Computer-Assited Proof of the Stability of the Eight

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N-body Problem

The Newton's equation for the movement of N particles

$$m_i \ddot{q}_i = \sum_{j \neq i} \frac{G m_i m_j (q_j - q_i)}{\|q_i - q_j\|^3}$$

where

- $q_i \in \mathbf{R}^n$,
- $i, j = 1, ..., N$,
- $G = 6.6732 \times 10^{-11} m^3 / s^2 kg$.

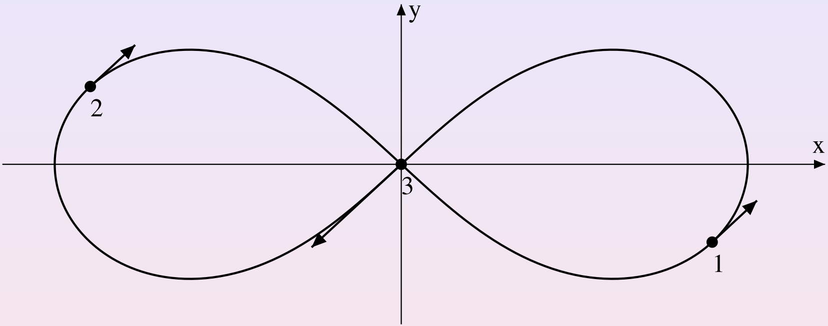
N-body Problem

The Newton's equation for the movement of N particles

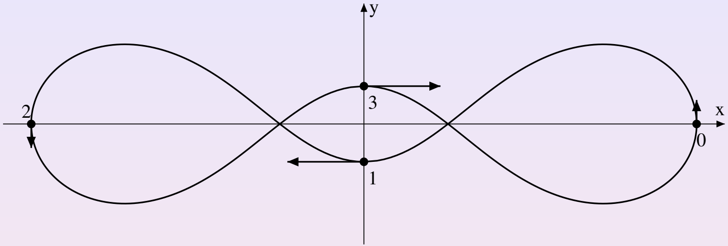
$$m_i\ddot{q}_i = \sum_{j\neq i} \frac{Gm_im_j(q_j - q_i)}{\|q_i - q_j\|^3}$$

In the following we assume that:

- $G = 1$
- $m_i = 1$
- $n = 2 \implies q_i = (x_i, y_i),\ \dot{q}_i = p_i = (\dot{x}_i, \dot{y}_i)$

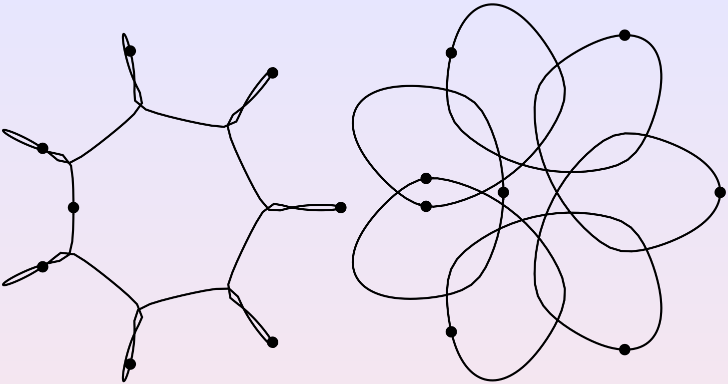


The Eight - famous eight shaped orbit



Gerver's orbit - Super Eight

Choreographies: starting from 2000 - C. Simó found numerically a lot of choreographies for various N,

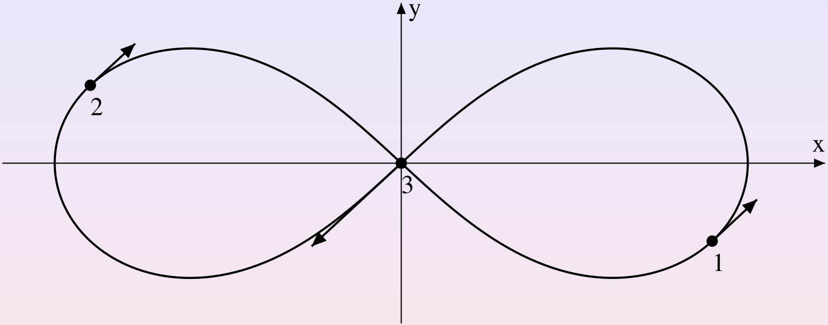


8 bodies choreographies

Definition

By a *choreography* we mean a collision-free solution of the N-body problem in which all masses move on the same curve with a constant phase shift.

Stability of the Eight - Outline of the method



- Rigorous estimates for initial conditions
- Computing the Eight monodromy matrix $\frac{\partial \varphi}{\partial x}(Z, T)$.
- Estimation of the eigenvalues of the matrix $\frac{\partial \varphi}{\partial x}(Z, T)$.

