Verified Enclosure of Invariant Manifolds of Planar Diffeomorphisms and Application to Homoclinic Phenomena

Johannes Grote, Kyoko Makino,
Martin Berz, Sheldon Newhouse

Dept. of Physics and Astronomy and Dept. of Mathematics,
Michigan State University, East Lansing, MI 48824, USA

Taylor Model Workshop, Boca Raton, FL
December 19, 2006

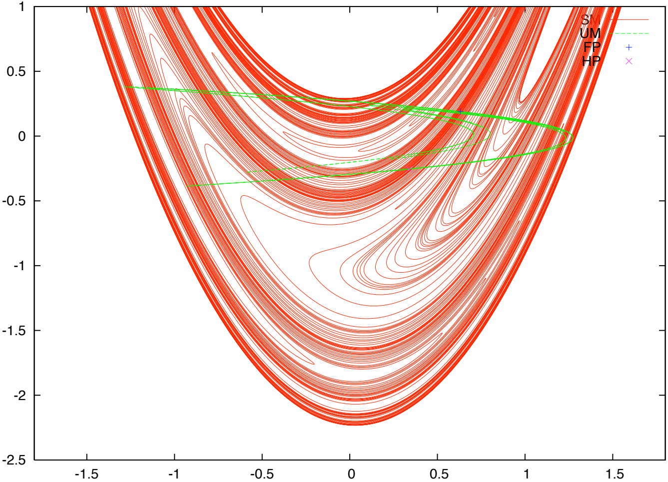
Some tidbits about invariant manifolds:

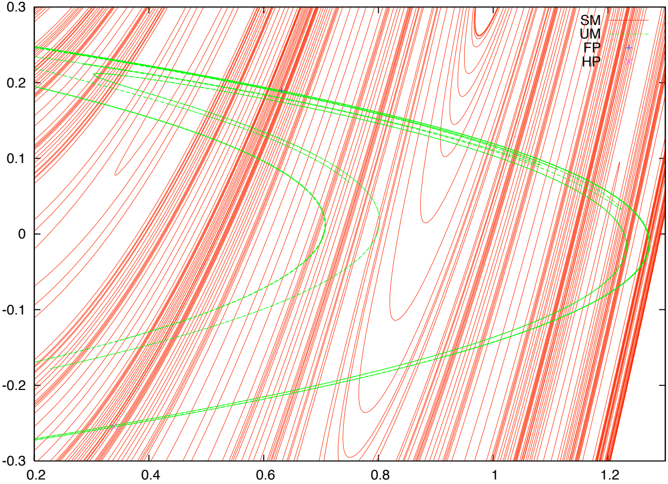
- under quite general assumptions, existence of local invariant manifolds near hyperbolix fixed (periodic) points is guaranteed
- invariant manifolds are global objects, the global manifolds are obtained as images/preimages of the local manifolds
- govern the long-term behavior of the system
- possible applications: spacecraft mission design, stability of nuclear fusion reactors

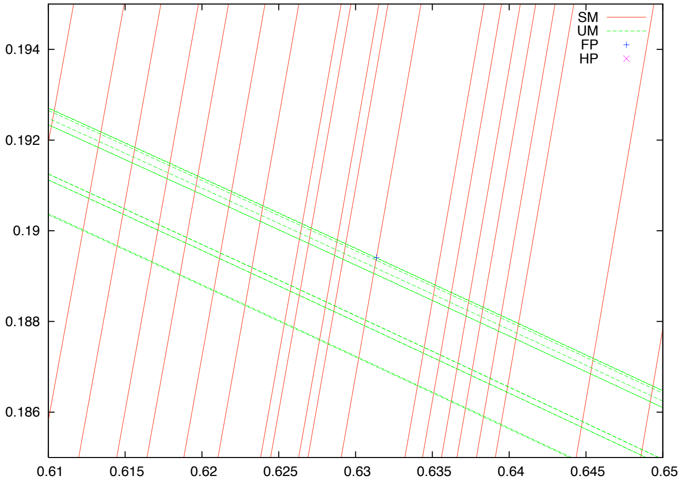
In the further discussion, we consider the Henon map

$$\mathcal{H}(x,y) = \mathcal{H}_{a,b}(x,y) = \begin{pmatrix} 1 + y - a \cdot x^2 \\ b \cdot x \end{pmatrix}$$

