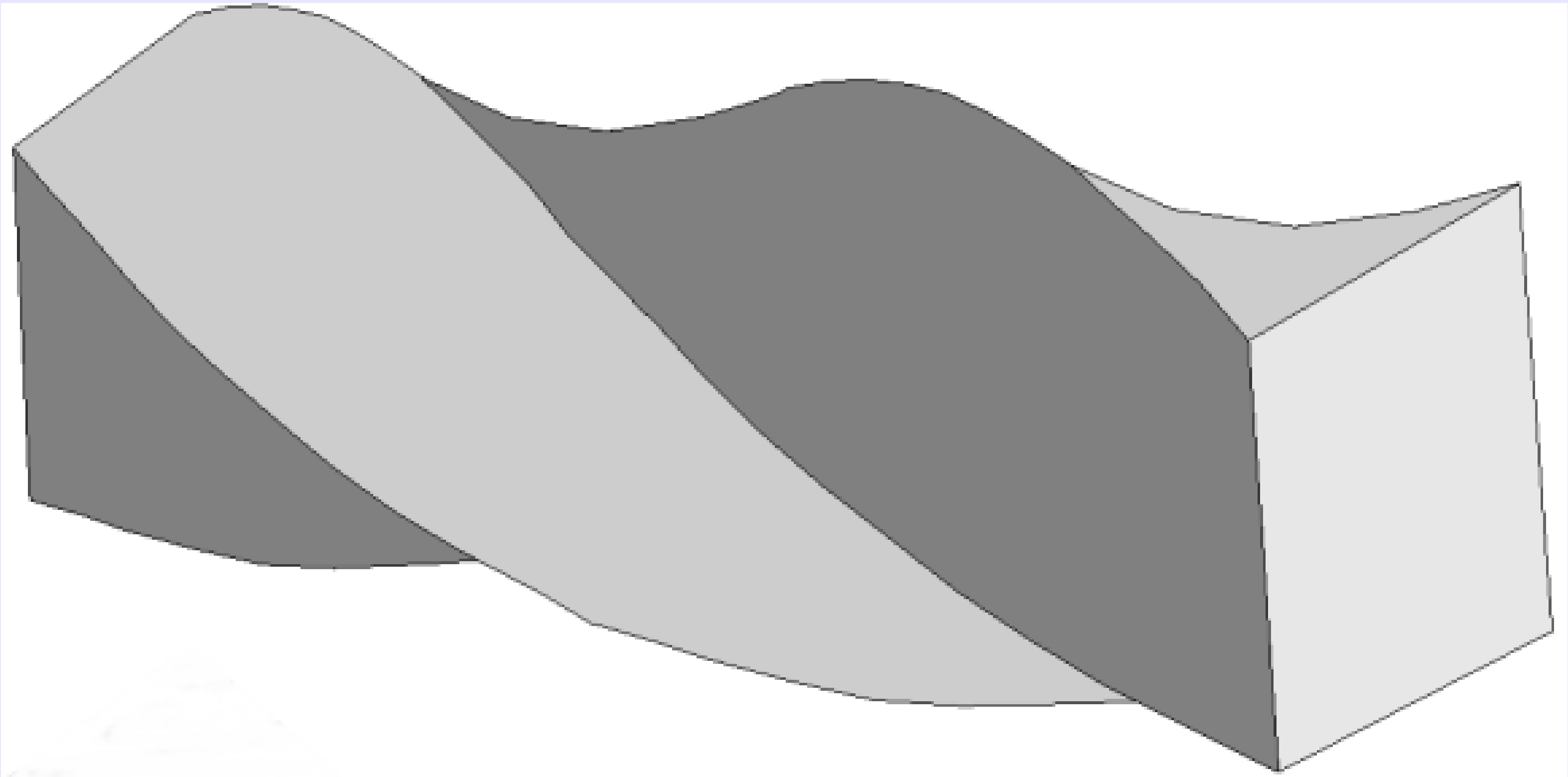
Poincaré map 5



Poincaré map







Comparing the two proofs one can guess that the analytical complexity of the proof grows as φ grows.

Question: Is it possible to provide a computer assisted proof, for instance for $\varphi = 1$?