Rigorous estimates of initial conditions

$$\mathbf{G}(\mathbf{x}) = \sigma\varphi\left(\frac{\mathbf{T}}{\mathbf{N}}, \mathbf{x}\right)$$

$x = (q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots, q_N, \dot{q}_N)$

$\varphi(\mathbf{t}, \mathbf{x})$ - flow generated by N-body equation.

$\sigma(\mathbf{x})$ cyclical shift of the particles $q_1 \to q_2 \to \cdots \to q_N \to q_1$

$$\mathbf{F}(\mathbf{x}) = \mathbf{G}(\mathbf{x}) - \mathbf{x}$$

$F(x_0) = 0 \Leftrightarrow G(x_0) = x_0 \Leftrightarrow$

$\Leftrightarrow x_0$ **initial condition for some choreography.**

Interval Krawczyk Method

- $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a C^1 function,
- $X \subset \mathbb{R}^n$ is an interval set, $\bar{x} \in X$
- $C \in \mathbb{R}^{n \times n}$ is a linear isomorphism.

Krawczyk operator

$$K(\bar{x}, X, F) := \bar{x} - CF(\bar{x}) + (Id - C[DF(X)])(X - \bar{x});$$

Krawczyk Theorem

With the assumptions introduced above, the following holds:

1. If $x^* \in X$ and $F(x^*) = 0$, then $x^* \in K(\bar{x}, X, F)$.
2. If $K(\bar{x}, X, F) \subset intX$, then there exists a unique $x^* \in X$ such that $F(x^*) = 0$.

Linear stability of the Eight

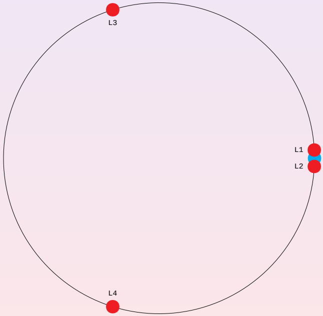
- In the case of Eight $\frac{\partial \varphi}{\partial x}(x_0, T)$ is an 8×8 matrix.
- At least 4 eigenvalues are equal 1
  (they correspond to the first integrals of the N-body equation).
- We will show that remaining "relevant" eigenvalues are on the unit circle.

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ denote "relevant" eigenvalues.

Computational challenge

Non-rigorous estimations of eigenvalues (C. Simó):

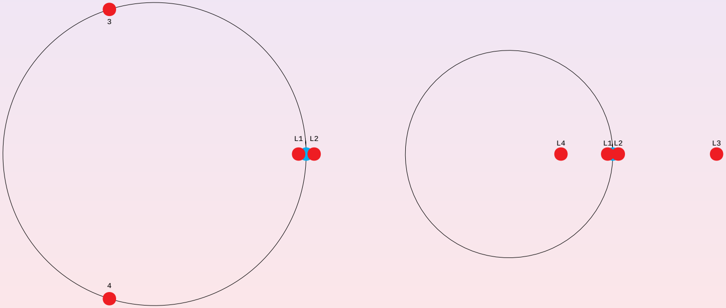
$$\lambda_{1,2} \approx 0.99859998 \pm 0.05289683i,$$
$$\lambda_{3,4} \approx -0.29759667 \pm 0.95469169i,$$



Possible cases

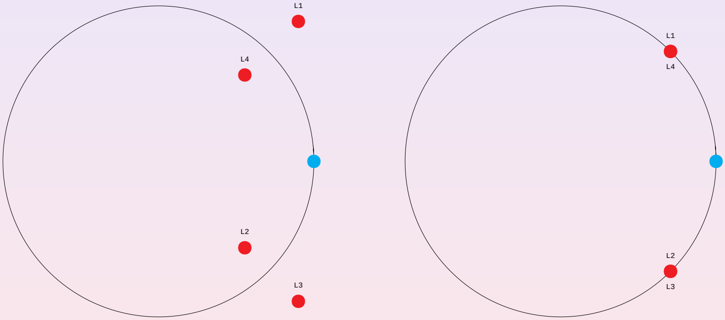
If λ is eigenvalue of simplectic matrix A then also $\bar{\lambda}$, λ^{-1}, $\bar{\lambda}^{-1}$ are eigenvalues of A.