- it has a hyperbolic fixed point at $\approx (0.63135, 0.18940)$
- it has a hyperbolic fixed point at $\approx (0.33885, -0.25511)$
- the determinant of the Jacobian is $-b$


SM
UM
FP
HP
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-1.5
-1
-0.5
0
0.5
1
1.5


0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2
SM
UM
FP
HP


SM
UM
FP
HP
0.194
0.192
0.19
0.188
0.186
0.61
0.615
0.62
0.625
0.63
0.635
0.64
0.645
0.65

- dynamical invariant
- measure for orbit complexity
- if positive, existence of homoclinic points and chaotic structure (exponentially growing number of periodic points etc.)
- find lower bound by finding symbolic dynamics in original dynamical system by finding regions that overlap under iteration $\Longrightarrow$ find incidence matrices, associated SFTs, lower bound for entropy by log of spectral radius

Strategy:

- find nonverified polynomial approximation of local manifolds near hyperbolic fixed point, using DA
- heuristically outfit polynomial with remainder bounds to obtain a TM-enclosure of the local manifold
- obtain enclosures of significant parts of the global manifolds as iterated images/preimages of the local manifold enclosures

Various techniques exist to obtain local polynomial parametrizations of manifold, all more or less normal form transformation based.

1. Hubbard's method

- consider a hyperbolic fixed point x_0
- let v_u, v_s be the eigenvectors to the un/stable eigenvalues λ_u and λ_s at x_0
- consider test functions

$$\gamma_n^u(t) := \mathcal{H}^n(x_0 + \frac{t}{\lambda_u^n}) \cdot v_u$$
$$\gamma_n^s(t) := \mathcal{H}^{-n}(x_0 + t \cdot \lambda_s^n \cdot v_s$$

- Thm.(Hubbard): the functions γ_n^u and γ_n^s converge to the true unstable manifolds W^u and W^s around x_0

2. Formulation as operator equation

Consider the unstable manifold for now (stable one works analogously for the preimage), near the fixed point x_0 with unstable eigenvalue λ_u.

A parametrization $\gamma(t)$ for the true unstable manifold W^u satisfies

$$\mathcal{H}(\gamma(t)) = \gamma(\lambda_u \cdot t)$$

This can be transformed into an operator equation for the coefficients of $\gamma(t)$ in DA-arithmetic:

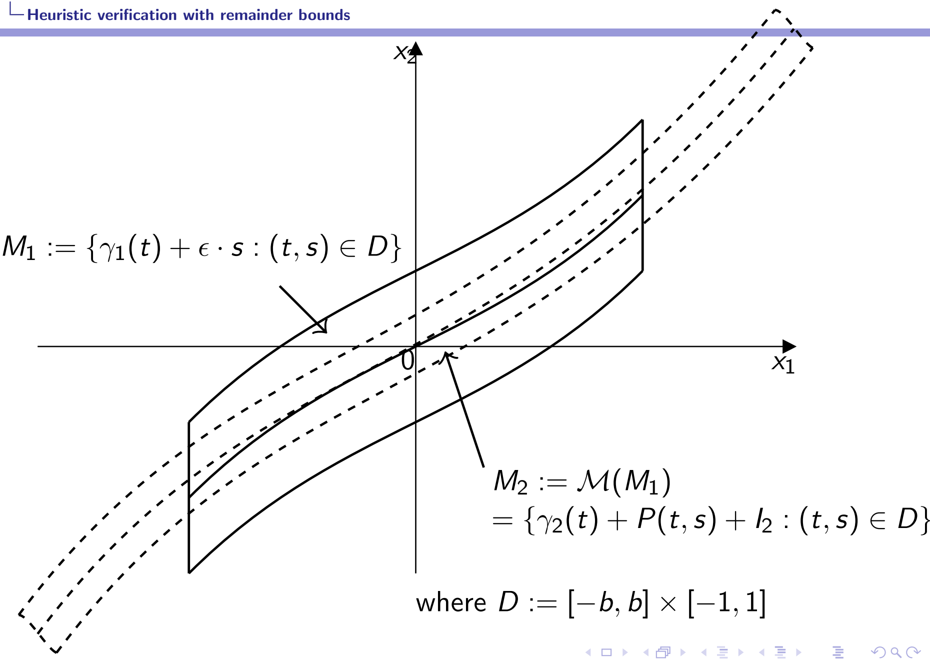
$$\gamma_n = -(H - \lambda_u^n \cdot I)^{-1} R_n,$$