- Bad news: it is difficult to find useful algorithms constructing isolating blocks for flows
- Good news: The topological criterion used in the Srzednicki-Wójcik proof does have a counterpart for maps

Isolating neighborhoods and index maps

Let $f : X \to X$ be a map and let $N \subset X$ be compact. The set N is an isloating neighborhood if

$$\{x \in N \mid \forall n \in \mathbb{Z}\ f^n(x) \in N\} \subset \operatorname{int} N.$$

A pair of compact $P = (P_1, P_2)$ subsets of N is an index pair if

$$x \in P_i,\ f(x) \in N \ \Rightarrow\ f(x) \in P_i, \quad i = 1, 2$$

$$x \in P_1,\ f(x) \notin N \ \Rightarrow\ x \in P_2$$

$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2).$$

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Conley index

The associated indedx map is

$$I_P := H^*(f_P) \circ H^*(i_P)^{-1} : H^*(P_1, P_2) \longrightarrow H^*(P_1, P_2)$$

where

$$f_P : (P_1, P_2) \ni x \longrightarrow f(x) \in (P_1 \cup f(P_2), P_2 \cup f(P_2))$$
$$i_P : (P_1, P_2) \ni x \longrightarrow x \in (P_1 \cup f(P_2), P_2 \cup f(P_2))$$

The generalized kernel of I_P is

$$\operatorname{gker}(I_P) := \bigcup_{n \in \mathbb{N}} \ker I_P^n.$$

The Conley index is

$$(CH^*(S, f), \chi(S, f)) := (H^*(P_1, P_2)/\operatorname{gker}(I_P), [I_P]) .$$

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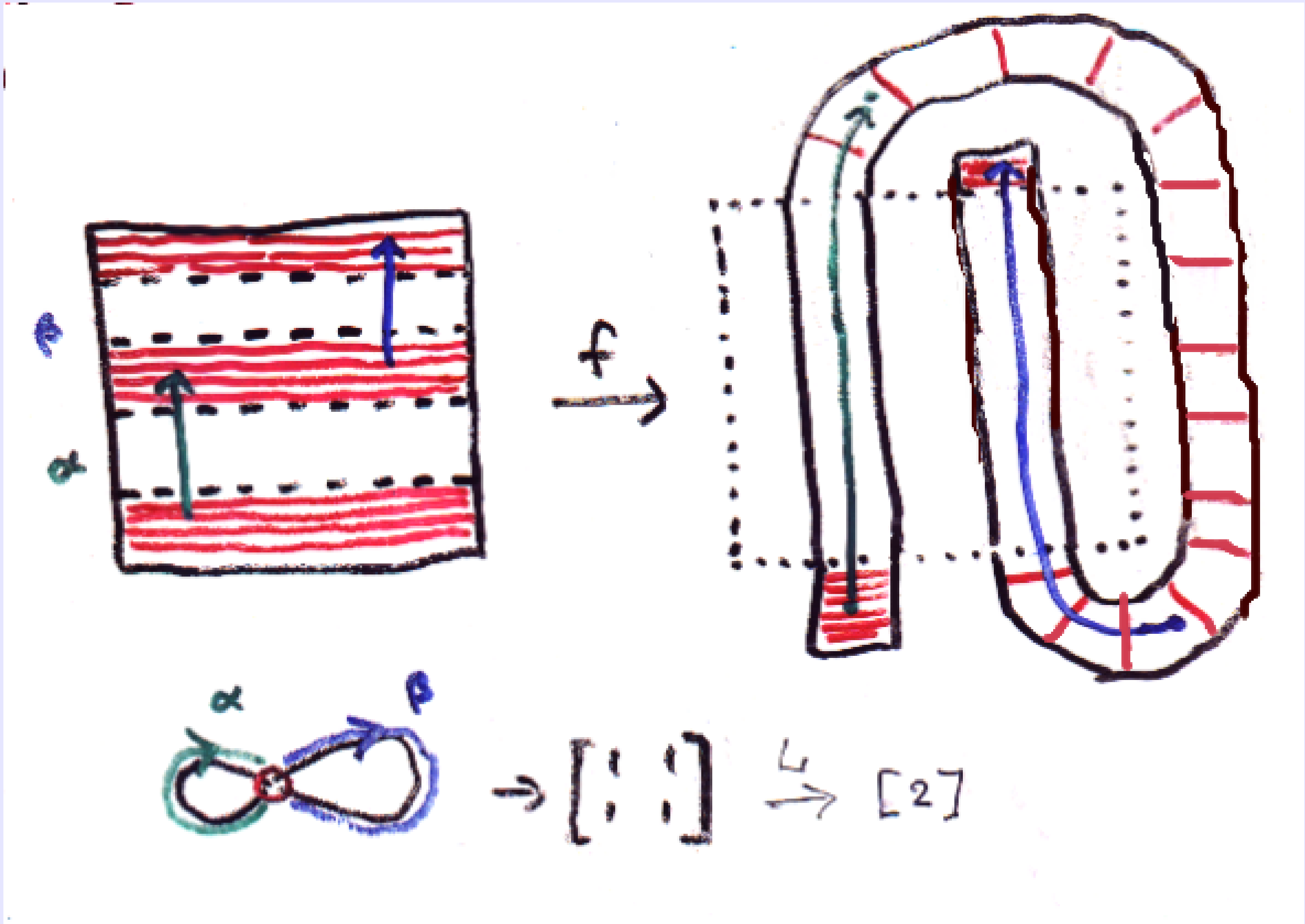
G-horseshoe example