(S1) some of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are real and $\lambda_1\lambda_2 = 1$, $\lambda_3\lambda_4 = 1$,



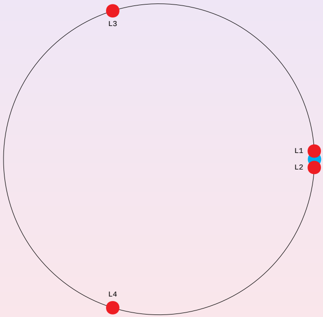
Possible cases

(S2) $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are not real and $\lambda_1 = \lambda_2^{-1} = \bar{\lambda}_3 = \bar{\lambda}_4^{-1}$,



Possible cases

(S3) $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are different, not real and $\lambda_1 = \lambda_2^{-1} = \bar{\lambda}_2$, $\lambda_3 = \lambda_4^{-1} = \bar{\lambda}_4$,



Characteristic polynomial

In the all above situations the characteristic polynomial is in the form:

$$
\begin{aligned}
&(\lambda - 1)^4(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) \\
&= \lambda^8 - (T_1 + T_2 + 4)\lambda^7 + (T_1 T_2 + 4(T_1 + T_2) + 8)\lambda^6 + \dots
\end{aligned}
$$

where $T_1 = \lambda_1 + \lambda_2$, $T_2 = \lambda_3 + \lambda_4$.

Characteristic polynomial

For any 8×8 matrix $A = (a_{ij})$ we have

$$det(A - \lambda I) \;=\; \lambda^8 - \alpha\lambda^7 + \beta\lambda^6 + \dots \tag{1}$$

where

$$\alpha = \text{ trace } (A) = \sum_{i=1}^{8} a_{ii},$$

$$\beta = \sum_{1 \leq i < j \leq 8} (a_{ii}a_{jj} - a_{ij}a_{ji}).$$

Characteristic polynomial

From equations

$$\begin{aligned} & det(A - \lambda I) \\ = \ & \lambda^8 - \alpha\lambda^7 + \beta\lambda^6 + \dots \\ = \ & \lambda^8 - (T_1 + T_2 + 4)\lambda^7 + (T_1 T_2 + 4(T_1 + T_2) + 8)\lambda^6 + \dots \end{aligned}$$

we get that T_1, T_2 are solution to:

$$T^2 - (\alpha - 4)T + \beta - 4\alpha + 8 = 0.$$

Main Theorem

Theorem

Let A be symplectic matrix having at least four eigenvalues equal to 1. Let T_1, T_2 be solutions of the equation

$$T^2 - (\alpha - 4)T + \beta - 4\alpha + 8 = 0.$$

If

$$\Delta = (\alpha - 4)^2 - 4(\beta - 4\alpha + 8) > 0, \tag{2}$$

$$|T_1| < 2, |T_2| < 2, \tag{3}$$

then all eigenvalues of matrix A belong to unit circle.

Method summary

For well chosen set X and $\bar{x} \in X$ we:

- check that $Z := K(\bar{x}, X, F) \subset intX$ (computation of $F(\bar{x}), [DF(X)]$)
- compute $A = \frac{\partial \varphi}{\partial x}(Z, T)$
- check that
  - $\Delta = (\alpha - 4)^2 - 4(\beta - 4\alpha + 8) > 0$
  - $|T_1| < 2, |T_2| < 2$

Rigorous Computations

We use:

- CAPD package: C^n Lohner algorithm, set representation,
- MPFR package: multi-precision floating points numbers with correct rounding,
- MPFR++ - C++ envelope class for MPFR.

Rigorous Computations

Set size $2 \cdot 10^{-9}$			
	double	MP 53	MP 100
F(x0)	4.584e-12	4.223e-12	3.013e-26
F(X)	1.413e-3	1.41299e-3	1.41291e-3
K(x0,X,F)	2.643e-10	2.537E-10	1.291e-10
time :	33.4 sec	914 sec	999 sec

Table: Comparison of results for various precision.

Rigorous Computations

Set size $2 \cdot 10^{-12}$			
	double	MP 56	MP 100
F(x0)	4.584e-12	5.273E-13	3.013e-26
diam F(X)	1.502e-06	1.423E-6	1.412E-6
K(x0,X,F)	1.352e-10	1.556E-11	1.291E-16
time:	32.9 sec	922 sec	999 sec

Table: Comparison of results for various precision.

Rigorous Computations

Set size $2 \cdot 10^{-12}$			
	MP 100	MP 200	MP 400
F(x0)	3.013e-26	6.451E-31	6.451E-31
diam F(X)	1.412E-6	1.412E-6	1.412E-6
K(x0,X,F)	1.291E-16	1.291E-16	1.291E-16
time:	999 sec	1041 sec	1150 sec

Table: Comparison of results for various precision.

Theorem

All eigenvalues of the Eight monodromy matrix belong to unit circle.

Initial conditions	
set size Z	1.29e-16
precision	100 mantissa bits
Results of computation	
Δ	6.720^{458965}_{547070}
T_1	1.997^{195667}_{204261}
T_2	-0.5951^{97631}_{89038}

Summary

- we can verify numeric simulations
- having rigorous estimates for initial conditions we can proof additional properties of the given orbit: stability, symmetries, ...
- good estimates needed:
  - multiprecision interval arithmetics
  - set representation: Taylor Models
- full stability $\rightarrow$ normal forms calculations