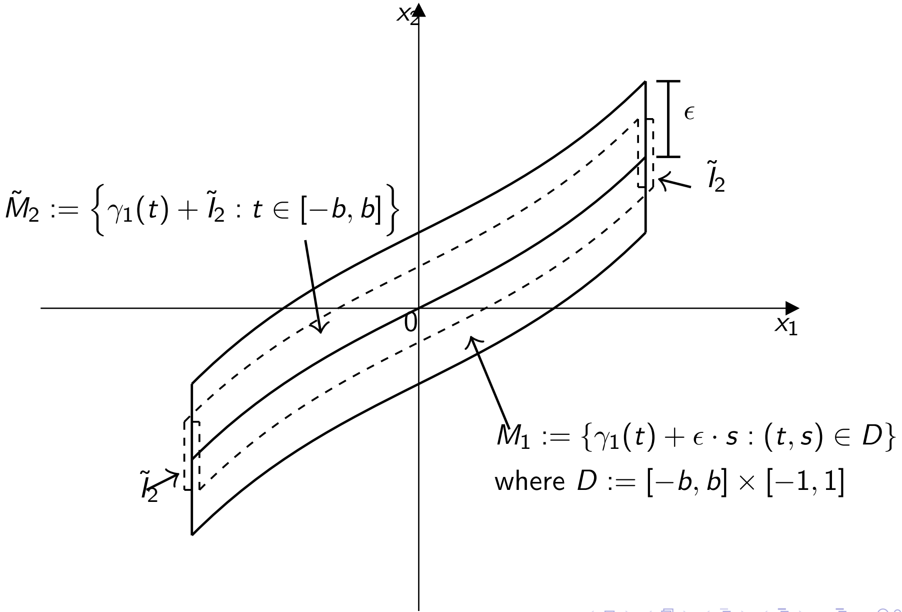
where $\mathcal{H} = H + \mathcal{N}$, and R_n is the n-th order part of $\mathcal{N}$

3. Complete normal form transformation

Under certain nonresonance assumptions, perform a NFT of $\mathcal{H}$ around x_0, s.t. in new coordinates $\mathcal{H}$ is fully linearized.

- find the NFT ψ s.t. $\psi^{-1} \circ \mathcal{H} \circ \psi(x) = \begin{pmatrix} \lambda_u & 0 \\ 0 & \lambda_s \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$
- in this picture, $W^u_{NFT} = \mathbb{R} \times 0$ and $W^s_{NFT} = 0 \times \mathbb{R}$
- obtain $W^u = \psi(W^u_{NFT})$ and $W^s = \psi(W^s_{NFT})$ in original coordinates


x_2
$M_1 := \{\gamma_1(t) + \epsilon \cdot s : (t, s) \in D\}$
0
x_1
$M_2 := \mathcal{M}(M_1)$
$= \{\gamma_2(t) + P(t, s) + I_2 : (t, s) \in D\}$
where $D := [-b, b] \times [-1, 1]$