A discrete analog of Srzednicki's criterion

Let X be an ENR. Assume that $M \subset N$ are isolating blocks with respect to f such that

(a): $\chi_M = \mathrm{id}_{\mathbb{Q}}$, $\chi_N = -\mathrm{id}_{\mathbb{Q}}$,
(b): $f(N) \cap M \cap f^{-1}(N) \subset \mathrm{int}(M)$,
(c): $f(N \setminus f^{-1}(\mathrm{int}(M))) \cap M \subset N^-$,
(d): all inclusions in the diagram

$$\begin{array}{ccc} (M, M^-) & \longrightarrow & (N, N \setminus f^{-1}(\mathrm{int}(M))) \\ \downarrow & & \downarrow \\ (M, M \cap N^-) & \longrightarrow & (N, N^-) \end{array}$$

induce isomorphisms in the Alexander-Spanier cohomology.

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A discrete analog of Srzednicki's criterion

Put $I = \mathrm{inv}_f N = \mathrm{inv}_f(\mathrm{cl}(N \setminus N^-))$. Let $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ and $\sigma : \Sigma_2 \longrightarrow \Sigma_2$ be a shift map.

Theorem. (K. Wójcik, MM, 2003)
There is a continuous, surjective map $g : I \longrightarrow \Sigma_2$ such that f restricted to I is semiconjugated by g to the shift σ i.e. $g \circ f = \sigma \circ g$. Moreover, for any n-periodic sequence of symbols $c \in \Sigma_2$ its counterimage $g^{-1}(c)$ contains an n-periodic point for f.

