On Taylor Model Based Integration of ODEs

Markus Neher
Universität Karlsruhe (TH)
Institute for Applied and Numerical Mathematics
76128 Karlsruhe, Germany
markus.neher@math.uni-karlsruhe.de

K. R. Jackson
Computer Science Department
University of Toronto
10 King's College Rd, Toronto, ON, M5S 3G4, Canada
krj@cs.toronto.edu

N. S. Nedialkov
Department of Computing and Software
McMaster University
Hamilton, ON, L8S 4L7, Canada
nedialk@mcmaster.ca

Abstract

The numerical solution of initial value problems (IVPs) for ODEs is one of the fundamental problems in computation. Today, there are many well-established algorithms for solving IVPs. However, traditional integration methods usually provide only approximate values for the solution. Precise error bounds are rarely available. The error estimates, which are sometimes delivered, are not guaranteed to be accurate and are sometimes unreliable.

In contrast, verified integration aims at computing guaranteed bounds for the flow of an ODE, including all discretization and roundoff errors in the computation. Originated by Moore in the 1960s [3], interval computations have become a particularly useful tool for this purpose. Unfortunately, the results of interval arithmetic computations are often impaired by overestimation caused by the dependency problem and by the wrapping effect. In verified integration, overestimation may degrade the computed enclosure of the flow, enforce miniscule step sizes, or even bring about premature abortion of an integration.

Berz and his co-workers have developed Taylor model methods, which combine interval arithmetic with symbolic computations [1, 2]. In Taylor model methods, the basic data type is not a single interval, but a *Taylor model*,

$$\mathcal{U} := p_n(x) + \boldsymbol{i}$$

consisting of a multivariate polynomial $p_n(x)$ of order n in m variables, and a remainder interval $\boldsymbol{i}$. In computations that involve $\mathcal{U}$, the polynomial part is propagated by symbolic calculations wherever possible, and thus hardly affected by the dependency problem or the wrapping effect. Only the interval remainder term and polynomial terms of order higher than n, which are usually small, are bounded using interval arithmetic.

Taylor model arithmetic is an extension of interval arithmetic with a comprehensive variety of applicable enclosure sets. In our talk, we analyze Taylor model methods for the verified integration of ODEs and compare these methods with existing interval methods. Taylor models are better suited for integrating ODEs than interval methods, whenever richness in available enclosure sets and reduction of the dependency problem is an advantage. This is usually the case for IVPs for nonlinear ODEs, especially in combination with large initial sets or with large integration domains.

This advantage is less obvious for linear ODEs, where interval methods should perform equally well. Nevertheless, we concentrate on a comparison of traditional interval methods and Taylor model methods for linear ODEs for two reasons. First, the discussion is simpler for linear ODEs than for nonlinear ones. Second, if Taylor model methods failed on linear ODEs, they would likely fail on nonlinear ODEs as well.

A nonlinear model problem is used to explain preconditioned Taylor model methods for ODEs. Numerical examples for linear and nonlinear ODEs are also given.

References

[1] M. Berz. From Taylor series to Taylor models. In *AIP Conference Proceedings 405*, pages 1–23, 1997.

[2] K. Makino and M. Berz. Taylor models and other validated functional inclusion methods. *Int. J. Pure Appl. Math.*, 4:379–456, 2003.

[3] R. E. Moore. *Interval Analysis.* Prentice Hall, Englewood Cliffs, N.J., 1966.

Keywords: Taylor model methods, verified integration, ODEs, IVPs.