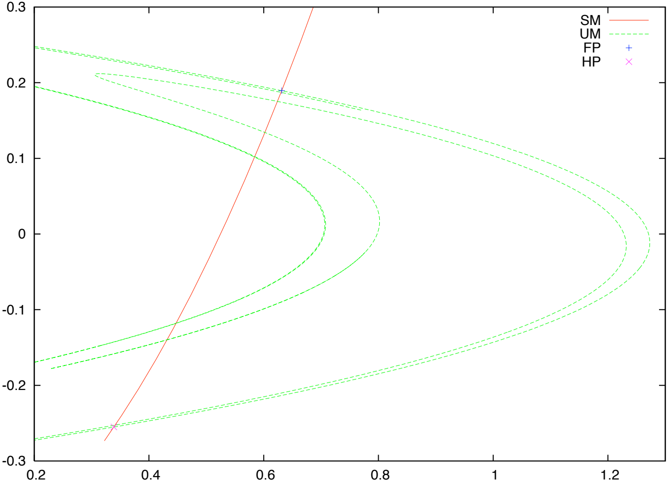

x_2
x_1
0
ϵ
$\tilde{I}_2$
$\tilde{M}_2 := \left\{\gamma_1(t) + \tilde{I}_2 : t \in [-b, b]\right\}$
$M_1 := \{\gamma_1(t) + \epsilon \cdot s : (t, s) \in D\}$
where $D := [-b, b] \times [-1, 1]$
$\tilde{I}_2$

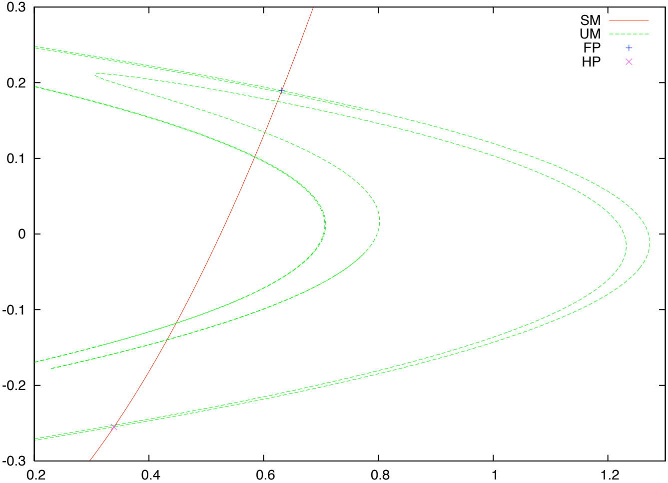
General idea: easy. TM-enclosure of local manifold will iteratively yield TM-enclosure of global manifolds, if images/preimages are computed in TM-arithmetic.
In practice, there are problems:

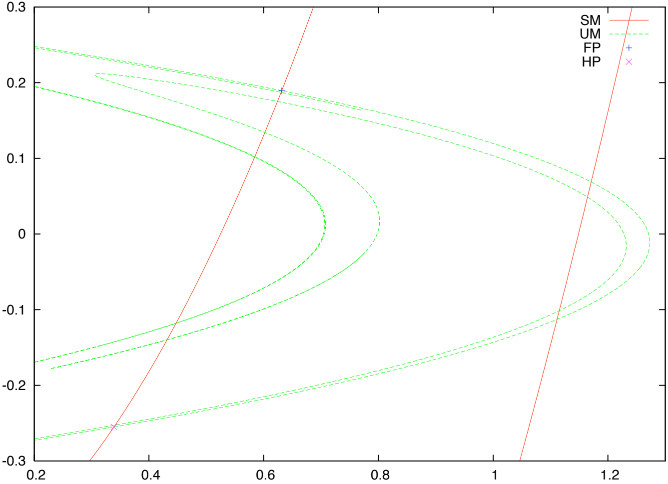
- blow-up of remainder bounds through strong expansion (Lipschitz constant of maps) $\Longrightarrow$ curves grow exponentially in length
- blow-up of remainder bound because manifolds take 'sharp turns' $\Longrightarrow$ challenging polynomial approximation

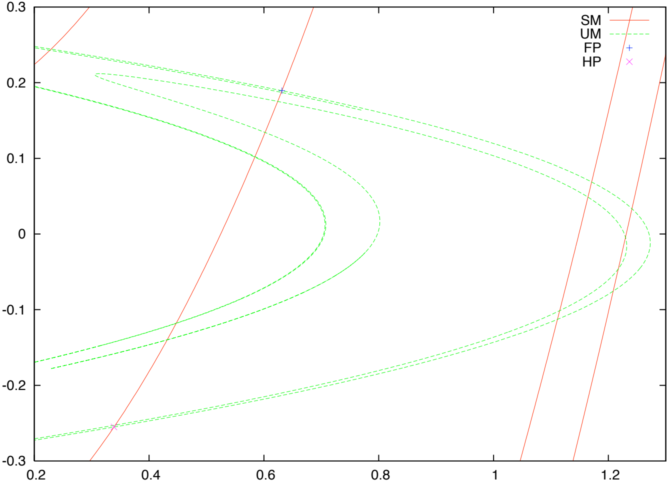
Solution: Step-size control/Dynamic Domain Decomposition