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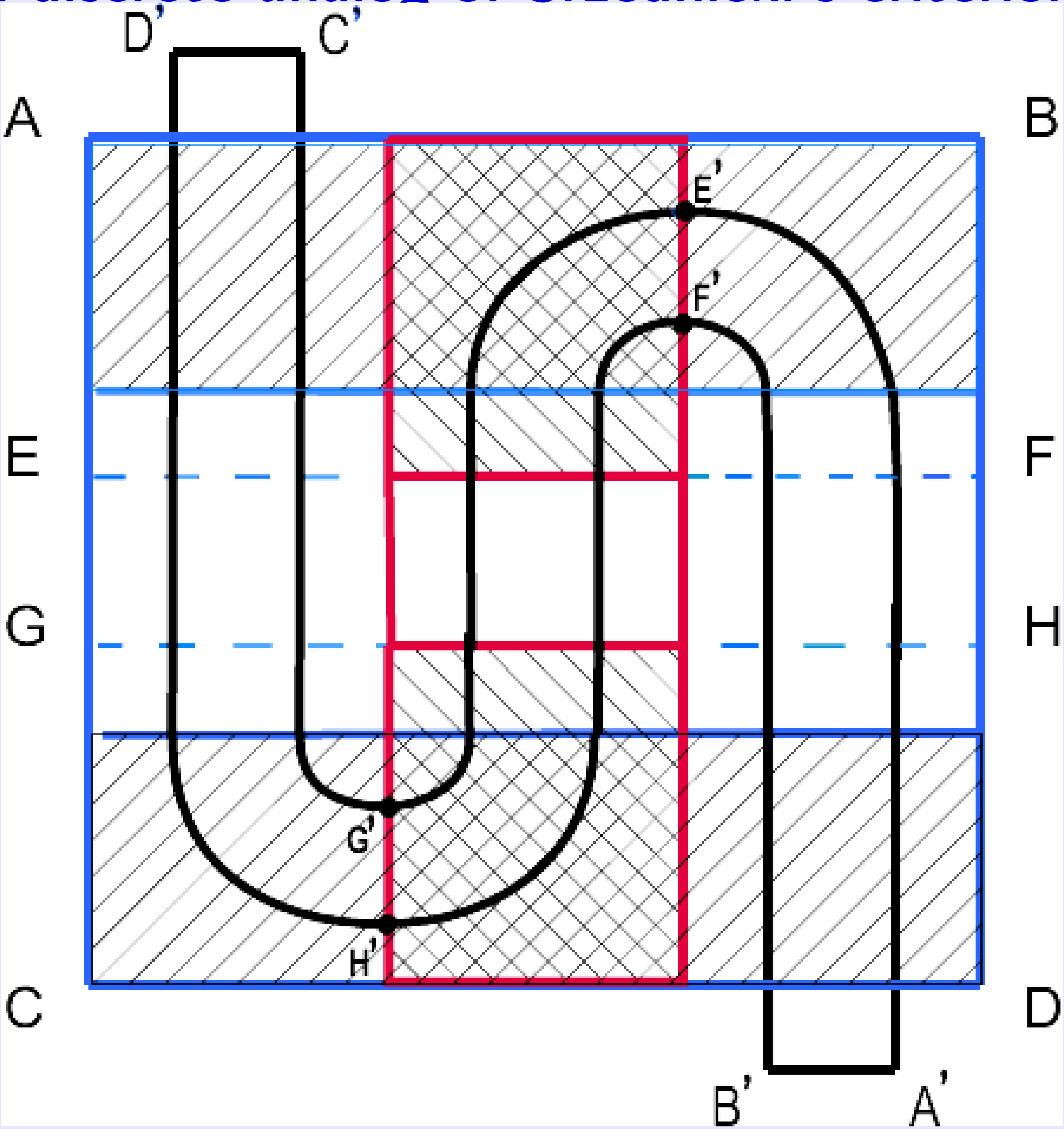
A discrete analog of Srzednicki's criterion



Problem: extremly strong expansion

- Almost every trajectory of this equation escapes in a short time to infinity.
- The expansion of the Poincareé map is extremly strong