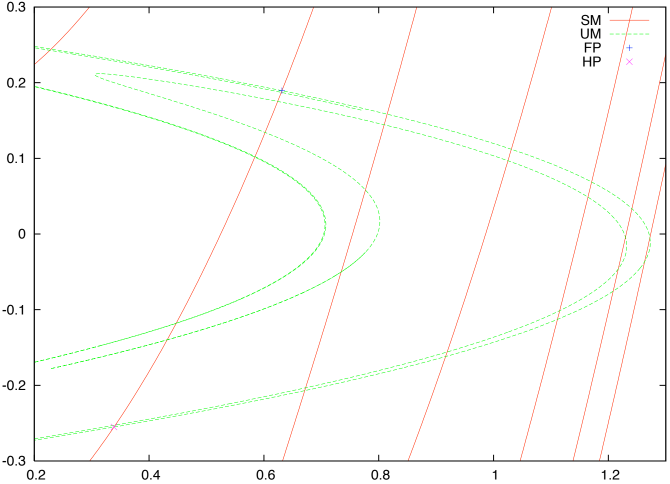
- set target remainder bound size δ
- if image of curve piece has remainder bound bigger than δ, go back, bisect the original curve piece, and repeat remainder bound check on the two images. Repeat as necessary until both new remainder bounds are smaller than δ
- $\Longrightarrow$ in every iteration we obtain an ordered list of Taylor Model curve pieces, enclosing the true manifold to within δ