- Escape time computation
- Expansion

Intermediate sections

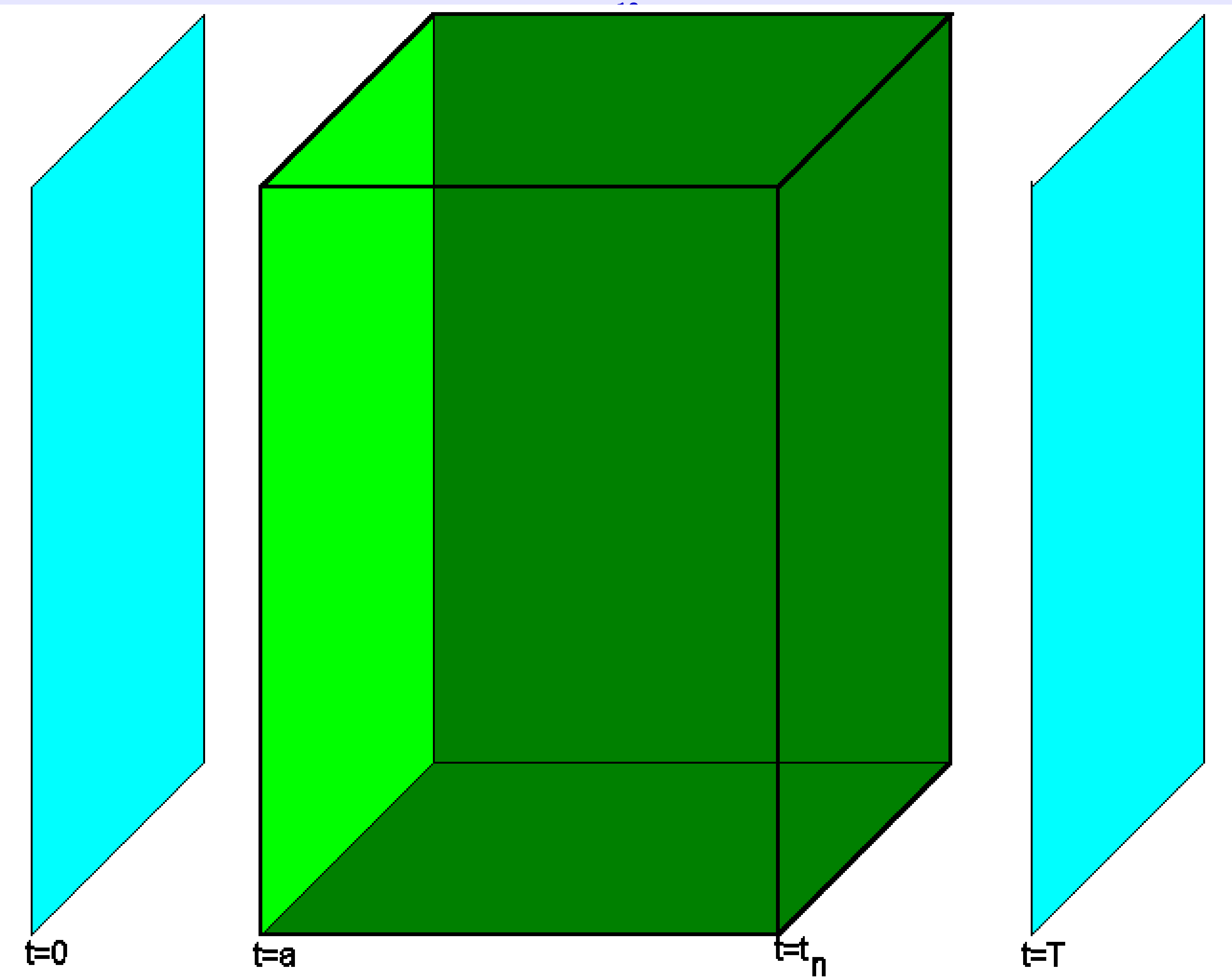
- intermediate sections $->$ compose intermediate multivalued maps to get the resulting multivalued enclosure of the Poincaré map
- intermediate topological sections $->$ find the index map from section to section and compose maps in homology

A special choice of sections

Assume $0 < a < t_1 < t_2 < \ldots < t_{n-1} < t_n = T$ and $R > 3$. Put

$$\begin{aligned} X_i \; := \; & [-R, R] \times [-R, -R] \times [a, t_i] \cup \\ & [R, R] \times [-R, R] \times [a, t_i] \cup \\ & [-R, R] \times [R, R] \times [a, t_i] \cup \\ & [-R, -R] \times [-R, R] \times [a, t_i] \cup \\ & [-R, R] \times [R, -R] \times [t_n, t_i] \end{aligned}$$

and $X_0 := X_n$. $\odot$


t=0
t=a
t=t
n
t=T

A special choice of sections

For $i = 2, 3, \dots n$ we have well defined Poincaré maps

$$f_i : X_{i-1} \to X_i$$

and for some small $\epsilon > 0$ the Poincaré map

$$f_0 : [-R + \epsilon, R - \epsilon] \times [-R + \epsilon, R - \epsilon] \times [0, 0] \to X_1$$

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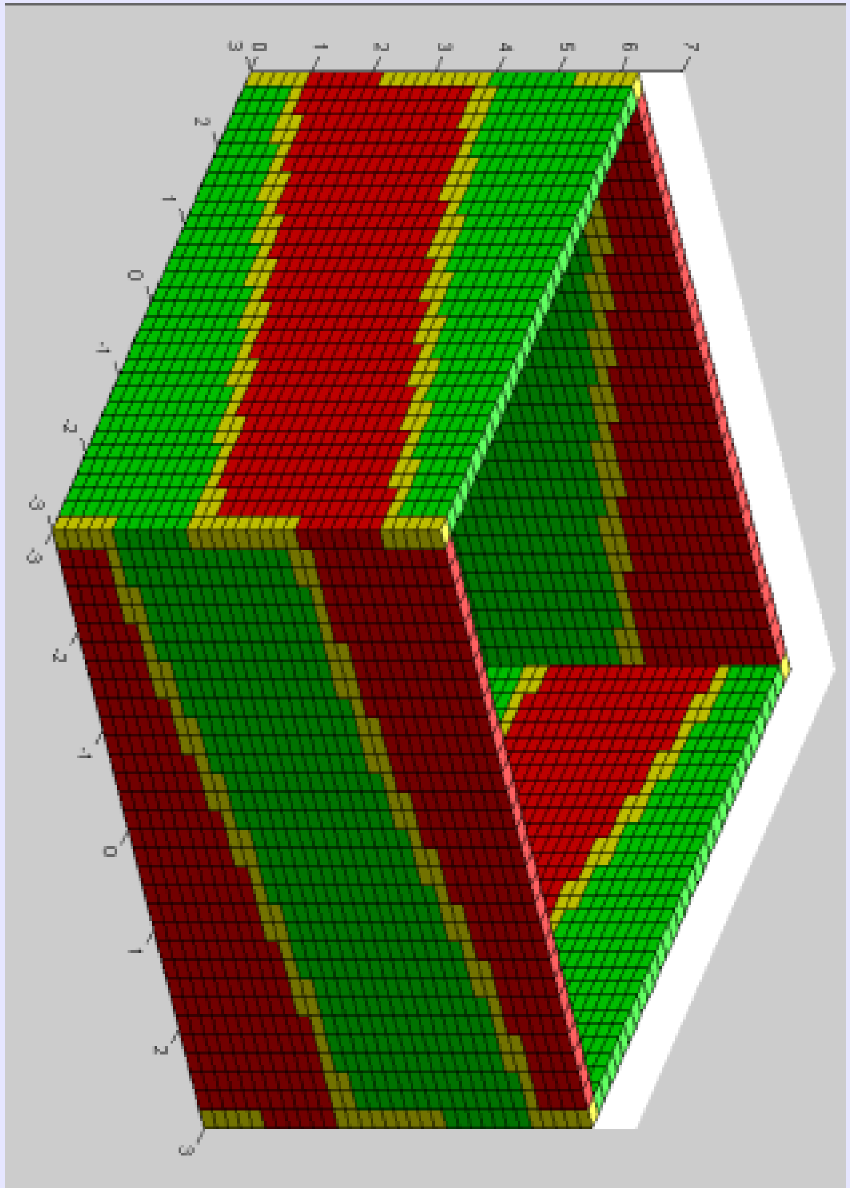
Proof of continuity



Topological intermediate sections

Define

$$X := \bigcup_{i=1}^{n} X_n$$

$$f := \bigcup_{i=1}^{n} f_n$$

For an isloating neighboorhood $N \subset X$ the index map χ decomposes as

$$\chi = \chi_1 \oplus \chi_2 \oplus \cdots \oplus \chi_n.$$

It turns out that the requested index map of the Poincaré map is

$$\chi_n \circ \chi_{n-1} \circ \cdots \circ \chi_1.$$

- Computation of the id index map
- Computation of the $-\mathrm{id}$ index map
- $-\mathrm{id}$ map - section 0
- $-\mathrm{id}$ map - section 20
- $-\mathrm{id}$ map - section 40

Theorem. (MM 2004)
For $\varphi = 1$ the Poincaré map of the equation

$$z' = (1 + e^{i\varphi t}|z|^2)\bar{z}.$$

admits a chaotic invariant set, which is semiconjugate to symbolic dynamics on two symbols.

Homological "intermediate sections"

It is possible to get rid of intermediate sections entirely and get the required index map directly from the index pair for the flow.

Theorem. (MM, R. Srzednicki, 2005)
Assume $(W, W^*) \subset \mathbb{R} \times \mathbb{C}$ is an isolating segment over $[0, T]$. Let $c \in C_q(W)$ be such that

$$\partial c = c_0 + c^- + c_T$$

for some $c_0 \in Z_{q-1}(W_0, W_0^*)$, $c_T \in Z_{q-1}(W_T, W_T^*)$ and $c^- \in C_{q-1}(W^*)$. Then