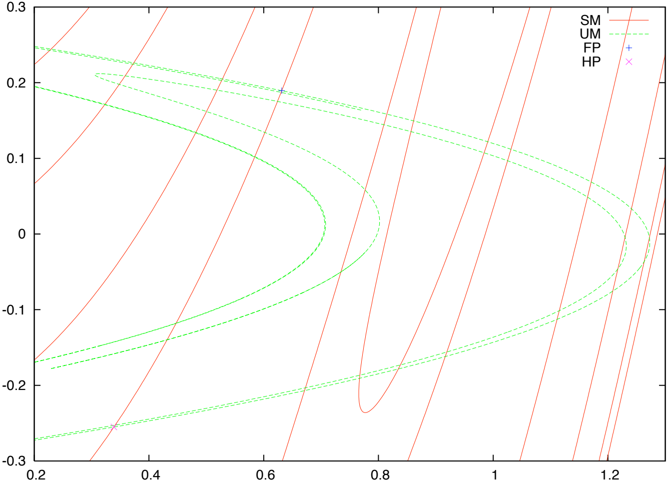

SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2

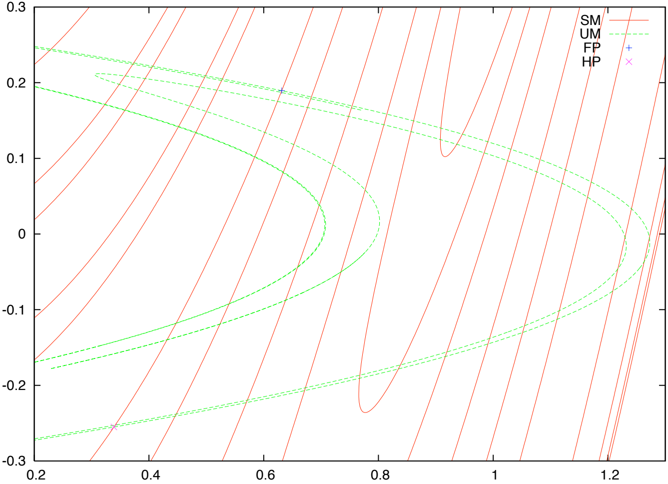

SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2

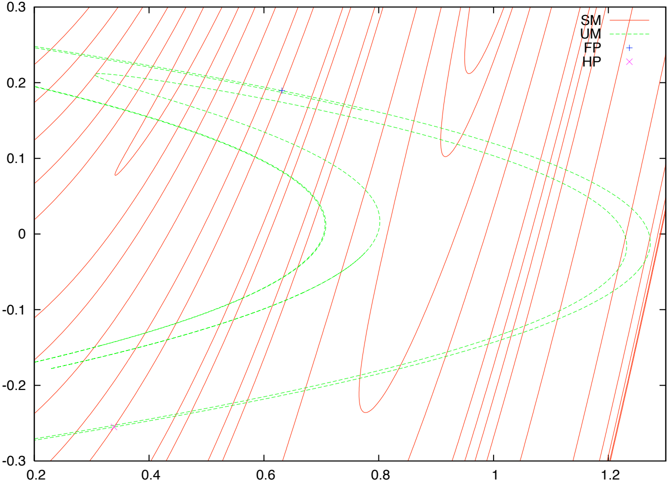

SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2

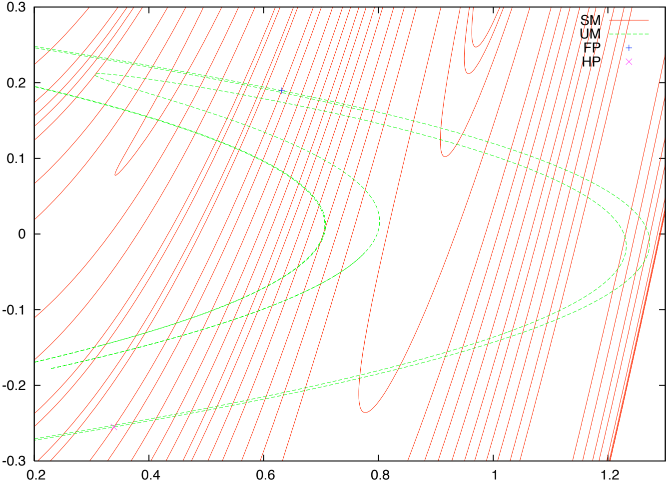

SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2

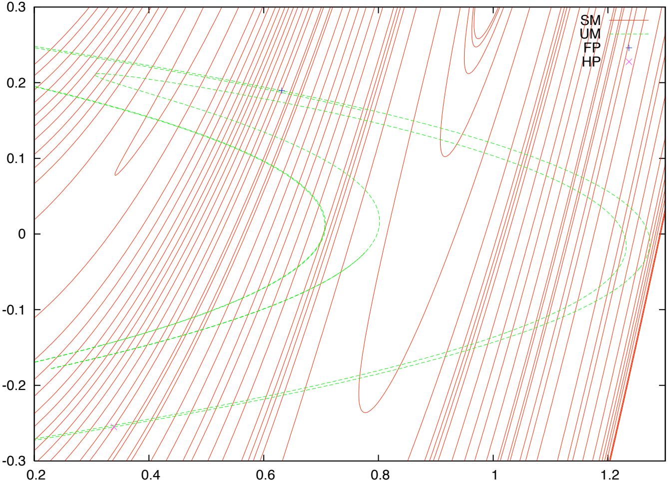

SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2

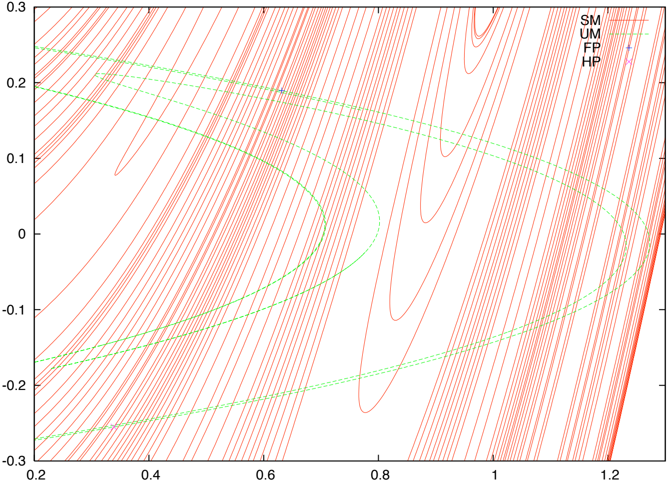

SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2

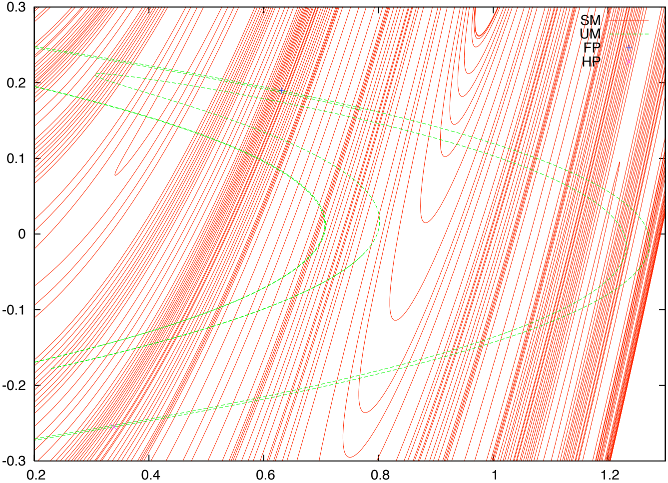

SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2


SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2


SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2

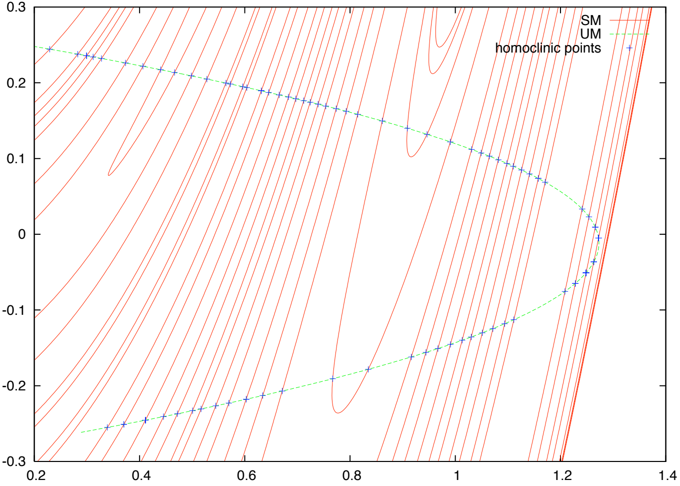

SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2

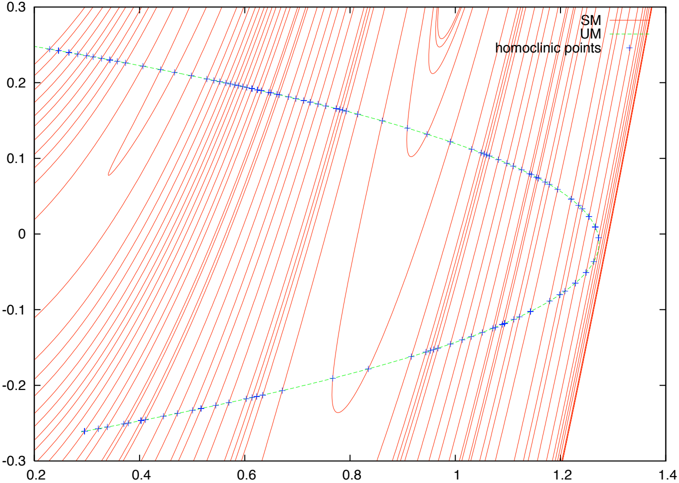

0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2
SM
UM
FP
HP

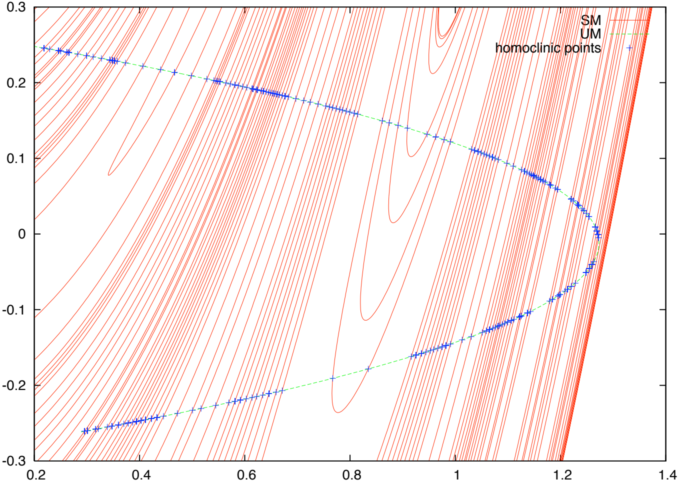

SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2

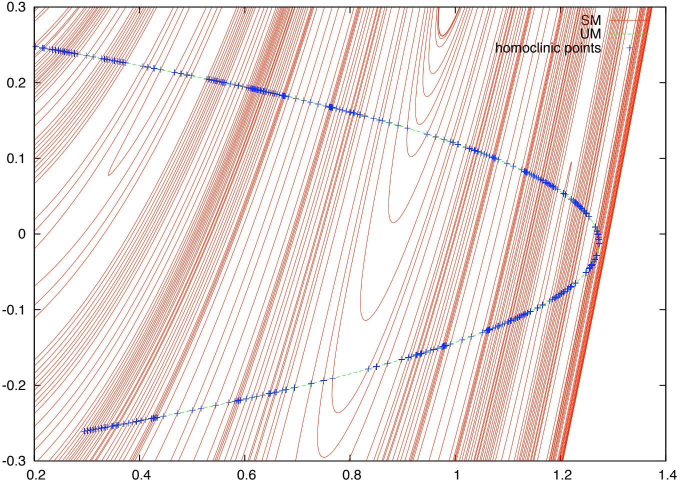
First observe: a transverse intersection of TM-pieces of the list contains (at least) one true homoclinic point.