$$\mu_W([c_0]) = [c_T].$$

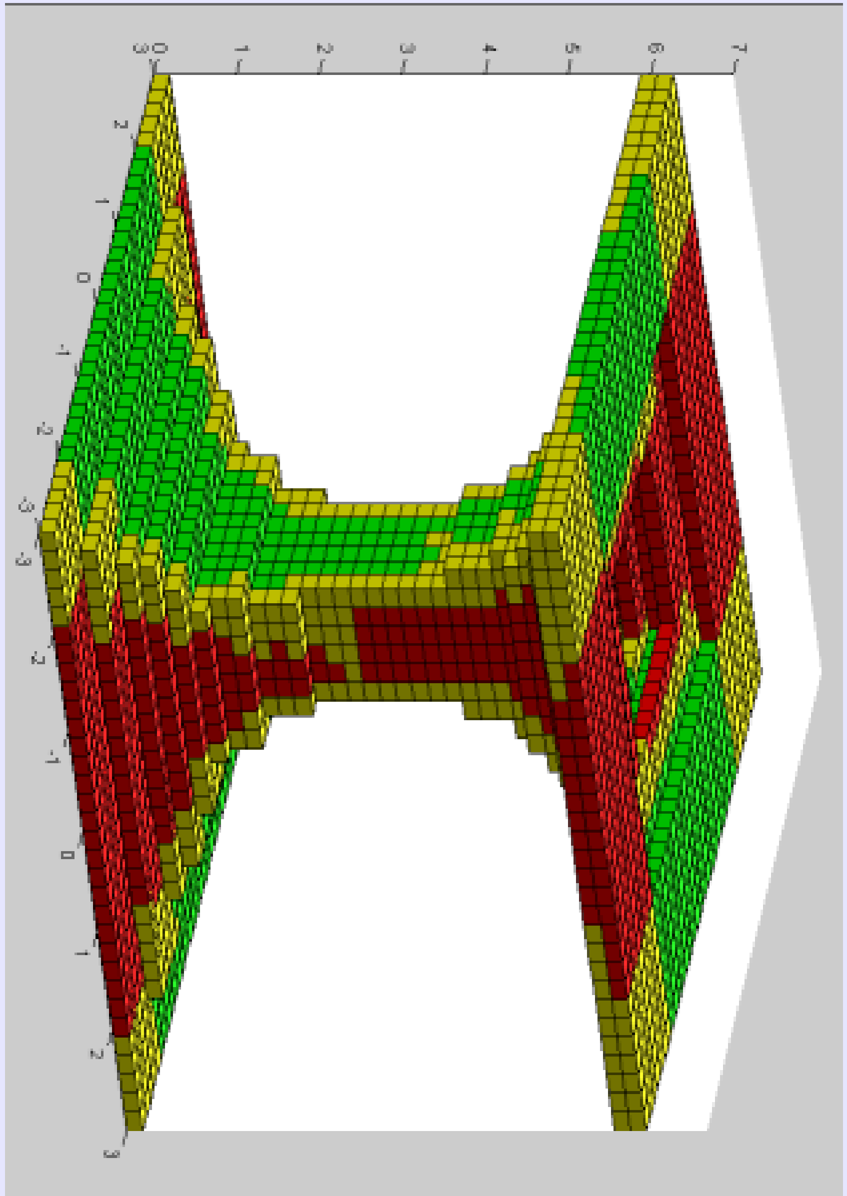
Homological "intermediate sections"

The theorem shows that to find the Conley index of the Poincaré map it is enough to:

- find a candidate for an isolating segment
- verify isolation
- find a sufficiently large subset of the exit set, so that the chains c in the above theorem may be constructed for all homology generators in $H_*(W_0, W_0^*)$.

There is no need to find the whole exit set.

New developments



For $\varphi \in [0.495, 0.5] \cup [0.997, 1.003]$ the Poincaré map of the equation

$$z' = (1 + e^{i\varphi t}|z|^2)\bar{z}.$$

admits a chaotic invariant set, which is semiconjugate to symbolic dynamics on two symbols.

Conclusions

- Strong expansion in a dynamical system does not necessarily mean that rigorous numerics of the system will not be helpful.
- Transfering information to topological level as soon as possible may be extremly helpful in solving problems, where other approaches fail becasue of rapid growth of error estimates
- The presented methods may be applied not only to Poincaré maps in time periodic non-autonomous differential equations, but also to Poincaré maps in autonomous equations and t translations.