- we can formulate the search for homoclinic points as a global optimization problem
- assume you have lists $L_u := \left\{\gamma_j^u(t),\, 1 \leq j \leq N_u, t \in [-1,1]\right\}$ and $L_s := \{\gamma_k^s(s),\, 1 \leq j \leq N_s, s \in [-1,1]\}$ of N_u and N_s TM-pieces enclosing finite pieces of W^u and W^s . Assume these TM-pieces to be parametrized in one longitudinal variable.
- set up a global minimization of a function of the type $\left|\gamma_j^u - \gamma_k^s\right|^2$ over the domain $[0, 2 \cdot N_u] \times [0, 2 \cdot N_s]$
- the remaining boxes have are candidates to contain homoclinic points


SM
UM
homoclinic points
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2
1.4


SM
UM
homoclinic points
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2
1.4


SM
UM
homoclinic points
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2
1.4


SM
UM
homoclinic points
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2
1.4

Some subtle intricacies connected to the homoclinic point search:

- the accuracy of the GO is limited. We can resolve boxes of size 10^{-5} in the parameter space, hence we get box enclosures of the HPs in phase space of size much bigger than the remainder bounds. We want box enclosures of the HPs not significantly bigger than the remainder bounds
- we will not only pick up transverse HPs, but alse homoclinic tangencies or near-tangencies
- we cannot guarantee that there is one and only one transverse HP in the box

Recall:

- we wish to compute lower bounds of the topological entropy $h_{\mathcal{H}}$ of the Henon map $\mathcal{H} = \mathcal{H}_{a,b}$
- find symbolic dynamics by considering regions (curvilinear rectangles) R_j that overlap each other under iteration
- compute incidence matrix A for rectangles that Markov-cross: $A_{ij} := 1 iff \mathcal{H}_{a,b}(R_i) \cap R_j$ Markov , $A_{i,j} := 0$ else
- compute lower bound for $h_{\mathcal{H}}$ as the log of the largest real eigenvalue of A

We have found the sets of homoclinic points. Additionally, we can find

- their order along both stable and unstable manifold
- their 'orientation' (tangent vectors to manifolds t the homoclinic points)
- how they map into each other (image-preimage-pairs of HPs)

This info will enable us to automatically construct curvilinear rectangles with boundaries in the un/stable mfds. and homoclinic points as cornerpoints.

Freestyle!